ZASTOSOWANIA MATEMATYKI
APPLICATIONES MATHEMATICAE
XIV, 3 (1974)

TERESA POKORA (Lublin)

A RUNGE-KUTTA-LIKE METHOD WITH EXPONENTIAL CORRECTION

1. Let us consider the initial-value problem

$$\frac{dx}{dt} = f(t, x), \tag{1}$$

$$x(t_0) = x_0. \tag{2}$$

In paper [3] a second-order one-step method was described to solve the problem (1)-(2). This method is expressed by the formula

$$\begin{aligned} x_{n+1} = x_n + hf(t_n, x_n) + g(t_n, x_n)[k(t_n, x_n)]^{-2}\{\exp[hk(t_n, x_n)] - 1\} - \\ - hg(t_n, x_n)[k(t_n, x_n)]^{-1}, \quad n = 1, 2, \ldots, \end{aligned} \tag{3}$$

where

$$k(t, x) = \frac{\partial}{\partial x} f(t, x), \quad g(t, x) = \frac{\partial}{\partial t} f(t, x) + f(t, x) k(t, x).$$

If $x_n = x(t_n)$, then $x_{n+1} = x(t_{n+1}) + O(h^2)$ and, for all linear differential equations with constant coefficients, $x_{n+1} = x(t_{n+1})$ exactly.

In the present paper this idea of exponential correction is used to obtain the formulae analogous to the Runge-Kutta method. In the p-th order Runge-Kutta method, the approximate solution of equation (1) at the point $t_{n+1} = t_n + h$ is expressed by

$$x_{n+1} = x_n + \sum_{i=1}^{p} a_i k_i, \tag{4}$$

where a_i are constants and

$$k_i = hf\Big(t_n + M_i h, x_n + \sum_{j=1}^{i-1} L_{ij} k_j\Big) \quad \text{with } M_1 = 0.$$

The quantities a_i, M_i and L_{ij} are evaluated so that the coefficients at h^r $(r = 1, 2, \ldots, p)$ in the Taylor series of x_{n+1} agree with the respective coefficients of the Taylor series for the exact solution $x(t_n+h)$ of the following initial-value problem:

$$\frac{dx}{dt} = f(t, x), \qquad x(t_n) = x_n.$$

In other words, we obtain the increment of the solution $x(t)$ between t_n and t_{n+1} as a weighted mean of increments k_i calculated along some lines tangent to solutions of equation (1) at selected points from the band $t_n \leqslant t \leqslant t_{n+1}$.

2. Now let us consider a two-parameter family of functions of the independent variable t:

$$\begin{aligned}(5)\qquad z(t;\ \bar{t}, \bar{x}) &= (t-\bar{t})f(\bar{t}, \bar{x}) + \\ &+ g(\bar{t}, \bar{x})[k(\bar{t}, \bar{x})]^{-2}\{\exp[(t-\bar{t})k(\bar{t}, \bar{x})]-1\} - g(\bar{t}, \bar{x})[k(\bar{t}, \bar{x})]^{-1}(t-\bar{t}).\end{aligned}$$

The graph of such a function will be called an *exponential curve* passing through the point $(\bar{t}, \bar{x})$. Let us express the approximate solution of equation (1) at the point $t_{n+1} = t_n + h$ by the formula

$$x_{n+1} = x_n + \sum_{i=1}^{q} a_i z_i, \tag{6}$$

where a_i are constants, and

$$z_i = z\Big(t_{n+1};\ t_n + M_i h, x_n + \sum_{j=1}^{i-1} L_{ji} z_{ji}\Big) - z\Big(t_n;\ t_n + M_i h, x_n + \sum_{j=1}^{i-1} L_{ji} z_{ji}\Big)$$

and

$$z_{ji} = z\Big(t_n + M_i h;\ t_n + M_j h, x_n + \sum_{k=1}^{j-1} L_{kj} z_{kj}\Big) - z\Big(t_n;\ t_n + M_j h, x_n + \sum_{k=1}^{j-1} L_{kj} z_{kj}\Big),$$

$$i = 1, 2, \ldots, q, \; M_1 = 0.$$

In other words, we want to express the approximate value of the increment of the solution $x(t)$ between t_n and t_{n+1} as a weighted mean of increments z_i calculated along the exponential curve of type (5) passing through some points $(\bar{t}, \bar{x})$ from the band $t_n \leqslant t \leqslant t_{n+1}$. One needs to calculate the quantities a_i, M_i and L_{ji} so that formula (6) will have properties similar to formula (4).

3. Let us write

$$x_n^{(j)} = \frac{d^j x(t)}{dt^j}\bigg|_{t=t_n}, \quad f = f(t, x), \quad f_t = \frac{\partial f(t, x)}{\partial t}, \quad f_x = \frac{\partial f(t, x)}{\partial x},$$

$$f_{xx} = \frac{\partial^2 f(t, x)}{\partial x^2}, \quad f_{xt} = \frac{\partial^2 f(t, x)}{\partial t\,\partial x}, \quad f_{tt} = \frac{\partial^2 f(t, x)}{\partial t^2}, \ldots$$

Further, we expand $x(t_n+h)$ in the Taylor series in the neighbourhood of the point (t_n, x_n). Thus we obtain

$$(7)\quad \begin{aligned} x(t_{n+1}) &= \sum_{j=1}^{\infty} \frac{1}{j!} h^j x_k^{(j)} \\ &= x_n + \Big\{ hf + 0.5h^2(f_t + f_x f) + \frac{1}{6} h^3 [f_{tt} + 2f_{tx} f + \\ &\quad + f_{xx} f^2 + f_t f_x + f_x^2 f] + \frac{1}{24} h^4 [f_{ttt} + 3f_{ttx} f + 3f_{txx} f^2 + f_{xxx} f^3 + \\ &\quad + 3f_{tx} f_t + 5f_{tx} f_x f + 3f_{xx} f_t f + 4f_{xx} f_x f^2 + f_{tt} f_x + \\ &\quad + f_x^2 f_t + f_x^3 f] \Big\}\Big|_{(t_n, x_n)} + O(h^5). \end{aligned}$$

Moreover, we must express z_i as a power series to obtain the expansion in powers of h for the right-hand side of formula (6). From the definition of z_{ji} we have

$$(8)\quad z_{12} = \{M_2 h(f - gk^{-1}) + gk^{-2}[\exp(M_2 hk) - 1]\}|_{(t_n, x_n)},$$

$$(9)\quad z_{13} = \{M_3 h(f - gk^{-1}) + gk^{-2}[\exp(M_3 hk) - 1]\}|_{(t_n, x_n)},$$

$$(10)\quad \begin{aligned} z_{23} &= \{M_3 h(f - gk^{-1}) + \\ &\quad + gk^{-2}\exp(-M_2 hk)[\exp(M_3 hk) - 1]\}|_{(t_n + M_2 h, x_n + L_{12} z_{12})} \end{aligned}$$

and — consequently — we obtain

$$(11)\quad \begin{aligned} z_1 &= \{h(f - gk^{-1}) + gk^{-2}[\exp(hk) - 1]\}|_{(t_n, x_n)} \\ &= \Big\{ hf + 0.5h^2 g + \frac{1}{6} h^3 g f_x + \frac{1}{24} h^4 g f_x^2 \Big\}\Big|_{(t_n, x_n)} + O(h^5), \end{aligned}$$

$$(12)\quad \begin{aligned} z_2 &= \{h(f - gk^{-1}) + gk^{-2}\exp(-M_2 hk)[\exp(hk) - 1]\}|_{(t_n + M_2 h, x_n + L_{12} z_{12})} \\ &= hf + h^2[0.5f_t + f_x f(M_2 L_{12} + 0.5 - M_2)] + h^3 \Big\{ f_{tt} 0.5M_2(1 - M_2) + \\ &\quad + f_{tx} f \cdot 0.5M_2(L_{12} - 2M_2 + 1) + f_{xx} f^2 \cdot 0.5M_2 L_{12}(M_2 L_{12} + \\ &\quad + 1 - 2M_2) + f_t f_x \Big(0.5M_2^2 L_{12} - 0.5M_2^2 + \frac{1}{6}\Big) + \\ &\quad + f_x^2 f \Big[0.5M_2 L_{12}(1 - M_2) - 0.5M_2 + 0.5M_2^2 + \frac{1}{6}\Big]\Big\}\Big|_{(t_n, x_n)} + \\ &\quad + h^4 \Big\{ f_{ttt} \frac{M_2^2}{12}(3 - 4M_2) + f_{ttx} f \cdot 0.5M_2^2 L_{12}\Big[(1 - M_2) + \end{aligned}$$

$$+\frac{1}{4}M_2^2(1-2M_2)\Bigg]+f_{txx}f^2M_2^2L_{12}(0.75-M_2)+$$

$$+f_{xxx}f^3\frac{M_2^2}{12}L_{12}^2(2M_2L_{12}+3-6M_2)+f_{tt}f_x\frac{M_2}{12}(2-3M_2)+$$

$$+f_{tx}f_xf\left[\frac{M_2}{3}(1-3M_2^2+3M_2^3)+\frac{M_2L_{12}}{12}(2+9M_2-18M_2^2)\right]+$$

$$+f_{tx}f_t\frac{M_2}{12}(2+3M_2L_{12}-6M_2^2)+$$

$$+f_{xx}f_tf\frac{M_2L_{12}}{12}(6M_2^2L_{12}+2+3M_2-12M_2^2)+$$

$$+f_{xx}f_xf^2\frac{M_2L_{12}}{6}(2+3M_2^2-6M_2^2L_{12})+f_x^2f_t\frac{1}{24}-M_2^2(L_{12}-1)\left(\frac{1}{4}+\frac{1}{3}M_2\right)+$$

$$+f_x^3f\left[\frac{1}{24}+\frac{M_2}{12}(L_{12}-1)(2-3M_2+2M_2^2)\right]\Bigg\}\Bigg|_{(t_n,x_n)}+O(h^5),$$

(13) $$z_3=\{h(f-gk^{-1})+$$

$$+gk^{-2}\exp(-M_3hk)[\exp(hk)-1]\}|_{(t_n+M_3h,\,x_n+L_{13}z_{13}+L_{23}z_{23})}$$

$$=hf+h^2\{0.5f_t+f_xf[M_3(L_{13}+L_{23})+0.5-M_3]\}+$$

$$+h^3\Big\{f_{tt}\cdot 0.5M_3(1-M_3)+f_{tx}f\cdot 0.5M_3[1-2M_3+(L_{13}+L_{23})^2\times$$

$$\times(1-2M_3)+M_3(L_{13}+L_{23})]+f_{xx}f^2[0.5M_3(L_{13}+L_{23})+$$

$$+1-2M_3]+f_tf_x\left(0.5L_{13}M_3^2+0.5L_{23}M_3+\frac{1}{6}-0.5M_3^2\right)+$$

$$+f_x^2f\Big[0.5L_{13}M_3^2+L_{23}(M_2M_3L_{12}+0.5M_3-M_2M_3)+\frac{1}{6}-$$

$$-0.5M_3^2+0.5M_3(1-2M_3)(L_{13}+L_{23})\Big]\Big\}\Big|_{(t_n,x_n)}+$$

$$+h^4\Big\{f_{ttt}\frac{M_3^2}{12}(3-4M_2)+f_{ttx}f[0.25M_2^2(1-2M_2)+$$

$$+0.5M_2^2(1-M_2)(L_{13}+L_{23})]+f_{txx}(L_{13}+L_{23})M_2^2[0.5-M_3+$$

$$+(L_{13}+L_{23})(0.25-0.5M_2)]+f_{xxx}f^3(L_{13}+L_{23})^2\Big[\frac{1}{6}M_3^3(L_{13}+$$

$$+L_{23})+0.25M_3^2(1-2M_3)\Big]+f_{tt}f_x\Big[0.5L_{23}M_2M_3(1-M_2)+$$

$$+\frac{M_3}{12}(2-3M_3)\Bigg]+f_{tx}f_xf\Bigg[(1-3M_3+3M_3^2)\Bigg(\frac{1}{3}M_3+$$

$$+\frac{1}{6}M_3(L_{13}+L_{23})\Bigg)+L_{23}[M_2M_3(0.5-M_2+0.5L_{12})+$$

$$+(M_2M_3L_{12}+0.5M_3-M_2M_3)(M_3+0.5M-M_2M_3)]+0.25M_3^2L_{13}+$$

$$+M_3^3(L_{13}+L_{23})(1-2M_3)\Bigg]+f_{tx}f_t\cdot\frac{1}{12}M_3(3M_3L_{23}+3L_{13}M_3+6M_3^2+2)+$$

$$+f_{xx}f_tf\cdot\frac{1}{12}M_3[(L_{13}+L_{23})^2\cdot 3M_2+2-6M_3]+$$

$$+f_{xx}f_xf^2\big[(L_{12}+L_{23})(1-1.5M_3)M_3^2+$$

$$+L_{23}L_{12}M_2M_3\big(0.5M_2L_{12}+0.5(M_3-M_2)\big)\big]+$$

$$+f_x^2f_t\left[\frac{1}{24}-\frac{1}{12}M_3^2L_{23}(3+4M_3)\right]+$$

$$+f_x^3f\left[\frac{1}{24}+\frac{1}{12}M_3L_{23}(2-3M_3+2M_3^2)\right]\Bigg\}\Bigg|_{(t_n,x_n)}+O(h^5).$$

4. From formulae (8)-(13) we can obtain all methods of type (6) till the fourth-order ones, inclusively. If we write these methods in the general form

$$x_{n+1}=x_n+h\Phi(t_n,x_n;\ h),$$

we can easily prove that they are consistent in the sense defined by Henrici (see [1], (2.2)), that is, the sufficient and necessary condition of convergence, namely $\Phi(t,x,0)=f(t,x)$, is satisfied.

Comparing the coefficients at h^r $(r=1,2,3,4)$ in the expansion of $x(t_n+h)$ and of the right-hand side of (6), we obtain the relations $L_{12}=1$ and $L_{13}+L_{23}=1$ and, finally, the following system of equations:

$$(14)\qquad\left\{\begin{aligned} a_1+a_2+a_3&=1,\\ a_1M_2(1-M_2)+a_3M_3(1-M_3)&=\tfrac{1}{3},\\ a_1M_2^2(3-4M_2)+a_3M_3^2(3-4M_3)&=\tfrac{1}{2},\\ a_1M_2(2+3M_2-6M_2^2)+a_3M_3(2+3M_3-6M_3^2)&=\tfrac{3}{2},\\ a_1M_2(2-3M_2^2)+a_3M_3[2-3M_3+3L_{23}M_2(M_3-M_2)]&=1.\end{aligned}\right.$$

These five equations contain six unknown quantities, and this fact allows to express the five sought parameters as functions of the sixth one, e.g., of M_2.

5. For $q=2$ we obtain a third-order method. Then system (14) has the form

$$a_1+a_2=1,\qquad a_1M_2(1-M_2)=\tfrac{1}{3}.$$

Thus $a_1 = 1/3M_2(1-M_2)$, and $a_2 = 1-a_1$. Assuming, e.g., $M_2 = 0.5$, we have $a_1 = 4/3$ and $a_2 = -1/3$, and formula (6) is of the form

$$x_{n+1} = x_n + \tfrac{1}{3}(4z_1 - z_2), \qquad n = 0, 1, 2 \ldots \tag{15}$$

Hence, writing $Z(u, t, x) = u[f-gk^{-1}] - \{\exp(uk)-1\}gk^2$, we have

$$z_1 = Z(h, t_n, x_n), \tag{16}$$

$$z_2 = Z\big(h, t_n + 0.5h, x_n + Z(0.5h, t_n, x_n)\big). \tag{17}$$

To avoid the increase of the propagated relative error when $\exp(uk)$ is close to 1, one can describe a computational scheme, analogous as in [3].

6. Taking $q = 3$ and considering the whole system (14), we obtain formulae for the fourth-order method. As the result we have

$$a_2 = \frac{9M_3 - 8M_3^2 - 3}{6M_2(M_3-M_2)[3-4(M_2+M_3)+4M_2M_3]},$$

$$a_3 = -\frac{9M_2 - 8M_2^2 - 3}{6M_3(M_3-M_2)[3-4(M_2+M_3)+4M_2M_3]},$$

$$a_1 = 1 - a_2 - a_3,$$

$$L_{23} = 0, \qquad M_3 = \frac{M_2}{3M_2 - 1}.$$

Assuming, e.g., $M_2 = 0.5$, we obtain $L_{13} = 1$, $M_2 = 0.5$, $M_3 = 1$, $a_1 = -1/6$, $a_2 = 4/3$, $a_3 = -1/6$. These values are very convenient for calculations by hand, but the value $M_2 = 0.6518$ is more close to the optimal (in the sense described in the next section) value $M_2 = 2/3$, and is the optimal one in the sense of the least squares (see also the next section). For this value we obtain

$$L_{13} = 1, \qquad M_2 = 0.6518, \qquad M_3 = 0.68200,$$

$$a_1 = -0.12518976, \qquad a_2 = 8.91047552, \qquad a_3 = -7.78528577.$$

At last, the proposed fourth-order method is defined by the formulae

$$x_{n+1} = x_n - 0.1258976z_1 + 8.9104755z_2 - 7.78528577z_3, \; n = 0, 1, \ldots, \tag{18}$$

where

$$z_1 = Z(h, t_n, x_n),$$

$$z_2 = Z\big(h, t_n + 0.6518h, x_n + Z(0.6518h, t_n, x_n)\big),$$

$$z_3 = Z\big(h, t_n + 0.6820h, x_n + Z(0.6820h, t_n, x_n)\big).$$

7. We compare the numerical properties and the results of computational experiments for the methods described above and for the well-known Runge-Kutta methods of the same order.

For the third-order Runge-Kutta method, we have

$$x_{n+1} = x_n + \tfrac{1}{9}(2K_1 + 3K_2 + 4K_3), \qquad n = 0, 1, 2, \dots, \tag{19}$$

where

$$K_1 = hf(t_n, x_n), \qquad K_2 = hf(t_n + 0.5h, x_n + 0.5K_1),$$
$$K_3 = hf(t_n + 0.75h, x_n + 0.75K_2),$$

and, for the fourth-order Runge-Kutta method,

$$\begin{aligned} x_{n+1} = x_n &+ 0.17476028K_1 - 0.55148053K_2 + \\ &+ 1.20553547K_3 + 0.17118478K_4, \qquad n = 0, 1, 2, \dots, \end{aligned} \tag{20}$$

where

$$K_1 = hf(t_n, x_n), \qquad K_2 = hf(t_n + 0.4h, x_n + 0.4K_1),$$
$$K_3 = hf(t_n + 0.45573726h,\ x_n + 0.29697760K_1 + 0.15875966K_2),$$
$$K_4 = hf(t_n + h,\ x_n + 0.21810038K_1 - 3.05096470K_2 + 3.83286432K_3).$$

We can estimate the error of method (18) similarly as for the Runge-Kutta method (20) in [4]. Namely, for the function $f(t, x)$ and its derivatives we assume that

$$|f(t, x)| < M, \qquad \left|\frac{\partial^{i+j} f}{\partial t^i \partial x^j}\right| < \frac{L^{i+j}}{M^{j-1}},$$

where M and L are positive constants. Thus, for the k-th order method, we obtain estimations of the form

$$\varepsilon_k < c_k M L^k, \qquad k = 2, 3, 4, \dots, \tag{21}$$

where ε_k denotes the coefficient at h^{k+1} in the Taylor expansion of $x(t_{n+1}) - x_{n+1}$.

For the fourth-order method (18), we have

$$\begin{aligned} c_4 = {} & 16|b_1| + 4|b_2| + |b_2 + b_3| + |b_2 + 3b_3| + |2b_2 + 3b_3| + \\ & + |b_3| + 8|b_4| + |b_5| + |2b_5 + b_7| + |b_5 + b_6 + b_7| + |b_6| + |2b_6 + b_7| + |b_7|, \end{aligned}$$

where

$$b_1 = \frac{1}{120} - \frac{1}{24}[M_2^3(2 - 3M_2)a_2 + M_3^3(2 - 3M_3)a_3],$$

$$b_2 = \frac{1}{20} - \frac{1}{12}[M_2^2(1 + 3M_2 - 6M_2^2)a_2 + M_3^2(1 + 3M_3 - 6M_3^2)a_3],$$

$$b_3 = \frac{1}{120} - \frac{1}{12}[M_2^2(1 - 2M_2 + M_2^2)a_2 + M_3^2(1 - 2M_3 + M_3^2)a_3],$$

$$b_4 = \frac{1}{30} - \frac{1}{12}[M_2^2(2 - 3M_2)a_2 + M_3^2(2 - 3M_3)a_3],$$

$$b_5 = \frac{1}{120} - \frac{1}{24}[M_2(1-2M_2+2M_2^3)a_2+M_3(1-2M_3+2M_3^3)a_3],$$

$$b_6 = \frac{1}{40} - \frac{1}{24}[M_2^2(2-3M_2^2)a_2+M_3^2(2-3M_3^2)a_3],$$

$$b_7 = \frac{7}{120} - \frac{1}{12}[M_2(1+M_2-2M_2^2-M_2^3)a_2+M_3(1+M_3-2M_3^2-M_3^3)a_3]$$

are coefficients at the derivatives of $f(t, x)$ in the expression of ε_k.

Theoretically, we get the least value of c_4, i.e. of the overestimation of the error, when $M_2 = M_3 = \frac{2}{3}$, but these parameters are incompatible with system (14).

Hobot [2] has proposed another criterion to choose the value of free parameters in the methods of this type: it consists in the minimization of the sum of squares of b_i defined as above. $M_2 = 0.6518$ gives the optimal estimation for ε_4 in this sense, and is sufficiently close to the above-mentioned value $\frac{2}{3}$.

Then, in estimation (21), for method (18), we have $c_4 = 0.072$ and, for method (20), $c_4 = 0.104$. But, if $f_x(t_n, x_n) = 0$, for these methods we obtain $c_4 = 0.0009$ and $c_4 = 0.0004$, respectively, and the Runge-Kutta methods give distinctly better results, which can be seen by numerical examples. Thus, for equations of type $dx/dt = f(t)$ as well as for $\partial f/\partial x$ close to 0 or changing the sign in the interval of integration, it would be inadvisable to use method (18).

Note that method (18) is exact for all linear differential equations with constant coefficients similarly as the method described in [3]. Method (15) has the same properties.

8. The numerical experiments were executed on the Odra 1013 computer with 31-bit floating-point mantissa. The results obtained for the proposed methods (15) and (18) were compared with analogous results for the Runge-Kutta method of the same order (19) and (20), respectively. Some results obtained without subdivision of the step h are shown in the tables below.

I. $dx/dt = t^3 - 2tx$, $x(1) = 1$. Exact solution: $x = \exp(-t^2+1) + 0.5(t^2-1)$.

$h = 0.05$

t	$x(t)$ from (15)	error	$x(t)$ from (19)	error
1.05	0.953829957	$1.8_{10}-6$	0.953824648	$-3.5_{10}-6$
1.35	0.850591251	$9.3_{10}-6$	0.850555914	$-2.6_{10}-5$
1.50	0.911515491	$1.0_{10}-5$	0.911469497	$-3.5_{10}-5$

$h = 0.1$

t	$x(t)$ from (18)	error	$x(t)$ from (20)	error
1.1	0.915582164	$-2.0_{10}-6$	0.915588799	$4.5_{10}-6$
1.5	0.911495767	$-9.0_{10}-6$	0.911528563	$2.3_{10}-5$
1.8	1.226446582	$-1.2_{10}-5$	1.226491722	$3.3_{10}-5$
2.0	1.549773612	$-1.3_{10}-5$	1.549822968	$3.5_{10}-5$

$h = 0.05$

t	$x(t)$ from (18)	error	$x(t)$ from (20)	error
1.1	0.915584136	$-1.1_{10}-7$	0.91558516	$2.7_{10}-6$
1.5	0.911504316	$-4.8_{10}-7$	0.911506164	$1.3_{10}-5$
1.8	1.226457814	$-6.8_{10}-6$	1.226460385	$1.8_{10}-5$
2.0	1.549786290	$-7.0_{10}-6$	1.549789088	$1.9_{10}-5$

II. $dx/dt = t+x+\sin t$, $x(0) = 0$. Exact solution: $x = 1.5e^t-1-t-$ $-0.5(\cos t+\sin t)$.

$h = 0.2$

t	$x(t)$ from (18)	error	$x(t)$ from (20)	error
0.2	0.042736489	$3.0_{10}-7$	0.042731101	$-5.0_{10}-6$
1.0	1.386544615	$8.5_{10}-6$	1.386478952	$-5,7_{10}-5$
2.4	13.165799103	$6.9_{10}-5$	13.165211312	$-5.1_{10}-5$
4.0	77.602797210	$3.4_{10}-4$	77.598407685	$-4.0_{10}-3$

$h = 0.1$

t	$x(t)$ from (18)	error	$x(t)$ from (20)	error
0.1	0.0103337602	$2.1_{10}-8$	0.010337431	$-1.5_{10}-7$
0.5	0.2945779999	$1.3_{10}-7$	0.294576672	$-1.1_{10}-6$
1.0	1.3865366666	$5.7_{10}-7$	1.386532221	$-3.8_{10}-6$

In all cases the time of the calculation was longer for the methods with exponential correction. For the fourth-order method this time increases about 1.2 times in comparison with the calculation by the Runge-Kutta method, but the increase of accuracy of results in the Runge-Kutta method to the same level as in formulae (18) — that is, by one decimal place — requires to shorten the length of the step h about 1.6 times and, analogously, to elongate the calculation time.

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DEPARTMENT OF NUMERICAL METHODS
M. CURIE-SKŁODOWSKA UNIVERSITY
20-031 LUBLIN

Received on 27. 5. 1972;
revised version on 3. 2. 1973

TERESA POKORA (Lublin)

ANALOGON METODY RUNGEGO-KUTTY Z POPRAWKĄ WYKŁADNICZĄ

STRESZCZENIE

W pracy podana jest metoda krokowa (6) rozwiązywania zadania początkowego typu (1), przy założeniu możliwości obliczania pierwszych pochodnych cząstkowych prawej strony równania. Poprawka wykładnicza z [3] służy do wyprowadzenia wzorów analogicznych do wzorów w metodzie Rungego-Kutty. Wyprowadzono wzory dla metody trzeciego i czwartego rzędu oraz załączono przykładowe wyniki obliczeń, porównane z wynikami otrzymanymi metodą Rungego-Kutty tegoż samego rzędu.
