ZASTOSOWANIA MATEMATYKI
APPLICATIONES MATHEMATICAE
XV, 4 (1977)

R. MAGIERA (Wrocław)

ON SEQUENTIAL MINIMAX ESTIMATION FOR THE EXPONENTIAL CLASS OF PROCESSES

1. Introduction. In the paper, a sequential estimation problem for stochastic processes is considered in the case where the loss incurred by the statistician is due not only to the error of estimation but also to the cost of observation. In such a problem, the statistician must decide at each moment whether to stop observing the process and to suffer a specified stopping risk or to continue the observation at some specified additional observing cost. We use the Bayesian and minimax sequential estimation methods for the solution determining his optimal behaviour.

It is proved by Dvoretzky et al. [1] that in case of the Poisson, negative-binomial, gamma and Wiener processes the fixed-time plan is minimax if a weighted quadratic loss function is used. In the present paper we generalize the result of Dvoretzky et al. for the case of the exponential class of processes satisfying some additional assumptions. This result was presented by the author at the 2-nd Conference on Mathematical Statistics in Wisła (Poland), December 1974.

2. Preliminaries. Let $\xi(t) = X(t, \omega)$, $t \in T$, $\omega \in \Omega$, be a stochastic process defined on a probability space $(\Omega, \mathscr{F}, P)$, with values in a measurable space $(\mathscr{X}, \mathscr{B})$. By T we denote the half-line $[0, \infty)$. We assume that $\mathscr{X} \subseteq R$, where R denotes the real line, and $\mathscr{B}$ is a σ-algebra of Borel subsets of $\mathscr{X}$. The state of the process $\xi(t)$ at time t is denoted by $X_t(\omega)$ or, more simply, by X_t.

Let, for every $t \in T$, $\mathscr{F}_t$ denote a sub-σ-algebra of $\mathscr{F}$, generated by the random variables $X_s(\omega)$, $s \leqslant t$. A function $\tau(\omega)$, $\mathscr{F}$-measurable, with values in $T \cup \{\infty\}$ is said to be a *Markov stopping time* if it satisfies the following condition:

$$\{\omega: \tau(\omega) \leqslant t\} \in \mathscr{F}_t \quad \text{for every } t \in T.$$

Let $\{\xi_\vartheta(t) = X(t, \omega), t \in T, \omega \in \Omega, \vartheta \in \Theta\}$ be a family of stochastically continuous processes, each of which is defined on an appropriate space $(\Omega, \mathscr{F}, P_\vartheta)$, $\vartheta \in \Theta$.

One-dimensional probability distributions induced on the space $(\mathscr{X}, \mathscr{B})$ by the random variables $X_t(\omega)$, $t \in T$, depend evidently on the parameter ϑ. The family of one-dimensional probability distributions on $(\mathscr{X}, \mathscr{B})$ corresponding to the family of processes $\{\xi_\vartheta(t), t \in T, \vartheta \in \Theta\}$ is denoted by $\{\mathscr{P}_\vartheta^{(t)}, t \in T, \vartheta \in \Theta\}$.

We assume that Θ is an open interval of the real line. Θ may consist of the entire line or of an entire half-line. $\vartheta \in \Theta$ plays the role of an unknown parameter of the process. To estimate this parameter we use the Bayesian and minimax sequential methods.

By $\mathrm{E}_\vartheta(\cdot)$ and $\mathrm{D}_\vartheta(\cdot)$ we denote the expected value and the variance, respectively, evaluated with respect to the measure P_ϑ.

In the consideration of optimal sequential estimation problems it is natural to assume that all measures P_ϑ, $\vartheta \in \Theta$, are mutually absolutely continuous. The random variable X_t is called a *sufficient statistic* for $\vartheta \in \Theta$ if for every system $0 \leqslant t_1 < t_2 < \ldots < t_n < t$ $(n = 1, 2, \ldots)$ and for every $B_i \in \mathscr{B}$ $(i = 1, 2, \ldots, n)$ the conditional probabilities

$$P_\vartheta(X_{t_1} \in B_1, X_{t_2} \in B_2, \ldots, X_{t_n} \in B_n \mid X_t = x_t)$$

are the same for each $\vartheta \in \Theta$.

Intuitively, it means that, given the values of t and X_t, we shall not obtain any additional information about the parameter ϑ from the values of random variables $X_{t_1}, X_{t_2}, \ldots, X_{t_n}$ at the previous moments. In the sequel we consider only such processes which have the above-given property.

All processes considered below are assumed to have right-continuous paths.

Let us note that if the paths of the process $\xi(t) = X(t, \omega)$ are right-continuous, then for every finite (with probability 1) Markov stopping time τ the function $X_\tau(\omega) = X(\tau(\omega), \omega)$ is a random variable and $\mathscr{F}_\tau$-measurable, where $\mathscr{F}_\tau$ denotes a σ-algebra consisting of all sets $A \in \mathscr{F}$ such that

$$A \cap \{\omega: \tau(\omega) \leqslant t\} \in \mathscr{F}_t \quad \text{for every } t \in T$$

(see, for instance, [3], p. 70).

Denote by U the product space $T \times \mathscr{X}$. Let $\mathscr{U}$ be a σ-algebra of Borel subsets of U (a σ-algebra of Borel subsets of T is considered).

Definition 1. A real-valued function f defined on U and $\mathscr{U}$-measurable is called an *estimator of the parameter* ϑ.

A Markov stopping time τ determines a time up to which the process $\xi(t)$ is observed. Knowing values of τ and X_τ we wish to estimate the value of the parameter ϑ. The problem is to find Markov stopping times τ and estimators $f(\tau, X_\tau)$ having some optimal properties.

Definition 2. By a *sequential plan* we mean any pair $\delta = (\tau, f)$, consisting of a Markov stopping time τ and an estimator f, such that the condition $P_\vartheta\ (0 < \tau < \infty) = 1$ is satisfied for each $\vartheta \in \Theta$.

The non-negative function $L(\vartheta, f)$, where ϑ is the true value of the parameter and f is the chosen estimator, determines the loss incurred by the statistician and is called the *loss function*.

Let $c(t)$, $t \in T$, be a given *cost function* which represents the cost to the statistician of observing the process up to time t. We assume that $c(t)$ is non-negative, lower semicontinuous, and satisfies the condition

$$\lim_{t\to\infty} c(t) = \infty.$$

The statistician observes the process $\xi(t)$ and decides when to stop the observation and what estimator to take when he does stop. He is interested in making such a choice of τ and f that the expected value of the over-all loss function $L(\vartheta, f) + c(\tau)$ be small.

The expected value of the over-all loss function is called the *risk function* and denoted by $R(\vartheta, \delta)$, where $\delta = (\tau, f)$ is a chosen sequential plan and ϑ is the true value of the parameter. We consider only such sequential plans δ for which

$$R(\vartheta, \delta) = \mathrm{E}_\vartheta[L(\vartheta, f) + c(\tau)]$$

exists and is finite for all $\vartheta \in \Theta$. The set of all such sequential plans is denoted by $\mathscr{D}$.

A sequential plan $\delta^* = (\tau^*, f^*)$ is said to be *minimax* if

$$\sup_{\vartheta\in\Theta} R(\vartheta, \delta^*) = \inf_{\delta\in\mathscr{D}} \sup_{\vartheta\in\Theta} R(\vartheta, \delta).$$

Let $\mathscr{G}$ be a σ-algebra of Borel subsets of Θ. Suppose that the prior probability distribution of the parameter ϑ on $(\Theta, \mathscr{G})$ is defined by the distribution function $\Phi(\vartheta)$. Let $R(\vartheta, \delta)$ be a $\mathscr{G}$-measurable function of the variable ϑ. Then for a given sequential plan δ the *expected risk* with respect to Φ is defined by

$$r(\Phi, \delta) = \int_\Theta R(\vartheta, \delta) d\Phi(\vartheta).$$

A sequential plan $\hat{\delta} = (\hat{\tau}, \hat{f})$ is said to be *Bayes* for Φ if

$$r(\Phi, \hat{\delta}) = \inf_{\delta\in\mathscr{D}} r(\Phi, \delta).$$

A sequential plan (τ, f) for which τ is equal, with probability 1, to a constant $T_0 > 0$ is called the *fixed-time plan* and denoted by

$$\delta^0 = (T_0, f^0) = (T_0, f^0(X_{T_0})), \quad \text{where } f^0(X_{T_0}) = f(T_0, X_{T_0}).$$

Further, for the fixed-time plan $\delta^0 = (T_0, f^0)$ we write

$$\tilde{R}(\vartheta, f^0) = \mathrm{E}_\vartheta[L(\vartheta, f^0)] \tag{1}$$

and

$$\tilde{r}(\Phi, f^0) = \int_{\Theta} \tilde{R}(\vartheta, f^0)\, d\Phi(\vartheta). \tag{2}$$

An estimator $\hat{f}^0$ is called a *δ^0-Bayes estimator* for Φ if functional (2) attains its minimum for $f^0 = \hat{f}^0$.

Suppose that, for a fixed-time plan $\delta^0 = (T_0, f^0)$ and the prior distribution function $\Phi(\vartheta)$, given $X_{T_0} = x$, the posterior distribution function $\Phi(\vartheta|x)$ is defined. Then the conditional expected loss, given the observation, corresponding to Φ and f^0 is defined by

$$\tilde{r}(\Phi, f^0|x) = \int_{\Theta} L(\vartheta, f^0)\, d\Phi(\vartheta|x). \tag{3}$$

The conditional expected loss $\tilde{r}(\Phi, f^0|x)$ is called the *posterior δ^0-risk* corresponding to Φ and f^0. A δ^0-Bayes estimator for Φ minimizes the posterior δ^0-risk corresponding to Φ and f^0.

The next lemma, which describes a well-known method of solving for minimax rules in decision theory (see also [1]), is used in further considerations.

LEMMA. *Suppose that, for every $T_0 > 0$, there exists a sequence of distribution functions Φ_n $(n = 1, 2, \ldots)$ for which there exist corresponding δ^0-Bayes estimators $\hat{f}^0_n$ with the property that the posterior δ^0-risk corresponding to Φ_n and $\hat{f}^0_n$ is independent of the value of the random variable X_{T_0}. Moreover, suppose that there exists an estimator f^0 for which*

$$\tilde{R}(T_0) = \sup_{\vartheta\in\Theta} \tilde{R}(\vartheta, f^0) = \lim_{n\to\infty} \tilde{r}(\Phi_n, \hat{f}^0_n).$$

If there exists a T_0^* $(0 < T_0^* < \infty)$ for which

$$c(T_0^*) + \tilde{R}(T_0^*) = \min_{T_0>0} [c(T_0) + \tilde{R}(T\ )]$$

holds, then the fixed-time plan $\delta^0 = (T_0, f^0)$ for $T_0 = T_0^*$ is minimax.

3. Exponential class of processes.

Definition 3. By the *exponential class of processes* we mean the class of homogeneous processes with independent increments and satisfying the following conditions:

(a) $P_\vartheta(X_0 = 0) = 1$ for each $\vartheta \in \Theta$;

(b) $\mathrm{E}_\vartheta(X_t^2) < \infty$ for every $t \in T$ and all $\vartheta \in \Theta$;

(c) for all $t > 0$ and $\vartheta \in \Theta$ the distributions $\mathscr{P}_\vartheta^{(t)}$ are absolutely continuous with respect to a σ-finite measure ν on $(\mathscr{X}, \mathscr{B})$ and their Radon-Nikodym derivatives are of the form

$$\frac{d\mathscr{P}_\vartheta^{(t)}}{d\nu}(x) = p(t, x; \vartheta) = s(t, x)\exp[w_1(\vartheta)t + w_2(\vartheta)x], \tag{4}$$

where $x \in \mathscr{X}$, $s(t, x)$ denotes a (non-negative) function defined on U, and $w_1(\vartheta)$ and $w_2(\vartheta)$ are some functions defined on Θ.

We assume that the functions $w_1(\vartheta)$ and $w_2(\vartheta)$ are twice continuously differentiable in the interval Θ and, moreover, that the derivatives $w_1'(\vartheta)$ and $w_2'(\vartheta)$ satisfy the following condition: $w_2'(\vartheta) > 0$ for all $\vartheta \in \Theta$ and the function $w_1'(\vartheta)/w_2'(\vartheta)$ is strictly decreasing in the whole interval Θ.

The expected value and the variance of processes belonging to the exponential class are given by

$$\mathrm{E}_\vartheta(X_t) = -\frac{w_1'(\vartheta)}{w_2'(\vartheta)}\, t \tag{5}$$

and

$$\mathrm{D}_\vartheta(X_t) = -\frac{1}{w_2'(\vartheta)}\frac{d}{d\vartheta}\left[\frac{w_1'(\vartheta)}{w_2'(\vartheta)}\right] t, \tag{6}$$

respectively.

The paths of processes from the exponential class are assumed to be right-continuous.

For processes belonging to the exponential class the random variable X_t is a sufficient statistic for the parameter ϑ.

Let us remark that the Poisson, negative-binomial, gamma and Wiener (with linear drift) processes belong to the considered class.

4. Bayes and minimax sequential estimation for the exponential class of processes. We consider processes belonging to the exponential class for which the relation

$$\vartheta = -\frac{w_1'(\vartheta)}{w_2'(\vartheta)} \tag{7}$$

holds for all $\vartheta \in \Theta$. Moreover, let us assume that there exists a constant $\beta \geqslant 0$ such that the relation

$$\int_\Theta \exp[w_1(\vartheta)t + w_2(\vartheta)x]\, d\vartheta = \frac{1}{s(t, x)(t-\beta)} \tag{8}$$

is valid for all $t > \beta$ and $x \in \mathscr{X}$ for which $s(t, x) > 0$.

Let Θ be an open interval (a, b). Suppose that

$$\lim_{\vartheta\to a^+} \exp[w_1(\vartheta)t + w_2(\vartheta)x] = \lim_{\vartheta\to b^-} \exp[w_1(\vartheta)t + w_2(\vartheta)x] \tag{9}$$

and

$$\lim_{\vartheta\to a^+} \vartheta \exp[w_1(\vartheta)t + w_2(\vartheta)x] = \lim_{\vartheta\to b^-} \vartheta \exp[w_1(\vartheta)t + w_2(\vartheta)x] \tag{10}$$

for every $t > \beta$ and each $x \in \mathscr{X}$ except perhaps $x = 0$ if $\mathscr{X} = \{x\colon\ x \geqslant 0\}$.

All of relations (7)-(10) are valid for processes with independent increments most frequently involved in mathematical statistics; for example, for the Poisson (with $a = 0$, $b = \infty$, $\beta = 0$), negative-binomial and gamma (with $a = 0$, $b = \infty$, $\beta = 1$), and Wiener (with $a = -\infty$, $b = \infty$, $\beta = 0$) processes.

Integration by parts, in view of (7), (9) and (10), leads to the following relations:

(11)
$$t\int_{\Theta} \vartheta w_2'(\vartheta)\exp[w_1(\vartheta)t+w_2(\vartheta)x]\,d\vartheta = x\int_{\Theta} w_2'(\vartheta)\exp[w_1(\vartheta)t+w_2(\vartheta)x]\,d\vartheta,$$

(12)
$$\int_{\Theta} \vartheta(x-\vartheta t)w_2'(\vartheta)\exp[w_1(\vartheta)t+w_2(\vartheta)x]\,d\vartheta = -\int_{\Theta} \exp[w_1(\vartheta)t+w_2(\vartheta)x]\,d\vartheta.$$

By equations (8), (11) and (12) we obtain

$$\int_{\Theta} (x-\vartheta t)^2 w_2'(\vartheta)\exp[w_1(\vartheta)t+w_2(\vartheta)x]\,d\vartheta = \frac{t}{s(t,x)(t-\beta)}. \tag{13}$$

In wiew of (7) the expected value and the variance of the processes to be considered, according to (5) and (6), are given by

$$\mathrm{E}_{\vartheta}(X_t) = \vartheta t \tag{14}$$

and

$$\mathrm{D}_{\vartheta}(X_t) = \frac{1}{w_2'(\vartheta)}\,t, \tag{15}$$

respectively.

As the loss function we take the weighted quadratic loss function

$$L(\vartheta, f) = w_2'(\vartheta)(f-\vartheta)^2, \tag{16}$$

i.e. the squared error measured in terms of the variance.

Let, for every $n = 1, 2, \ldots,$ the prior probability distribution of the parameter ϑ, determined by the distribution function $\Phi(\vartheta)$, have the density

$$\begin{aligned} \varphi_n(\vartheta) &= \frac{1}{n}\,p\left(\frac{1}{n}+\beta, \gamma; \vartheta\right) \\ &= \frac{1}{n}\,s\left(\frac{1}{n}+\beta, \gamma\right)\exp\left[\left(\frac{1}{n}+\beta\right)w_1(\vartheta)+\gamma w_2(\vartheta)\right], \end{aligned} \tag{17}$$

where γ is a positive constant. We prove that $\varphi_n(\vartheta)$ is really a density of a probability distribution on Θ, i.e. we show that

$$\int_{\Theta} \varphi_n(\vartheta)\,d\vartheta = 1.$$

In view of (17) we obtain

$$\int_{\Theta} \varphi_n(\vartheta)\,d\vartheta = \frac{1}{n}\,s\left(\frac{1}{n}+\beta,\gamma\right)\int_{\Theta} \exp\left[\left(\frac{1}{n}+\beta\right)w_1(\vartheta)+\gamma w_2(\vartheta)\right]d\vartheta,$$

and the desired result follows from (8).

Let us suppose that the process is realized up to time $T_0 > 0$. The density of the posterior probability distribution of the parameter ϑ, given $X_{T_0} = x$, is determined by

$$\varphi_n(\vartheta\,|\,x) = \frac{\varphi_n(\vartheta)p(T_0,x;\vartheta)}{\int_{\Theta}\varphi_n(\vartheta)p(T_0,x;\vartheta)\,d\vartheta}.$$

Substituting (17) into this formula and taking into account (4) we obtain

$$\varphi_n(\vartheta\,|\,x) = \frac{\exp[w_1(\vartheta)(T_0+1/n+\beta)+w_2(\vartheta)(x+\gamma)]}{\int_{\Theta}\exp[w_1(\vartheta)(T_0+1/n+\beta)+w_2(\vartheta)(x+\gamma)]\,d\vartheta}.$$

Now, using relation (8), we have

$$\varphi_n(\vartheta\,|\,x) = \left(T_0+\frac{1}{n}\right)s\left(T_0+\frac{1}{n}+\beta,\,x+\gamma\right)\exp\left[w_1(\vartheta)\left(T_0+\frac{1}{n}+\beta\right)+\right.$$
$$\left.+w_2(\vartheta)(x+\gamma)\right]. \tag{18}$$

According to formulae (3) and (16), the posterior δ^0-risk takes the form

$$\tilde{r}(\Phi_n,f^0\,|\,x) = \int_{\Theta} w_2'(\vartheta)(f^0-\vartheta)^2\varphi_n(\vartheta\,|\,x)\,d\vartheta. \tag{19}$$

It is easily seen that this risk attains its minimum for

$$\hat{f}_n^0(x) = \frac{\int_{\Theta}\vartheta w_2'(\vartheta)\varphi_n(\vartheta\,|\,x)\,d\vartheta}{\int_{\Theta} w_2'(\vartheta)\varphi_n(\vartheta\,|\,x)\,d\vartheta}.$$

By substituting (18) into this formula we have

$$\hat{f}_n^0(x) = \frac{\int_{\Theta}\vartheta w_2'(\vartheta)\exp[w_1(\vartheta)(T_0+1/n+\beta)+w_2(\vartheta)(x+\gamma)]\,d\vartheta}{\int_{\Theta} w_2'(\vartheta)\exp[w_1(\vartheta)(T_0+1/n+\beta)+w_2(\vartheta)(x+\gamma)]\,d\vartheta}.$$

Finally, taking into account (11), we obtain

$$\hat{f}_n^0(x) = \frac{x+\gamma}{T_0+1/n+\beta}. \tag{20}$$

Thus the δ^0-Bayes estimator for Φ_n is of the form

$$\hat{f}_n^0 = \frac{X_{T_0}+\gamma}{T_0+1/n+\beta}. \tag{21}$$

Substituting (20) into (19) and using (18) we obtain the following formula which determines the posterior δ^0-risk corresponding to estimator (21):

$$\tilde{r}(\Phi_n, \hat{f}_n^0 | x) = \left(T_0+\frac{1}{n}\right) s\left(T_0+\frac{1}{n}+\beta, x+\gamma\right) \times$$
$$\times \int_{\Theta} w_2'(\vartheta)\left(\frac{x+\gamma}{T_0+1/n+\beta}-\vartheta\right)^2 \exp\left[w_1(\vartheta)\left(T_0+\frac{1}{n}+\beta\right)+w_2(\vartheta)(x+\gamma)\right] d\vartheta$$
$$= \frac{(T_0+1/n)s(T_0+1/n+\beta, x+\gamma)}{(T_0+1/n+\beta)^2} \times$$
$$\times \int_{\Theta} \left[x+\gamma-\vartheta\left(T_0+\frac{1}{n}+\beta\right)\right]^2 w_2'(\vartheta) \exp\left[w_1(\vartheta)\left(T_0+\frac{1}{n}+\beta\right)+w_2(\vartheta)(x+\gamma)\right] d\vartheta.$$

Taking into account relation (13) we obtain

$$\tilde{r}(\Phi_n, \hat{f}_n^0 | x) = \frac{1}{T_0+1/n+\beta}.$$

We then see that the posterior δ^0-risk is independent of X_{T_0}.

With reference to the Lemma let us now take into consideration the estimator

$$f^0 = \frac{X_{T_0}}{T_0+\beta}. \tag{22}$$

Let us evaluate the risk corresponding to this estimator. According to formula (1), in view of (16) and (22), we have

$$\tilde{R}(\vartheta, f^0) = \frac{w_2'(\vartheta)}{(T_0+\beta)^2} \mathrm{E}_\vartheta\{[X_{T_0}-\vartheta(T_0+\beta)]^2\}.$$

Using (14) and (15) we get

$$\tilde{R}(\vartheta, f^0) = \frac{T_0+\beta^2\vartheta^2 w_2'(\vartheta)}{(T_0+\beta)^2}.$$

Let us remark that if $\beta = 0$, then $\tilde{R}(\vartheta, f^0) = 1/T_0$. Moreover, if

$$\sup_{\vartheta\in\Theta} \vartheta^2 w_2'(\vartheta) = \frac{1}{\beta} \quad \text{for } \beta > 0,$$

then

$$\sup_{\vartheta\in\Theta} \tilde{R}(\vartheta, f^0) = \frac{1}{T_0+\beta}.$$

We then have

$$\tilde{R}(T_0) = \frac{1}{T_0+\beta} = \sup_{\vartheta\in\Theta} \tilde{R}(\vartheta, f^0) = \lim_{n\to\infty} \tilde{r}(\Phi_n, \hat{f}_n^0 \mid x) \quad \text{for } \beta \geqslant 0.$$

The above consideration allows us to assert what follows:

THEOREM. *Suppose that, for* $\beta > 0$,

$$\sup_{\vartheta\in\Theta} \vartheta^2 w_2'(\vartheta) = \frac{1}{\beta}. \tag{23}$$

Then, for processes of the exponential class satisfying the above-given assumptions and for the loss function (16), *the fixed-time plan* $\delta^0 = (T_0, f^0)$ *with* T_0 *for which the expression*

$$c(T_0) + \frac{1}{T_0+\beta}$$

attains its minimum is minimax.

Condition (23) is fulfilled, for example, for the negative-binomial and gamma processes ($\beta = 1$). Moreover, let us remark that for the Poisson and Wiener processes ($\beta = 0$) f^0 is an unbiased and efficient estimator of the parameter ϑ (as concerns the efficiency of sequential plans for the exponential class of processes see [2]).

References

[1] A. Dvoretzky, J. Kiefer and J. Wolfowitz, *Sequential decision problems for processes with continuous time parameter. Problems of estimation*, Ann. Math. Statist. 24 (1953), p. 403-415.

[2] R. Magiera, *On the inequality of Cramér-Rao type in sequential estimation theory*, Zastosow. Matem. 14 (1974), p. 227-235.

[3] P. A. Meyer, *Probability and potentials*, Blaisdell, Waltham, Mass., 1966.

INSTITUTE OF MATHEMATICS
TECHNICAL UNIVERSITY
50-370 WROCŁAW

Received on 5. 2. 1976

R. MAGIERA (Wrocław)

MINIMAKSOWA ESTYMACJA SEKWENCYJNA DLA WYKŁADNICZEJ KLASY PROCESÓW

STRESZCZENIE

W pracy rozpatruje się problem estymacji sekwencyjnej nieznanych parametrów procesów stochastycznych, gdy strata poniesiona przez statystyka związana jest nie tylko z błędem estymacji, ale również z kosztami za obserwację procesu. Uogólniając wynik Dvoretzky'ego et al. [1] udowodniono, że dla wykładniczej klasy procesów spełniających pewne dodatkowe założenia i dla kwadratowej funkcji strat plan sekwencyjny o stałym czasie obserwacji jest minimaksowy.