ZASTOSOWANIA MATEMATYKI
APPLICATIONES MATHEMATICAE
XVIII, 2 (1984), p. 211-224

MARIA JANKIEWICZ (Wrocław)

STATIONARY INEQUALITIES FOR DAMS WITH CONTENT-DEPENDENT RELEASE RATE

1. Introduction. Let us consider the content process Z with state space $[0, \infty)$, defined by the stochastic equation

$$Z(t) = z + A(t) - \int_0^t r(Z(u))\,du, \qquad t \geqslant 0, \tag{1}$$

where $z \geqslant 0$ denotes the initial content of the dam, $A(t)$ the total input in the interval $(0, t]$, and $r(x)$ the output rate if the content is x.

Assume that the process A is a compound Poisson process with jump rate $\lambda > 0$ and jump size distribution function (d.f.) H ($H(x) = 0$, $x \leqslant 0$). Assume that the release function r satisfies the following conditions: $r(0) = 0$, r is strictly positive and continuous on $(0, \infty)$, and

$$R(x) = \int_0^x \frac{1}{r(u)}\,du < \infty \qquad \text{for } x \geqslant 0. \tag{2}$$

In this model we are interested in the bounds for the stationary content distribution of the dam. Namely, consider two content processes Z^0, Z, defined by (1), with input processes A^0, A and release functions r^0, r, respectively. Denote by F^0, F the appropriate d.f.'s of the stationary content distributions of the dams and by $\hat{F}^0, \hat{F}$ their second order d.f.'s. For an arbitrary d.f. G of the positive random variable we put

$$\hat{G}(x) = \int_0^x \big(1 - G(u)\big)\,du \Big/ \int_0^\infty \big(1 - G(u)\big)\,du.$$

In Theorems 2 and 4, respectively, we shall give sufficient conditions for the inequalities $F^0(x) \leqslant (\geqslant) F(x)$, $x \geqslant 0$, and $\hat{F}^0(x) \leqslant (\geqslant) \hat{F}(x)$, $x \geqslant 0$, to hold. Next, applying these theorems we show that — under some assumptions concerning, among others, the identity of the first two moments of inputs and of the release functions — the deterministic distribution of inputs is maximal, in a certain sense, among the stationary d.f.'s and the second order stationary d.f.'s (Theorems 6 and 7), and the exponential distribution of inputs is minimal among these d.f.'s but only in the class of IFR input distributions (Theorems 8 and 9).

The results obtained in this paper are an extension of some results given by Harrison [3] (see also [6]) who has considered the model with additive input process and constant release function. We show that Theorem 2 may easily be proved also for the additive input process under the assumption that $r(0+) > 0$.

2. Stationary distribution. The problem of the existence of the stationary d.f. F in the above-defined model has been solved by Harrison and Resnick [4] who have considered even a somewhat more general model. In [4] the construction of the Markov process Z (for the proof of the Markov property see [2]) satisfying equation (1) has been shown and the necessary and sufficient condition for the existence of the stationary density of this process and also its explicit form have been given. For our purpose we express the Harrison and Resnick theorem in the form adapted for the stationary d.f. Therefore, we introduce the necessary notation.

Let k_n and K_n $(n \geqslant 1)$ denote the nonnegative functions defined by

$$k_1(x, y) = \lambda\big(1-H(x-y)\big)/r(x), \quad 0 \leqslant y < x, \tag{3}$$

$$k_n(x, y) = \int_y^x k(x, u)k_{n-1}(u, y)\,du, \quad 0 \leqslant y < x,\ n \geqslant 2, \tag{4}$$

$$K_n(x, y) = \int_y^x k_n(u, y)\,du, \quad 0 \leqslant y \leqslant x,\ n \geqslant 1. \tag{5}$$

We put $k(x, y) = k_1(x, y)$ and $K(x, y) = K_1(x, y)$. Furthermore, let

$$k^*(x, y) = \sum_{n=1}^{\infty} k_n(x, y), \quad K^*(x, y) = \sum_{n=1}^{\infty} K_n(x, y),$$

and let k denote the constant given by the formula

$$k = \int_0^{\infty} k^*(x, 0)\,dx. \tag{6}$$

Remark 1. The series $k^*(x, y)$ and $K^*(x, y)$ are convergent because by (2) the following inequalities hold:

$$k_n(x, y) \leqslant \lambda^n \big(R(x)-R(y)\big)^{n-1}/\big(r(x)(n-1)!\big), \quad 0 \leqslant y < x, \tag{7}$$

$$K_n(x, y) \leqslant \lambda^n \big(R(x)-R(y)\big)^{n}/n!, \quad 0 \leqslant y \leqslant x,\ n \geqslant 1. \tag{8}$$

THEOREM 1. *The content process Z defined by* (1) *has a stationary distribution iff $k < \infty$. Then the stationary d.f. F fulfils the equation*

$$F(x) = F(0) + \int_{[0,x)} K(x, u)\,dF(u), \quad x \geqslant 0, \tag{9}$$

which has a unique solution of the form

$$F(x) = F(0) + F(0)K^*(x, 0), \quad x \geqslant 0, \tag{10}$$

where $F(0) = 1/(1+k)$.

The proof of the theorem is an adaptation of the proof of Theorem 1 in paper [4] with application of Proposition 6 of that paper.

Remark 2. If (2) is not fulfilled, then there exist stationary distributions being not of the form (10). For example, in the case $r(x) = x$ and $H(x) = 1 - \exp(-\mu x)$, $x \geqslant 0$, we have $R(x) = \infty$, $k = \infty$, and the stationary distribution exists and the Laplace-Stieltjes transform of this distribution can be obtained (see [1] or [5]).

3. First order bounds. In the next part of the paper we suppose that the assumptions of Theorem 1 are fulfilled and let F be the stationary d.f. defined by (10). Define a chain of functions $\{F_n, n \geqslant 1\}$ of the form

$$F_{n+1}(x) = F(0) + \int_{[0,x)} K(x, u)\,dF_n(u), \quad x \geqslant 0,\ n \geqslant 0, \tag{11}$$

where F_0 is a given, nonnegative and nondecreasing function, and $F(0) = 1/(1+k)$. It is easy to notice that the functions F_n are also nonnegative and nondecreasing, and $F_n(0) = F(0)$. Iterating (11) we have

$$F_n(x) = F(0) + F(0)\sum_{j=1}^{n} K_j(x, 0) + \int_{(0,x)} K_n(x, u)\,dF_0(u). \tag{12}$$

Taking in (12) the limit with $n \to \infty$, using (8) and the bounded convergence theorem, we obtain

$$\lim_{n\to\infty} F_n(x) = F(x), \quad x \geqslant 0, \tag{13}$$

for an arbitrary function F_0.

Assume now that F_0 is of the form

$$F_0(x) = F(0) + F(0)\sum_{n=1}^{\infty} K_n^0(x, 0), \quad x \geqslant 0, \tag{14}$$

where the functions K_n^0 are defined by (4) and (5) with upper index zero and $k_1^0 = k^0$ is some nonnegative function.

Remark 3. The obvious candidate for F_0 of the form (14) is the stationary d.f. F^0 of the content process Z^0 defined by (1) with the input process A^0 and release function r^0. If A^0 is a compound Poisson process with jump rate λ^0 and jump size d.f. H^0, then

$$k^0(x, y) = \lambda^0\big(1-H^0(x-y)\big)/r^0(x).$$

Furthermore, if

$$k^0 = \int_0^\infty \sum_{n=1}^\infty k_n^0(u, 0)\,du = k,$$

then $F^0 = F_0$. However, in the following theorem, stating the monotone convergence of the chain $\{F_n, n \geqslant 1\}$ to F, the form of the function k^0 is not important.

THEOREM 2. *Assume that the function F_0 is of the form* (14), *the release rate r does not decrease, and the following condition is fulfilled*:

$$K^0(x, y) \leqslant (\geqslant) K(x, y) \quad \textit{for } 0 \leqslant y \leqslant x. \tag{15}$$

Then the chain $\{F_n, n \geqslant 1\}$ converges monotonically to F and

$$F_n(x) \nearrow (\searrow) F(x), \quad x \geqslant 0.$$

We precede the proof of the theorem by the following

LEMMA. *If the release rate r does not decrease, then for arbitrary $n = 1, 2, \ldots$, $x \geqslant 0$, the functions $K_n(x, y)$ defined by* (3)-(5) *are nonincreasing with respect to y.*

Proof. We show that for every fixed x the function

$$K(x, y) = \lambda \int_y^x \big[\big(1-H(u-y)\big)/r(u)\big]\,du$$

is nonincreasing. Let $0 \leqslant y_1 < y$. Since r is nondecreasing, we obtain

$$\begin{aligned} K(x, y_1) &= \lambda \int_y^{x+(y-y_1)} \big[\big(1-H(u-y)\big)/r\big(u-(y-y_1)\big)\big]\,du \\ &\geqslant \lambda \int_y^{x+(y-y_1)} \big[\big(1-H(u-y)\big)/r(u)\big]\,du \geqslant K(x, y). \end{aligned}$$

Assume by induction that $K_j(x, y)$, $j = 2, 3, \ldots, n$, are nonincreasing functions with respect to y. Using (3)-(5) we obtain

$$\begin{aligned} K_{n+1}(x, y_1) &= \lambda \int_{y_1}^x \big[K_n(x, u)\big(1-H(u-y_1)\big)/r(u)\big]\,du \\ &= \lambda \int_y^{x+(y-y_1)} \big[K_n\big(x, u-(y-y_1)\big)\big(1-H(u-y)\big)/r\big(u-(y-y_1)\big)\big]\,du \\ &\geqslant \lambda \int_y^{x+(y-y_1)} \big[K_n(x, u)\big(1-H(u-y)\big)/r(u)\big]\,du \geqslant K_{n+1}(x, y). \end{aligned}$$

Proof of Theorem 2. We prove the version without brackets. It is not difficult to see that the function F_0 of the form (14) satisfies equation (9) with $K = K^0$. Hence and from (11) we have

$$F_1(x) - F_0(x) = \int\limits_{[0,x)} \big(K(x,u) - K^0(x,u)\big)\,dF_0(u),$$

which by (15) gives $F_1(x) \geqslant F_0(x)$.

Assume by induction that $F_j(x) \geqslant F_{j-1}(x)$, $j = 2, 3, \ldots, n$. Integrating by parts the right-hand side of (11) we obtain

$$F_{n+1}(x) - F_n(x) = \int\limits_{[0,x)} \big(F_{n-1}(u) - F_n(u)\big)\,d_u K(x,u).$$

It follows from the Lemma and the inductive assumption that the right-hand side of the last inequality is nonnegative, which by (13) completes the proof.

Remark 4. By the assumptions of Theorem 2, we can prove directly the inequality $F_0(x) \leqslant (\geqslant) F(x)$ using (10) and (14) because (15) implies the inequality $K_n^0(x,y) \leqslant (\geqslant) K_n(x,y)$, $0 \leqslant y \leqslant x$, $n \geqslant 1$.

4. Second order bounds. Assume that the stationary content distribution with d.f. F satisfying (9) has a finite mean φ. Let $\hat{F}$ be the stationary second order d.f., i.e. let

$$\hat{F}(x) = \frac{1}{\varphi}\int_0^x \big(1 - F(u)\big)\,du, \qquad x \geqslant 0. \tag{16}$$

For the function K (see (3)-(5)) let

$$\hat{K}(x,y) = 1/\big(F(0)\varphi\big)\int_y^x \big(1 - F(0) - K(u,y)\big)\,du, \qquad 0 \leqslant y \leqslant x. \tag{17}$$

THEOREM 3. *The stationary second order d.f. $\hat{F}$ is of the form*

$$\hat{F}(x) = \int\limits_{[0,x)} \hat{K}(x,u)\,dF(u), \qquad x \geqslant 0. \tag{18}$$

Proof. Let $\hat{f}$ denote the density for $\hat{F}$. Formulas (16) and (9) imply the equalities

$$\begin{aligned}
\hat{f}(x) &= \varphi^{-1}\Big(1 - F(0) - \int\limits_{[0,x)} K(x,u)\,dF(u)\Big)\\
&= \big(1 - F(0)\big)/\big(F(0)\varphi\big)\Big(F(0) + \int\limits_{[0,x)} K(x,u)\,dF(u) -\\
&\qquad -1/\big(1 - F(0)\big)\int\limits_{[0,x)} K(x,u)\,dF(u)\Big)\\
&= 1/\big(F(0)\varphi\big)\int\limits_{[0,x)} \big(1 - F(0) - K(x,u)\big)\,dF(u).
\end{aligned}$$

Integrating both sides in the interval $(0, x)$ we obtain (18).

Consider again the chain of functions $\{F_n, n \geqslant 1\}$ defined by (11) and define the chain of functions $\{\hat{F}_n, n \geqslant 1\}$ of the form

$$\hat{F}_n(x) = \varphi^{-1} \int_0^x \big(1-F_n(u)\big)\,du, \qquad x \geqslant 0. \tag{19}$$

Assume that the function F_0 occurring in (11) for $n = 0$ is the d.f. with finite mean equal to φ and that $F_0(0) = F(0)$. Moreover, assume that the second order d.f. $\hat{F}_0$ is of the form

$$\hat{F}_0(x) = \int_{[0,x)} \hat{K}^0(x, u)\,dF_0(u), \qquad x \geqslant 0, \tag{20}$$

where $\hat{K}^0$ is some nonnegative function.

Remark 5. The candidate for $\hat{F}_0$ of the form (20) is the second order d.f. $\hat{F}^0$ of the stationary d.f. F^0 of the content process Z^0 defined by (1) with input process A^0 and release function r^0. If A^0 is a compound Poisson process with jump rate λ^0 and jump size d.f. H^0 and the equalities $k^0 = k$ and $\varphi^0 = \varphi$ are fulfilled, then it follows from Theorem 3 that $\hat{F}^0$ is of the form (20) with the function $\hat{K}^0$ defined by (17), where in place of K we should put

$$K^0(x, y) = \lambda^0 \int_y^x \big[\big(1-H(x-u)\big)/r^0(u)\big]\,du.$$

In the following theorem the form of the function $\hat{K}^0$ is not important.

THEOREM 4. *Assume that the function $\hat{F}^0$ is of the form* (20), *the function r is nondecreasing and $1/r$ is convex, there exists the density h for the d.f. H, and that the following condition is fulfilled:*

$$\hat{K}^0(x, y) \leqslant (\geqslant)\, \hat{K}(x, y) \quad \textit{for } 0 \leqslant y \leqslant x. \tag{21}$$

Then the chain $\{\hat{F}_n, n \geqslant 1\}$ converges monotonically to $\hat{F}$ and

$$\hat{F}_n(x) \nearrow (\searrow)\, \hat{F}(x), \qquad x \geqslant 0.$$

Proof. Using (19) and (11) we can verify the equality

$$\hat{F}_1(x) = \big(1-F(0)\big)\hat{F}_0(x) + F(0) \int_{[0,x)} \hat{K}(x, u)\,dF_0(u).$$

Hence and from (20) and (21) we obtain

$$\hat{F}_1(x) - \hat{F}_0(x) = F(0) \int_{[0,x)} \big(\hat{K}(x, u) - \hat{K}^0(x, u)\big)\,dF_0(u) \geqslant 0.$$

It follows from the existence of the density h that the derivative

$$\frac{\partial}{\partial y}K(x,y) = \lambda\int_y^x [h(u-y)/r(u)]du-\lambda/r(y), \qquad 0<y<x,$$

exists. Let us put

$$a(x,y) = \int_y^x \frac{\partial}{\partial y}K(u,y)du.$$

Using again (19) and (11) and also (17) and integrating by parts we obtain

$$\begin{aligned}\hat{F}_n(x) &= \varphi^{-1}\int_0^x \Big(1-F(0)-K(u,0)-\int_{(0,u)}\big(1-F_{n-1}(s)\big)d_sK(u,s)\Big)du\\ &= F(0)\hat{K}(x,0)-\int_0^x a(x,u)d\hat{F}_{n-1}(u)\\ &= F(0)\hat{K}(x,0)+\int_0^x \hat{F}_{n-1}(u)d_u a(x,u) \qquad \text{for } n\geqslant 2.\end{aligned}$$

By the equality of the extreme expressions we have

$$\hat{F}_{n+1}(x)-\hat{F}_n(x) = \int_0^x \big(\hat{F}_n(u)-\hat{F}_{n-1}(u)\big)d_u a(x,u).$$

Since by the inductive assumption the inequality $\hat{F}_n(u)\geqslant\hat{F}_{n-1}(u)$, $u\geqslant 0$, holds, it is sufficient to show that for fixed $x>0$ the function $a(x,u)$, $0<u\leqslant x$, is nondecreasing.

Define the functions g and G by

$$g(y) = \begin{cases} x-y, & 0\leqslant y\leqslant x,\\ 0, & y>x,\end{cases}$$

$$G(y) = g(y)\frac{1}{r(y)}, \qquad y>0.$$

Since g and $1/r$ are convex, the function G is also convex. Hence, for every $u\geqslant 0$ the function $G(u+y)-G(y)$ is nondecreasing for $y>0$. If h is a density, then the function $\int_0^\infty \big(G(u+y)-G(y)\big)h(u)du$ is also nondecreasing for $y>0$.

From the definition of the functions a and G we obtain the equalities

$$\begin{aligned}a(x,y) &= \lambda\int_y^x [(x-u)/r(u)h(u-y)]du-\lambda(x-y)/r(y)\\ &= \lambda\int_0^\infty \big(G(u+y)-G(y)\big)h(u)du\end{aligned}$$

which imply that the function $a(x,y)$, $0<y\leqslant x$, is nondecreasing. Thus we have shown that $\hat{F}_{n+1}(x)\geqslant\hat{F}_n(x)$, $x\geqslant 0$, $n\geqslant 0$.

It follows from (13) that for the pointwise convergence of $\hat{F}_n$ to $\hat{F}$ it is sufficient to show that there exists a constant $M<\infty$ such that $F_n(x)\leqslant M$ for arbitrary x and n. Since, for fixed n, the function $K_n(x,y)$ is nondecreasing with respect to x, using (12) and the monotone convergence theorem we have

$$\lim_{x\to\infty}F_n(x) = F(0)+F(0)\sum_{j=1}^{n}\int_0^\infty k_j(u,0)\,du+\int_0^\infty K_n(\infty,u)\,dF_0(u). \tag{22}$$

From the Lemma we get the inequality $K_n(x,y)\leqslant K_n(x,0)$ for arbitrary n and x, and hence also $K_n(\infty,y)\leqslant K_n(\infty,0)$. Further, since

$$k = \int_0^\infty\sum_{n=1}^{\infty}k_n(u,0)\,du = \sum_{n=1}^{\infty}\int_0^\infty k_n(u,0)\,du<\infty,$$

taking the limit in (22) with $n\to\infty$ we obtain

$$\lim_{n\to\infty}\lim_{x\to\infty}F_n(x) = 1 = \lim_{x\to\infty}F(x),$$

i.e. the chain $\{F_n(\infty),\,n\geqslant 1\}$ is bounded. This completes the proof.

Remark 6. From the inequalities for the second order d.f.'s we can obtain other useful inequalities. If we denote the random variables with d.f.'s G^0 and G by X^0 and X, respectively, and if we assume $\mathrm{E}X^0=\mathrm{E}X$, then the implication

$$\hat{G}^0(x)\geqslant\hat{G}(x)\;\Rightarrow\;\mathrm{E}g(X^0)\geqslant\mathrm{E}g(X)$$

is true for an arbitrary concave function g (for convex g we have a converse inequality). A simple proof of this implication can be found in [3].

5. Additive input. Consider now the case where the input process A occurring in (1) is a pure-jump Lévy process with nondecreasing, right-continuous sample paths, vanishing at zero. Such a process is called *additive* (see [2] or [5]). The Laplace transform of the random variable $A(t)$ for fixed t is of the form

$$\mathrm{E}\{e^{-sA(t)}\} = \exp\Big\{-t\int_0^\infty(1-e^{-su})\,\nu(du)\Big\},\qquad \mathrm{Re}\,s\geqslant 0, \tag{23}$$

where ν is a σ-finite measure satisfying the condition

$$\int_0^\infty(y\wedge 1)\,\nu(dy)<\infty$$

($a \wedge b = \min(a, b)$). It is well known (see [3]) that

$$\mathrm{E}\{A(t)\} = mt \quad \text{and} \quad \mathrm{D}^2\{A(t)\} = \sigma^2 t,$$

where

$$m = \int_0^\infty u\nu(du) \quad \text{and} \quad \sigma^2 = \int_0^\infty u^2\nu(du).$$

The compound Poisson process is the only one additive process whose measure ν is finite. Next, we suppose that $m < \infty$. Assume that the release function r satisfies the following conditions: $r(0) = 0$, r is strictly positive and continuous on $(0, \infty)$ and also that r is nondecreasing, and $r(0+) > 0$.

Let k_n and K_n ($n \geqslant 1$) be nonnegative functions defined by formulas (4) and (5) in which now $k_1(x, y) = k(x, y) = Q(x-y)/r(x)$, where $Q(x) = \nu(x, \infty)$.

If

$$G_1(x) = \frac{1}{r(0+)} \int_0^x Q(u)\,du, \qquad G_{n+1}(x) = \int_0^x G_1(x-u)\,dG_n(u),$$

then

(24) $$K_n(x, y) \leqslant G_n(x-y), \qquad n \geqslant 1,\ 0 \leqslant y \leqslant x.$$

In [2] it was shown that also for such a model the Markov process Z satisfying (1) can be constructed, and then the assumption $r(0+) > 0$ is not needed. Using the results of [1] we can prove the following theorem, analogous to Theorem 1.

THEOREM 5. *In the dam model with additive input the content process Z has a stationary distribution iff*

(*) $$\sup_{x \geqslant 0} r(x) > m.$$

Then the stationary d.f. fulfils equation (9) *which has a unique solution of the form* (10).

Proof. In [1] it was proved that condition (*) is necessary and sufficient for the existence of the stationary distribution and that it is equivalent to the condition $k < \infty$. Then the stationary density f fulfils the equation

$$f(x) = F(0)Q(x)/r(x) + 1/r(x) \int_0^x Q(x-u)f(u)\,du.$$

Hence, integrating by sides, we obtain (9). Iterating (9) N times we have

$$F(x) = F(0) + F(0)\sum_{n=1}^{N} K_n(x, 0) + \int_{[0,x)} K_{N+1}(x, u)\,dF(u).$$

From (24) and from a well-known fact in renewal theory we infer that $\sum_{n=1}^{\infty} G_n(A)$ is a measure which is finite on bounded Borel sets A. It follows that the series $K^*(x, y)$ is convergent for $0 \leqslant y \leqslant x$. Consequently, by the bounded convergence theorem, we obtain (10). Using the monotone convergence theorem we have

$$F(x) = F(0)+F(0)\int_0^x k^*(u, 0)\,du,$$

whence $k < \infty$.

On the other hand, it is easy to verify that, by the condition $k < \infty$, the d.f. F of the form (10) satisfies (9), which completes the proof.

Remark 7. Obviously, assumption (2) is fulfilled when r is nondecreasing and $r(0+) > 0$.

It is not difficult to show that in the case of additive input the Lemma remains true and one can obtain the proof by a modification of the proof given in Section 3, having in mind the new definition of $k(x, y)$. Then Theorem 2 remains also true and its proof does not change. Only in the proof of the convergence in (13), taking the limit in (12) with $n\to\infty$ we use (24) instead of (8).

Unfortunately, one cannot copy the proof of Theorem 4 because $\lambda = Q(0) = \infty$.

6. Bounds with deterministic jumps. Consider two content processes Z^0 and Z defined by (1). Assume that both input processes A^0 and A are compound Poisson processes with jump rates λ^0 and λ and with jump size d.f.'s H^0 and H ($H^0(x) = H(x) = 0$, $x \leqslant 0$), respectively. Let S denote the random variable with d.f. H. Moreover, assume that $H^0(x) = 0$ for $0 \leqslant x < 1/\mu^0$, $H^0(x) = 1$ for $x \geqslant 1/\mu^0$, $\mathrm{E}S = 1/\mu < \infty$, and that the function r is strictly positive, continuous, and nondecreasing on $(0, \infty)$ $(r(0) = 0)$.

We use again the notation for m, σ^2, and Q introduced in Section 5, taking into account that for a compound Poisson process we have $Q = \lambda(1-H)$. We attach the index zero to all quantities associated with the process Z^0. Applying Theorems 2 and 4 we prove theorems about the inequalities for stationary d.f.'s F^0, F and also for stationary second order d.f.'s $\hat{F}^0$, $\hat{F}$.

First, we put

$$\hat{Q}(x) = m^{-1}\int_0^x Q(u)\,du, \qquad \beta(x, y) = K^0(x, y)-K(x, y),$$

$$\gamma(x, y) = F(0)\varphi\big(\hat{K}^0(x, y)-\hat{K}(x, y)\big),$$

where

$$K^0(x, y) = \lambda^0 \int_y^x \left[\left(1-H^0(x-u)\right)/r^0(u)\right] du$$

and $\hat{K}^0$ is of the form (17) in which we substitute K^0 for K. Since r is nondecreasing, using (3), (5), and (17), we can verify the inequalities

$$(25) \qquad \frac{1}{r(x)} \int_0^{x-y} \left(Q^0(u)-Q(u)\right) du \leqslant \beta(x, y) \leqslant \frac{1}{r(y)} \int_0^{x-y} \left(Q^0(u)-Q(u)\right) du,$$

$$(26) \qquad \frac{1}{r(x)} \int_0^{x-y} \left(\hat{Q}(u)-\hat{Q}^0(u)\right) du \leqslant \gamma(x, y) \leqslant \frac{1}{r(y)} \int_0^{x-y} \left(\hat{Q}(u)-\hat{Q}^0(u)\right) du.$$

THEOREM 6. *If $\lambda^0 = \lambda$, $\mu^0 = \mu$, $r^0 = r$, then for arbitrary y $(0 \leqslant y \leqslant x)$ the inequality $K^0(x, y) \geqslant K(x, y)$ holds true.*

Proof. By assumption we have $\lambda^0 = Q^0(0) = Q(0) = \lambda$ and $Q^0(x) = \lambda$ for $0 \leqslant x < 1/\mu^0$ and $Q^0(x) = 0$ for $x \geqslant 1/\mu^0$, while $Q = \lambda(1-H)$. Hence the integrand in the definition of the function $\beta(x, y)$ is nonnegative for $y \leqslant u < y+1/\mu^0$ and nonpositive for $u \geqslant y+1/\mu^0$. Then, for fixed y, the function $\beta(x, y)$ is nondecreasing for $y \leqslant x < y+1/\mu^0$ and nonincreasing for $x \geqslant y+1/\mu^0$. Moreover,

$$\lim_{x\to y} \beta(x, y) = 0$$

and it suffices to show that

$$\lim_{x\to\infty} \beta(x, y) = 0,$$

which follows from (25) and from the equalities

$$\int_0^\infty Q^0(u)\, du = \lambda^0/\mu^0 = \lambda/\mu = \int_0^\infty Q(u)\, du.$$

Now, we obtain the following obvious corollary:

COROLLARY 1. *Theorem 6 remains true for the function $r^0(x) \leqslant r(x)$, $x \geqslant 0$.*

In order to apply Theorem 2 to the stationary d.f.'s F^0 and F, assume that $F^0(0) = F(0)$, which is equivalent to the equality $k^0 = k$. Thus it follows from Remark 3 that F^0 is of the form (14) and we can formulate the following

COROLLARY 2. *Under the assumptions of Theorem 6 the inequality $F^0(x) \geqslant F(x)$, $x \geqslant 0$, holds true.*

To apply Theorem 4 we assume that the d.f. F^0 has a finite mean φ^0 and $\varphi^0 = \varphi$. Thus it follows from Remark 5 that $\hat{F}^0$ is of the form (20).

THEOREM 7. *Assume that $\sigma^2 < \infty$, the function $1/r$ is convex, and that there exists a density h for the d.f. H. Then, if $\lambda^0 = m^2/\sigma^2$, $\mu^0 = m/\sigma^2$, $r^0 = r$, then the following inequality holds: $\hat{F}^0(x) \geqslant \hat{F}(x)$, $x \geqslant 0$.*

Proof. Using Theorem 4 we show that $\hat{K}^0(x, y) \geqslant \hat{K}(x, y)$. From (17) we obtain

$$\gamma(x, y) = -\int_y^x \beta(u, y)\,du.$$

By the assumption $r^0 = r$ we have

$$\beta(x, y) = \int_y^x \left[\left(Q^0(u-y) - Q(u-y)\right)/r(u)\right]du.$$

Furthermore, we can verify that $\lambda^0 = \lambda(\mathrm{E}S)^2/(\mathrm{E}S^2) < \lambda$, i.e. $Q^0(0) < Q(0)$. Both functions Q^0 and Q are nonincreasing and

$$Q^0(x) = \begin{cases} \lambda^0, & 0 \leqslant x < 1/\mu^0, \\ 0, & x \geqslant 1/\mu^0. \end{cases}$$

Hence Q^0 may cross Q at most twice. If there are no crossings, then we obtain a contradiction because

$$m^0 = \int_0^\infty Q^0(u)\,du < \int_0^\infty Q(u)\,du = m$$

and $m^0 = m$ by assumption. If one crossing occurs, then by the equality

$$m\left(\hat{Q}^0(x) - \hat{Q}(x)\right) = \int_0^x \left(Q^0(u) - Q(u)\right)du$$

we have $\hat{Q}^0(x) \leqslant \hat{Q}(x)$ for all x with strict inequality for some x. This leads to a contradiction with the fact that d.f.'s $\hat{Q}^0$ and $\hat{Q}$ have here the same means equal to $\sigma^2/2m$. Thus there are exactly two crossings, the first one from below and the second from above. Simultaneously, it follows from (25) that

$$\lim_{x\to\infty} \beta(x, y) = 0 = \lim_{x\to y} \beta(x, y).$$

Thus, for fixed y the function $\beta(x, y)$ must cross zero exactly once, from below. Hence $\gamma(x, y)$ for fixed y is first nondecreasing, and then nonincreasing. Moreover, $\gamma(y, y) = 0$ and it suffices to show that

$$\lim_{x\to\infty} \gamma(x, y) = 0,$$

which follows from (26).

Remark 8. It follows from the proof of Theorem 7 that even for the additive input process A the inequality $\hat{K}^0(x, y) \geqslant \hat{K}(x, y)$ holds because then $Q(0) = \infty$.

7. Bounds with exponential jumps. Under the notation of Section 6 we assume that $H^0(x) = 1-\exp\{-\mu^0 x\}$, $x \geqslant 0$, and H belongs to the class IFR, i.e. the function $\log(1-H)$ is concave on $[0, \infty)$ (see [3]). The function r is again strictly positive, continuous, and nondecreasing on $(0, \infty)$ ($r(0) = 0$). In the next theorems we use the fact that the d.f. of the class IFR has a finite second moment and $\mathrm{E}S^2 \leqslant 2(\mathrm{E}S)^2$ with equality holding only for the exponential case.

THEOREM 8. *If $\lambda^0 = \lambda$, $\mu^0 = \mu$, and $r^0 = r$, then for arbitrary y $(0 \leqslant y \leqslant x)$ the inequality $K^0(x, y) \leqslant K(x, y)$ holds true.*

Proof. By assumption we have $Q^0(0) = Q(0)$. Furthermore, the function $\log Q$ is concave and the function $\log Q^0$ is linear. Thus $\log Q$ may cross $\log Q^0$ at most once and from above. The same is true for Q and Q^0 since the log function is monotone. The supposition that there are no crossings leads to a contradiction with the equality $m^0 = m$ following from the assumptions. Thus, for fixed y, the function $\beta(x, y)$ is first nonincreasing, and next nondecreasing. Simultaneously, from (25) we obtain

$$\lim_{x\to\infty} \beta(x, y) = 0 = \lim_{x\to y} \beta(x, y),$$

and thus $\beta(x, y) \leqslant 0$ for $0 \leqslant y \leqslant x$.

COROLLARY 3. *Under the assumptions of Theorem 8, if $k^0 = k$, then the inequality $F^0(x) \leqslant F(x)$, $x \geqslant 0$, holds true.*

THEOREM 9. *Assume that $\sigma^2 < \infty$, $k^0 = k$, $\varphi^0 = \varphi$, the function $1/r$ is convex, and that there exists a density h for the d.f. H. Then, if $\lambda^0 = 2\mu^2/\sigma^2$, $\mu^0 = 2\mu/\sigma^2$, and $r^0 = r$, then the following inequality holds: $\hat{F}^0(x) \leqslant \hat{F}(x)$, $x \geqslant 0$.*

Proof. It is easy to verify that $m^0 = m$ and $\sigma^0 = \sigma$. If the d.f. H is not exponential (for not to have the same model), then $\mathrm{E}S^2 < 2(\mathrm{E}S)^2$, i.e. $Q^0(0) > Q(0)$. Similarly as in the proof of Theorem 7, we can show that Q crosses Q^0 exactly twice. Thus $\beta(x, y)$ for fixed y crosses zero exactly once, from above. Hence, by (26), we obtain $\gamma(x, y) \leqslant 0$, i.e. $\hat{K}^0(x, y) \leqslant \hat{K}(x, y)$ and the theorem follows from Theorem 4.

Remark 9. If $r^0 = r = c$, then in Theorems 7 and 9 and in Corollaries 1 and 2 it is not necessary to assume that $k^0 = k$ and $\varphi^0 = \varphi$ because this follows from the remaining assumptions (see [3]).

References

[1] P. J. Brockwell, *Stationary distributions for dams with additive input and content-dependent release rate*, Advances in Appl. Probability 9 (1977), p. 645-663.

[2] E. Çinlar and M. Pinsky, *On dams with additive inputs and a general release rule*, J. Appl. Probability 9 (1972), p. 422-429.

[3] J. M. Harrison, *Some stochastic bounds for dams and queues*, Math. Opns. Res. 2 (1977), p. 54-63.
[4] — and S. I. Resnick, *The stationary distribution and first exit probabilities of a storage process with general release-rule*, ibidem 1 (1976), p. 347-358.
[5] M. Jankiewicz, *Elementy teorii tam*, Mat. Stosow. 18 (1980), p. 5-57.
[6] T. Rolski and D. Stoyan, *On the comparison of waiting times in GI/G/1 queues*, Operations Res. 24 (1976), p. 197-200.

MATHEMATICAL INSTITUTE
UNIVERSITY OF WROCŁAW
50-384 WROCŁAW

Received on 7. 12. 1981