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EXPRESSIONS FOR THE PROBABILITIES OF THE BIVARIATE HERMITE DISTRIBUTION AND RELATED PROPERTIES

The univariate Hermite distribution with probability generating function (p.g.f.)

$$g(s) = \exp\{\alpha_1(s-1) + \alpha_2(s^2-1)\}, \quad \alpha_1, \alpha_2 \geq 0,$$

was introduced by Kemp and Kemp [3]. They showed that the probabilities and the factorial moments can be expressed in terms of modified univariate Hermite polynomials.

Generalizations of the distribution to the bivariate case have been examined by Kemp [2] and Kemp and Papageorgiou [4]. One of the versions considered has five parameters and the p.g.f.

$$g(s, t) = \exp\{\alpha_1(s-1) + \alpha_2(s^2-1) + \alpha_3(t-1) + \alpha_4(t^2-1) + \alpha_5(st-1)\},$$
$$\alpha_i \geq 0, \quad i = 1, \dots, 5.$$

No closed expression for the probabilities and the factorial moments of this distribution has been given.

This paper defines the modified bivariate Hermite polynomials and considers the problem of expressing the probabilities and the factorial moments of the distribution in terms of them.

The univariate Hermite distribution is then characterized by the conditional distribution of one random variable (r.v.) given another and the regression of the second r.v. on the first. The characterization is extended to the bivariate case to characterize the five-parameter bivariate Hermite distribution.

1. Introduction. It is well known that

$$\exp\{tx - \frac{1}{2}t^2\} = \sum_{n=0}^{\infty} H_n(x) t^n / n!,$$

where $H_n(x)$ are the univariate Hermite polynomials given by

$$(1) \quad H_n(x) = \sum_{j=0}^{[n/2]} \frac{(-1)^j n! x^{n-2j}}{(n-2j)! j! 2^j}, \quad n \geq 0, \quad H_0(x) = 1;$$

here $[a]$ denotes the integer part of a .

Kemp and Kemp [3] defined the "modified" Hermite polynomials by

$$H_n^*(x) = i^{-n} H_n(ix), \quad n = 0, 1, 2, \dots,$$

and hence

$$(2) \quad \exp\{tx + \frac{1}{2}t^2\} = \sum_{n=0}^{\infty} H_n^*(x) t^n / n!.$$

The p.g.f. of the univariate Hermite distribution can be brought to this form by a simple transformation on the parameters, i.e., for $\alpha_1 = ab$, $\alpha_2 = \frac{1}{2}a^2$, $a, b \geq 0$, the p.g.f. becomes

$$\begin{aligned} g(s) &= \exp\{-(ab + a^2/2)\} \exp\{abs + a^2s^2/2\} \\ &= \exp\{-(ab + a^2/2)\} \sum_{n=0}^{\infty} H_n^*(b) (as)^n / n!, \end{aligned}$$

which implies

$$P_n = \exp\{-(ab + a^2/2)\} H_n^*(b) a^n / n!, \quad n \geq 0.$$

Similarly,

$$\mu_{(r)} = a^r H_r^*(a+b), \quad r = 0, 1, 2, \dots,$$

where $\mu_{(r)}$ denotes the r -th factorial moment about the origin.

It is natural now to ask whether the known theory on bivariate Hermite polynomials provides similar expressions for the probabilities and the factorial moments of the bivariate Hermite distribution.

In the next section the question is answered in the affirmative.

In Section 3 a characterization of the univariate Hermite distribution is considered. In particular, the non-negative r.v. X is characterized as Hermite when for another non-negative r.v. Y the distribution of $Y|X$ is binomial and the conditional expectation of X on Y satisfies a certain condition. A characterization of the Poisson, binomial and negative binomial distributions based on a similar argument has been given by Korwar [5].

Section 4 deals with the problem of providing a characterization for the five-parameter bivariate version following an analogous approach.

2. Expressions for the probabilities and the factorial moments. The bivariate Hermite polynomials are given by (see [1])

(3)

$$H_{m,n}(x, y) = a^{m/2} c^{n/2} \sum_{j=0}^{\min(m,n)} \frac{(-b)^j (-m)_{(j)} (-n)_{(j)}}{(ac)^{j/2} j!} H_{m-j} \left(\frac{u}{\sqrt{a}} \right) H_{n-j} \left(\frac{v}{\sqrt{c}} \right),$$

where $H_n(z)$ are the polynomials (1), $z_{(r)} = z(z+1) \dots (z+r-1)$, $z_{(0)} = 1$, and x, y are such that

$$u = ax + by, \quad v = bx + cy, \quad ac - b^2 > 0, \quad a, b, c > 0.$$

Define now the modified bivariate Hermite polynomials by

$$H_{m,n}^*(x, y) = i^{-m-n} H_{m,n}(ix, iy).$$

Multiplying both sides of (3) by i^{-m-n} and letting $x \rightarrow ix$, $y \rightarrow iy$, we can show that $H_{m,n}^*(x, y)$ relates to $H_i^*(z)$. Thus

$$(4) \quad H_{m,n}^*(x, y) = a^{m/2} c^{n/2} \sum_{j=0}^{\min(m,n)} \frac{b^j (-m)_{(j)} (-n)_{(j)}}{(ac)^{j/2} j!} H_{m-j}^* \left(\frac{u}{\sqrt{a}} \right) H_{n-j}^* \left(\frac{v}{\sqrt{c}} \right).$$

Consider now the p.g.f. of the bivariate Hermite distribution

$$(5) \quad g(s, t) = \exp\{a_1(s-1) + a_2(s^2-1) + a_3(t-1) + a_4(t^2-1) + a_5(st-1)\},$$

$$a_i \geq 0, \quad i = 1, \dots, 5.$$

Let $a_1 = a_1 b_1$, $a_2 = \frac{1}{2} a_1^2$, $a_3 = a_2 b_2$, $a_4 = \frac{1}{2} a_2^2$, $a_5 = a_0$, $a_1, a_2 > 0$, $a_0, b_1, b_2 \geq 0$. Then the p.g.f. (5) takes the form

$$g(s, t) = c \exp\{a_1 b_1 s + \frac{1}{2} a_1^2 s^2\} \exp\{a_2 b_2 t + \frac{1}{2} a_2^2 t^2\} \exp\{a_0 st\},$$

where $c = \exp\{-a_1 b_1 - \frac{1}{2} a_1^2 - a_2 b_2 - \frac{1}{2} a_2^2 - a_0\}$.

Using (2) we obtain

$$\begin{aligned} g(s, t) &= c \sum_{m,n,k} H_m^*(b_1) H_n^*(b_2) \frac{a_1^m}{m!} \frac{a_2^n}{n!} \frac{a_0^k}{k!} s^{m+k} t^{n+k} \\ &= c \sum_{k=0}^{\infty} \sum_{m=k}^{\infty} \sum_{n=k}^{\infty} \frac{H_{m-k}^*(b_1)}{(m-k)!} \frac{H_{n-k}^*(b_2)}{(n-k)!} \frac{(a_0/a_1 a_2)^k}{k!} (a_1 s)^m (a_2 t)^n \\ &= c \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a_1 s)^m}{m!} \frac{(a_2 t)^n}{n!} \sum_{k=0}^{\min(m,n)} \frac{(a_0/a_1 a_2)^k (-m)_{(k)} (-n)_{(k)}}{k!} \times \\ &\quad \times H_{m-k}^*(b_1) H_{n-k}^*(b_2). \end{aligned}$$

Then, from (4) it follows that

$$g(s, t) = c \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} H_{m,n}^*(h, l) \frac{(a_1 s)^m}{m!} \frac{(a_2 t)^n}{n!},$$

where (h, l) is the solution to the system

$$h + a_0 l / a_1 a_2 = b_1, \quad l + a_0 h / a_1 a_2 = b_2.$$

Therefore, the corresponding joint probabilities are given by

$$P_{00} = \exp\{-a_1 b_1 - \frac{1}{2} a_1^2 - a_2 b_2 - \frac{1}{2} a_2^2 - a_0\},$$

$$P_{m,n} = P_{00} \frac{a_1^m}{m!} \frac{a_2^n}{n!} H_{m,n}^*(h, l), \quad m, n > 0,$$

where

$$h = \frac{b_1 - a_0 b_2 / a_1 a_2}{1 - a_0^2 / a_1^2 a_2^2}, \quad l = \frac{b_2 - a_0 b_1 / a_1 a_2}{1 - a_0^2 / a_1^2 a_2^2}.$$

Using a similar argument we obtain the following formula for the (m, n) -factorial moment about the origin:

$$\mu_{(m,n)} = a_1^m a_2^n H_{m,n}^*(h + a_1, l + a_2).$$

3. Characterization of the Univariate Hermite Distribution (U.H.D.).

THEOREM 1. *Let X, Y be non-negative discrete r.v.'s with*

$$(6) \quad P(Y = y | X = x) = \binom{x}{y} p^y q^{x-y}, \quad 0 < p < 1, \quad p + q = 1, \quad y \leq x.$$

Then

$$(7) \quad E(X | Y = y) = a \frac{H_{y+1}^*(a+b)}{H_y^*(a+b)} + y, \quad a, b > 0,$$

iff $X \sim \text{U.H.D.}(ab/q, a^2/2q^2)$.

Proof. The necessity follows immediately by substituting $P(X = x) \sim \text{U.H.D.}(\lambda_1, \lambda_2)$ and $P[Y = y | X = x] \sim \text{binomial}(x, p)$ in

$$E(X | Y = y) = \sum_x x P(X = x) P(Y = y | X = x) / P(Y = y)$$

and making use of the identity

$$x \binom{x}{y} = (y+1) \binom{x}{y+1} + y \binom{x}{y}.$$

Sufficiency. Assume that (6) and (7) hold. Then it is easily seen that

$$E(X | Y = y) = (y+1) \frac{q}{p} \frac{P(Y = y+1)}{P(Y = y)} + y,$$

and hence, by (7), we have

$$P(Y = y + 1) - \frac{ap}{q(y + 1)} \frac{H_{y+1}^*(a + b)}{H_y^*(a + b)} P(Y = y) = 0.$$

Therefore,

$$\begin{aligned} P(Y = y) &= P(Y = 0) \prod_{i=0}^{y-1} \frac{ap}{q} \frac{H_{i+1}^*(a + b)}{H_i^*(a + b)} \frac{1}{i + 1} \\ &= P(Y = 0) \frac{(ap/q)^y}{y!} H_y^*(a + b). \end{aligned}$$

Summing both sides over y we obtain

$$P(Y = 0) = \exp\{-ap(a + b)/q - a^2p^2/q^2\},$$

i.e. $Y \sim \text{U.H.D.}(ap(a + b)/q, a^2p^2/2q^2)$.

But, if $G_Z(t)$ denotes the p.g.f. of Z , we have

$$\begin{aligned} \exp\left\{\frac{ap}{q}(a + b)(t - 1) + \frac{a^2p^2}{2q^2}(t^2 - 1)\right\} &\equiv G_Y(t) = \sum_x G_{Y|X=x}(t) P(X = x) \\ &= G_X(pt + q). \end{aligned}$$

Hence

$$G_X(t) = G_Y\left(\frac{t - q}{p}\right) = \exp\left\{\frac{ab}{q}(t - 1) + \frac{a^2}{2q^2}(t^2 - 1)\right\},$$

i.e. $X \sim \text{U.H.D.}(ab/q, a^2/2q^2)$.

4. Characterization of the Bivariate Hermite Distribution (B.H.D.).

THEOREM 2. Let $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$ be non-negative discrete random vectors with

$$P(Y = \mathbf{y} | X = \mathbf{x}) = \prod_{i=1}^2 \binom{x_i}{y_i} p_i^{y_i} q_i^{x_i - y_i},$$

$$0 < p_i < 1, p_i + q_i = 1, y_i \leq x_i, i = 1, 2.$$

Then

$$(8) \quad E(X_1 | Y = \mathbf{y}) = y_1 + l_1 \frac{H_{y_1+1, y_2}^*(c_1, c_2)}{H_{y_1, y_2}^*(c_1, c_2)},$$

$$E(X_2 | Y = \mathbf{y}) = y_2 + l_2 \frac{H_{y_1, y_2+1}^*(c_1, c_2)}{H_{y_1, y_2}^*(c_1, c_2)},$$

$$l_i, c_i > 0, i = 1, 2, l_1 + l_2 \leq \min(c_1, c_2),$$

iff $X \sim \text{B.H.D.}$

Proof. The necessity is straightforward.
For the sufficiency we observe that

$$\mathbb{E}(X_1 | \mathbf{Y} = \mathbf{y}) = (y_1 + 1) \frac{q_1}{p_1} \frac{P(Y_1 = y_1 + 1, Y_2 = y_2)}{P(\mathbf{Y} = \mathbf{y})} + y_1$$

and

$$\mathbb{E}(X_2 | \mathbf{Y} = \mathbf{y}) = (y_2 + 1) \frac{q_2}{p_2} \frac{P(Y_1 = y_1, Y_2 = y_2 + 1)}{P(\mathbf{Y} = \mathbf{y})} + y_2.$$

Then from (8) we obtain the system of equations

$$(9) \quad \begin{aligned} P(Y_1 = y_1 + 1, Y_2 = y_2) &= \frac{p_1 l_1 / q_1}{y_1 + 1} \frac{H_{y_1 + 1, y_2}^*(c_1, c_2)}{H_{y_1, y_2}^*(c_1, c_2)} P(\mathbf{Y} = \mathbf{y}), \\ P(Y_1 = y_1, Y_2 = y_2 + 1) &= \frac{p_2 l_2 / q_2}{y_2 + 1} \frac{H_{y_1, y_2 + 1}^*(c_1, c_2)}{H_{y_1, y_2}^*(c_1, c_2)} P(\mathbf{Y} = \mathbf{y}), \end{aligned}$$

which implies

$$P(\mathbf{Y} = \mathbf{y}) = P(\mathbf{Y} = \mathbf{0}) \prod_{i=0}^{y_1-1} h_1(i, 0) \prod_{j=0}^{y_2-1} h_2(y_1, j),$$

where $h_i(y_1, y_2)$, $i = 1, 2$, is the right-hand side of the i -th equation of system (9), i.e.,

$$P(\mathbf{Y} = \mathbf{y}) = P(\mathbf{Y} = \mathbf{0}) \frac{(p_1 l_1 / q_1)^{y_1}}{y_1!} \frac{(p_2 l_2 / q_2)^{y_2}}{y_2!} H_{y_1, y_2}^*(c_1, c_2).$$

This implies that

$$G_{\mathbf{Y}}(\mathbf{t}) = \exp\{a_1 b_1 (s-1) + \frac{1}{2} a_1^2 (s^2-1) + a_2 b_2 (t-1) + \frac{1}{2} a_2^2 (t^2-1) + a_0 (st-1)\},$$

where a_i, b_j ($i = 0, 1, 2$; $j = 1, 2$) satisfy

$$\begin{aligned} a_1 &= p_1 l_1 / q_1, & a_2 &= p_2 l_2 / q_2, \\ b_1 &= c_1 + a_0 c_2 / a_1 a_2, & b_2 &= c_2 + a_0 c_1 / a_1 a_2, \end{aligned}$$

$$\begin{aligned} &\exp\{-a_1 b_1 - \frac{1}{2} a_1^2 - a_2 b_2 - \frac{1}{2} a_2^2 - a_0\} \\ &= \left\{ \sum_{y_1, y_2} \frac{(p_1 l_1 / q_1)^{y_1}}{y_1!} \frac{(p_2 l_2 / q_2)^{y_2}}{y_2!} H_{y_1, y_2}^*(c_1, c_2) \right\}^{-1}. \end{aligned}$$

Hence

$$G_{\mathbf{X}}(\mathbf{t}) = G_{\mathbf{Y}}\left(\frac{t_1 - q_1}{p_1}, \frac{t_2 - q_2}{p_2}\right),$$

i.e.,

$X \sim$

$$\text{B.H.D.} \left(\frac{a_1 b_1}{p_1} - \frac{a_1^2 q_1}{p_1^2} - \frac{a_0 q_2}{p_1 p_2}, \frac{a_1^2}{2p_1^2}, \frac{a_2 b_2}{p_2} - \frac{a_2^2 q_2}{p_2^2} - \frac{a_0 q_1}{p_1 p_2}, \frac{a_2^2}{2p_2^2}, \frac{a_0}{p_1 p_2} \right)$$

Hence the theorem is established.

References

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