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PROSPECTIVE KOLMOGOROV EQUATION OF NON-MARKOVIAN STOCHASTIC PROCESSES

Let Y_t, for $t \geqslant 0$, be a stochastic process with real values from some interval I. Let F and H be the conditional probability distribution functions of this process, defined as

(1) $$F(t_0, y_0, t, y) = \mathrm{P}(Y_t < y \mid Y_{t_0} = y_0),$$

(2) $$H(t_0, y_0, w, z, t, y) = \mathrm{P}(Y_t < y \mid Y_{t_0} = y_0,\ Y_w = z), \quad 0 \leqslant t_0 < w < t.$$

Then we have

(3) $$F(t_0, y_0, t, y) = \int_I \mathrm{P}(Y_t < y \mid Y_{t_0} = y_0,\ Y_w = z)\, d_z \mathrm{P}(Y_w < z \mid Y_{t_0} = y_0) = \int_I H(t_0, y_0, w, z, t, y)\, d_z F(t_0, y_0, w, z).$$

Let us assume that there exist continuous derivatives

(4) $$\frac{\partial}{\partial y} F(t_0, y_0, t, y) = f(t_0, y_0, t, y),$$

(5) $$\frac{\partial}{\partial y} H(t_0, y_0, w, z, t, y) = h(t_0, y_0, w, z, t, y).$$

From (3)-(5) it follows that

(6) $$f(t_0, y_0, t, y) = \int_I h(t_0, y_0, w, z, t, y) f(t_0, y_0, w, z)\, dz.$$

It is evident that for a Markov process we have

$$h(t_0, y_0, w, z, t, y) = f(w, z, t, y),$$

and equation (6) takes the following form:

(7) $$f(t_0, y_0, t, y) = \int_I f(w, z, t, y) f(t_0, y_0, w, z)\, dz.$$

Formula (7) is the Chapman-Kolmogorov equation. Formula (6) is a generalization of the Chapman-Kolmogorov equation to the case of non-Markovian processes. The fact that the first factor of the integrend depends on t_0 and y_0 shows that the Markov property does not hold for the process under consideration ([1], p. 89).

It is known that if Y_t is a Markov process and for any $\delta > 0$ the function f satisfies the conditions

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{|y-y_0|\geqslant\delta} f(t-\Delta t, y_0, t, y)\,dy = 0, \tag{8}$$

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{|y-y_0|<\delta} (y-y_0)f(t-\Delta t, y_0, t, y)\,dy = a_1(t, y), \tag{9}$$

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{|y-y_0|<\delta} (y-y_0)^2 f(t-\Delta t, y_0, t, y)\,dy = a_2(t, y), \tag{10}$$

where the convergence in (8)-(10) is uniform in y_0, and the derivatives

$$\frac{\partial}{\partial t}f(t_0, y_0, t, y), \quad \frac{\partial^i}{\partial y_0^i}f(t_0, y_0, t, y), \tag{11}$$

$$\frac{\partial^i}{\partial y^i}[a_i(t, y)f(t_0, y_0, t, y)] \quad \text{for } i = 1, 2$$

exist and are continuous, then the function f satisfies the Kolmogorov equations

$$\frac{\partial}{\partial t}f(t_0, y_0, t, y)+\frac{\partial}{\partial y}[a_1(t, y)f(t_0, y_0, t, y)] = \frac{1}{2}\frac{\partial^2}{\partial y^2}[a_2(t, y)f(t_0, y_0, t, y)], \tag{12}$$

$$\frac{\partial}{\partial t_0}f(t_0, y_0, t, y)]+a_1(t_0, y_0)\frac{\partial}{\partial y_0}f(t_0, y_0, t, y) = -\frac{1}{2}a_2(t_0, y_0)\frac{\partial}{\partial y_0^2}f(t_0, y_0, t, y). \tag{13}$$

The proof of equations (12) and (13) is based on the fundamental formula (7).

This paper deal with a generalization of prospective equation (12) to the case of non-Markovian process.

We shall assume that the function h, defined by (5), satisfies for any $\delta > 0$ the conditions

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{|y-z|\geq\delta} h(t_0, y_0, w, z, t, y)\,dy = 0, \tag{14}$$

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{|y-z|<\delta} (y-z)h(t_0, y_0, w, z, t, y)\,dy = a_1(t_0, y_0, w, z), \tag{15}$$

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{|y-z|<\delta} (y-z)^2 h(t_0, y_0, w, z, t, y)\,dy = a_2(t_0, y_0, w, z), \tag{16}$$

where the convergence in (14)-(16) is uniform in z, and

$$\frac{\partial}{\partial t}f(t_0, y_0, t, y), \quad \frac{\partial^i}{\partial y^i}[a_i(t_0, y_0, t, y)f(t_0, y_0, t, y)] \quad \text{for } i = 1, 2 \tag{17}$$

exist and are continuous.

It is evident that conditions (14)-(16) are a generalization of conditions (8)-(10).

THEOREM. *If conditions* (14)-(17) *are satisfied, then the function* f *satisfies the following partial differential equation*:

$$\frac{\partial}{\partial t}f(t_0, y_0, t, y)+\frac{\partial}{\partial y}[a_1(t_0, y_0, t, y)f(t_0, y_0, t, y)] = \frac{1}{2}\frac{\partial^2}{\partial y^2}[a_2(t_0, y_0, t, y)f(t_0, y_0, t, y)]. \tag{18}$$

Proof. Let a and b be arbitrary real numbers such that $(a, b)\in I$. Let $R(y)$ denote an arbitrary non-negative function from class C^2 and let $R(y) = 0$ for $y < a$ and $y > b$.

Then

$$R(a) = R(b) = R'(a) = R'(b) = R''(a) = R''(b) = 0.$$

From the assumptions concerning the function f and h it follows that

$$\int_a^b \frac{\partial}{\partial t}f(t_0, y_0, t, y)R(y)\,dy = \frac{\partial}{\partial t}\int_a^b f(t_0, y_0, t, y)R(y)\,dy$$

$$= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_a^b [f(t_0, y_0, t+\Delta t, y)-f(t_0, y_0, t, y)]R(y)\,dy$$

$$= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_a^b\Big[\int_I h(t_0,y_0,w,z,t+\Delta t,y)f(t_0,y_0,w,z)\,dz - f(t_0,y_0,t,y)\Big]R(y)\,dy$$

$$= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\Big[\int_I dz\int_a^b h(t_0,y_0,w,z,t+\Delta t,y)f(t_0,y_0,w,z)R(y)\,dy - \\ - \int_a^b f(t_0,y_0,t,y)R(y)\,dy\Big]$$

$$= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\Big[\int_I dy\int_I h(t_0,y_0,t,y,t+\Delta t,z)f(t_0,y_0,t,y)R(z)\,dz - \\ - \int_I f(t_0,y_0,t,y)R(y)\,dy\Big]$$

$$= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_I f(t_0,y_0,t,y)\Big[\int_I h(t_0,y_0,t,y,t+\Delta t,z)R(z)\,dz - R(y)\Big]dy$$

$$= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_I f(t_0,y_0,t,y)\Big[\int_{|y-z|<\delta} h(t_0,y_0,t,y,t+\Delta t,z)R(z)\,dz - R(y)\Big]dy +$$

$$+ \lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_I f(t_0,y_0,t,y)\Big[\int_{|y-z|\geqslant\delta} h(t_0,y_0,t,y,t+\Delta t,z)R(z)\,dz\,dy\Big]$$

$$= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_I f(t_0,y_0,t,y)\Big[\int_{|y-z|<\delta} h(t_0,y_0,t,y,t+\Delta t,z)R(z)\,dz - R(y)\Big]dy.$$

Let us expand the function R into a Taylor series. Then we have

$$\int_{|y-z|<\delta} h(t_0,y_0,t,y,t+\Delta t,z)R(z)\,dz - R(y) = \int_{|y-z|<\delta} h(t_0,y_0,t,y,t+\Delta t,z)\times \\ \times\big[R(y)+(z-y)R'(y)+\tfrac{1}{2}(z-y)^2R''(y)+o(z-y)^2\big]dz - R(y)$$

$$= R'(y)\int_{|y-z|<\delta}(z-y)h(t_0,y_0,t,y,t+\Delta t,z)\,dz +$$

$$+\frac{1}{2}R''(y)\int_{|y-z|<\delta}(z-y)^2h(t_0,y_0,t,y,t+\Delta t,z)\,dz +$$

$$+\int_{|y-z|<\delta}o(z-y)^2h(t_0,y_0,t,y,t+\Delta t,z)\,dz + o(\Delta t).$$

Then

$$(19)\qquad \int_I \frac{\partial}{\partial t} f(t_0, y_0, t, y) R(y)\, dy$$

$$= \lim_{\Delta t\to 0} \frac{1}{\Delta t} \int_a^b f(t_0, y_0, t, y) \Big\{ R'(y) \int_{|y-z|<\delta} (z-y) h(t_0, y_0, t, y, t+\Delta t, z)\, dz +$$

$$+ \frac{1}{2} R''(y) \int_{|y-z|<\delta} (z-y)^2 h(t_0, y_0, t, y, t+\Delta t, z)\, dz +$$

$$+ \int_{|y-z|<\delta} o(z-y)^2 h(t_0, y_0, t, y, t+\Delta t, z)\, dz + o(\Delta t) \Big\} dy.$$

Now let us notice that

$$(20)\qquad \frac{1}{\Delta t} \int_a^b f(t_0, y_0, t, y) \Big[\int_{|y-z|<\delta} o(z-y)^2 h(t_0, y_0, t, y, t+\Delta t, t, z)\, dz \Big] dy$$

$$= \frac{1}{\Delta t} \int_a^b f(t_0, y_0, t, y) \Big[\int_{|y-z|<\delta} O(z-y)(z-y)^2 h(t_0, y_0, t, y, t+\Delta t, z)\, dz \Big] dy = W.$$

For $|z-y| < \delta$ the function $O(z-y)$ satisfies the inequality $-\delta \leqslant O(z-y) \leqslant \delta$.

In virtue of (16) it is evident that the function W, defined by (20), tends to 0 when $\delta \to 0$ and $\Delta t \to 0$.

Using (15), (16) and the fact that W tends to 0, we can write relation (19) in the following form:

$$\int_a^b \frac{\partial}{\partial t} f(t_0, y_0, t, y) R(y)\, dy$$

$$= \int_a^b f(t_0, y_0, t, y) [R'(y) a_1(t_0, y_0, t, y) + \tfrac{1}{2} R''(y) a_2(t_0, y_0, t, y)]\, dy$$

$$= \int_a^b \Big\{ -R(y) \frac{\partial}{\partial y} [a_1(t_0, y_0, t, y) f(t_0, y_0, t, y)] +$$

$$+ \frac{1}{2} R(y) \frac{\partial^2}{\partial y^2} [a_2(t_0, y_0, t, y) f(t_0, y_0, t, y)] \Big\} dy.$$

The assertion of the theorem follows directly from the last relation.

The reasoning which was used in proving the Theorem is similar to that used in proving the corresponding theorem for Markov processes ([2] Chap. VIII, § 1, and [3], § 54). Function (5) depends on t_0 and y_0, but these variables in our reasoning have the role of constant parameters.

In virtue of (18) we can determine the function f if the functions a_i are given. Note that without Markov property the function f does not determine the process (even if an initial condition is given). However, the function f gives some characteristics of the process.

Now we are going to give a comparison of different forms of prospective equations.

First let us notice that if the process Y_t is homogeneous in time and in space, then equation (18) has the following form:

$$\frac{\partial}{\partial t}f(t-t_0, y-y_0)+\frac{\partial}{\partial y}[a_1(t-t_0, y-y_0)f(t-t_0, y-y_0)]$$
$$=\frac{1}{2}\frac{\partial^2}{\partial y^2}[a_2(t-t_0, y-y_0)f(t-t_0, y-y_0)].$$

In this case a_1, a_2 and f can be treated as functions of two variables.

If Y_t is a homogeneous Markov process, then equation (12) has the following form:

$$\frac{\partial}{\partial t}f(t-t_0, y-y_0)+a_1\frac{\partial}{\partial y}f(t-t_0, y-y_0) = \frac{1}{2}a_2\frac{\partial^2}{\partial y^2}f(t-t_0, y-y_0).$$

In this case a_1 and a_2 are constant.

If Y_t is a Markov process and an initial distribution $P(Y_0 < y)$ is given, then it follows from (12) that the function

$$\tilde{f}(t, y) = \frac{\partial}{\partial y}[\mathrm{P}(Y_t < y)]$$

satisfies the following equation ([2], § 1):

$$\frac{\partial}{\partial t}\tilde{f}(t, y)+\frac{\partial}{\partial y}[a_1(t, y)\tilde{f}(t, y)] = \frac{1}{2}\frac{\partial^2}{\partial y^2}[a_2(t, y)\tilde{f}(t, y)].$$

All these discussed forms of the prospective equation for the densities are given in Table 1.

It is interesting to notice that

1° equations IV and V have an analogous form,

2° equations I and II have an analogous form if the functions $a_i(t_0, y_0, t, y)$ for $i = 1, 2$ do not depend on t_0 and y_0.

TABLE 1

	Markov process	non-Markovian process
Equation for the density of the conditional distribution	I $\frac{\partial}{\partial t} f(t_0, y_0, t, y) + \frac{\partial}{\partial y} [a_1(t, y) f(t_0, y_0, t, y)] = \frac{1}{2} \frac{\partial^2}{\partial y^2} [a_2(t, y) f(t_0, y_0, t, y)]$	II $\frac{\partial}{\partial t} f(t_0, y_0, t, y) + \frac{\partial}{\partial y} [a_1(t_0, y_0, t, y) f(t_0, y_0, t, y)] = \frac{1}{2} \frac{\partial^2}{\partial y^2} [a_2(t_0, y_0, t, y) f(t_0, y_0, t, y)]$
Equation for the density of the conditional distribution. The process homogeneous in time and in space	III $\frac{\partial}{\partial t} f(t-t_0, y-y_0) + a_1 \frac{\partial}{\partial y} f(t-t_0, y-y_0) = \frac{1}{2} a_2 \frac{\partial^2}{\partial y^2} f(t-t_0, y-y_0)$	IV $\frac{\partial}{\partial t} f(t-t_0, y-y_0) + \frac{\partial}{\partial y} [a_1(t-t_0, y-y_0) f(t-t_0, y-y_0)] = \frac{1}{2} \frac{\partial^2}{\partial y^2} [a_2(t-t_0, y-y_0) f(t-t_0, y-y_0)]$
Equation for the density of the unconditional distribution	V $\frac{\partial}{\partial t} \tilde{f}(t, y) + \frac{\partial}{\partial y} [a_1(t, y) \tilde{f}(t, y)] = \frac{1}{2} \frac{\partial^2}{\partial y^2} [a_2(t, y) \tilde{f}(t, y)]$	

References

[1] J. L. Doob, *Stochastic processes*, New York 1953.
[2] I. I. Gichman i A. W. Skorochod, *Wstęp do teorii procesów stochastycznych*, Warszawa 1968.
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PROSPEKTYWNE RÓWNANIE KOŁMOGOROWA DLA NIEMARKOWSKICH PROCESÓW STOCHASTYCZNYCH

STRESZCZENIE

Twierdzenie, udowodnione w tej pracy, podaje równanie różniczkowe cząstkowe (18), które musi spełniać warunkowa gęstość $f(t_0, y_0, t, y)$ procesu stochastycznego, jeżeli spełnione są warunki (14)-(17), a funkcja $h(t_0, y_0, w, z, t, y)$ jest określona równaniami (2) i (5). Rozważane procesy stochastyczne nie muszą być procesami Markowa.
