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A GENERALIZATION OF SMITH'S THEOREM

A *perfect matching* of a graph G is a regular spanning subgraph of degree 1. A *1-tree* (or a *unicyclic graph*) is a connected graph with exactly one cycle. 1-trees are used in some algorithms for solving the travelling salesman problem (see [6]). In this paper we prove the following theorem, where we assume that 0 is also an even number.

THEOREM 1. *Let* $G = (V, E)$ *be an undirected graph,* $\{u, v\} \in E$, *and* $\deg(u) = 3$. *Then the number of partitions of* E *into sets* M *and* T *is even if* M *and* T *satisfy the following conditions*:

(a) M *is a perfect matching in* G;

(b) T *is a spanning* 1*-tree of* G *containing* $\{u, v\}$ *in its cycle.*

As was proved by Smith (see [1]), the number of hamiltonian cycles in a cubic graph which contain a given edge is even. This result is a special case of Theorem 1 when G is a cubic graph. The proof in [1] is ineffective. A different, effective proof was given by Thomason in [8]. Our proof is based on some ideas taken from his paper.

Proof of Theorem 1. Let G, u, v be fixed. By a *u-partition* of E we mean a partition of E into sets M and T such that

(1) T is a spanning tree such that u is a leaf of T and $\{u, v\} \in T$;

(2) in the subgraph (V, M) of G, the vertex u and some other vertex t have degree 2, all other vertices have degree 1.

A u-partition defined above is denoted by (M, T). The vertex t from (2) is called an *end-vertex* of (V, M).

Let (M, T) be a u-partition with end-vertex t and let x and y be neighbours of t in M. Let us assume that $x \neq u$. If $\{t, x\}$ is added to T, then we obtain a cycle. Let z be the neighbour of x in this cycle different from t. Since u is a leaf of T, z cannot be equal to u. Suppose that we transfer $\{t, x\}$ from M to T and $\{x, z\}$ from T to M. Then we obtain another u-partition (M', T') with end-vertex z. The above operation is called a *switch*. We can perform a similar operation on y provided that y is not equal to u. Consider now a graph PG whose vertices are u-partitions and two u-partitions are joined by an edge if one can be obtained from another by a switch. All vertices in PG have degree at most 2. Thus PG is a union of disjoint cycles and paths. Obviously, the number of u-partitions which have degree 1 in PG is even. However, a u-partition (M, T) has degree 1 in PG iff its end-vertex t is joined to u in M. If we transfer $\{u, t\}$ from M to T, we obtain a partition of E satisfying (a) and (b). Thus there is a 1-1 correspondence between partitions of E satisfying (a) and (b) and u-partitions with degree 1 in PG. This completes the proof.

In [3] it was shown that Smith's theorem has some consequences for combinatorial optimization. Let $K = (V, E, d)$ be a complete weighted graph, where d: $E \to R^+$ is a distance function. Consider the following optimization problems:

(MPM) Given K, find a minimum perfect matching in K.

(TSP) Given K, find a minimum hamiltonian cycle in K (this is the travelling salesman problem).

(MST) Given K, find a minimum spanning tree in K.

Let P_1 and P_2 be two of the problems listed above (one can consider here also other problems of a similar form). We say that P_1 and P_2 are *k-dependent* if for each K the following condition holds:

($*$) For each solution X to P_1 (resp. P_2) in K there is a solution Y to P_2 (resp. P_1) in K such that X and Y have k edges in common.

In particular, if solutions of P_1 and P_2 are unique, then they must have k common edges.

In [3] it was shown that MPM and TSP are 2-dependent. From Theorem 1 we infer that MPM and MST are 1-dependent. First we prove the following

COROLLARY 1. *Let $G = (V, E)$ be a graph with at least 4 vertices. Suppose there is a partition of E into two sets T_1 and M_1, where T_1 is a spanning tree and M_1 is a perfect matching. Then there is another partition of E into a spanning tree $T_2 \neq T_1$ and a perfect matching $M_2 \neq M_1$.*

Proof. Consider any leaf u of T_1. Let v be any vertex not adjacent to u in G. Add the edge $\{u, v\}$ to T_1. Let T be the resulting 1-tree. Then, by Theorem 1, E can be partitioned into T' and M_2, where T' is a 1-tree different from T and containing $\{u,v\}$ in its cycle, and M_2 is a perfect matching. Deleting $\{u,v\}$ from T' we obtain a spanning tree $T_2 \neq T_1$.

COROLLARY 2. *MPM and MST are 1-dependent.*

Proof. Consider solutions M_1 to MPM and T_1 to MST in a complete weighted graph $K = (V, E, d)$. Let $F = M_1 \cup T_1$ and suppose that M_1 and T_1 are disjoint. Consider the graph $G = (V, F)$. By Corollary 1 there is another partition of F into a perfect matching M_2 and a spanning tree T_2. We have

$d(M_1)+d(T_1)=d(M_2)+d(T_2)$. Clearly, M_1 and T_2 must have an edge in common. Since $d(M_2) \geqslant d(M_1)$, we have $d(T_2) \leqslant d(T_1)$, so T_2 is also a solution to MST. The other part of the proof is symmetric.

The following example shows that Corollary 2 is the best possible.

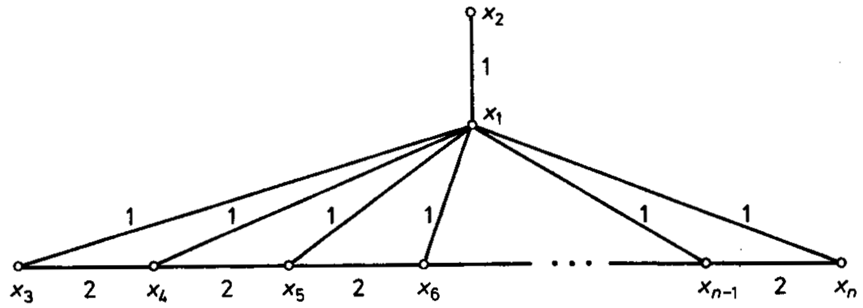

Fig. 1

EXAMPLE 1. Let us consider a weighted graph shown in Fig. 1. The edges not drawn have infinite length. Then a unique solution to MPM is

$$M = \{\{x_1, x_2\}, \{x_3, x_4\}, \ldots, \{x_{n-1}, x_n\}\},$$

a unique solution to MST is

$$T = \{\{x_1, x_2\}, \{x_1, x_3\}, \ldots, \{x_1, x_n\}\},$$

and M and T have exactly one edge in common.

There is an open problem related to Smith's theorem. It is well known that the problem whether a given graph contains a hamiltonian cycle is NP-complete, even for cubic graphs [5]. Suppose now that we are given a cubic graph G, a hamiltonian cycle H in G and we want to find another hamiltonian cycle. The corresponding decision problem is trivial by Smith's theorem, so we cannot prove NP-completeness here. On the other hand, no polynomial-time algorithm is known. The proof of Thomason [8] suggests an algorithm, but its time complexity is not known. The problem is to bound the number of switching operations (similar to those in the proof of Theorem 1). In [7] it was shown that the order of magnitude of this number is at least n^2, which was recently improved to n^3 in [4]. We believe that Theorem 1 is a special case of some general phenomena whose investigation may give a deeper insight into the nature of this problem and lead to its solution.

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