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THE EVOLUTION OF AN ITERATIVE MATRIX OF ORDER 2 IN A TWO-ELEMENT POLATA

In this paper we shall consider a problem which is a continuation of the discussion in paper [1].

Let a passive asymmetrical four-terminal network be described by an iterative matrix of order 2, as

$$A = \begin{Vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{Vmatrix} \tag{1}$$

with the determinant $\det A = A_{11}A_{22} - A_{12}A_{21} = 1$.

The elements of the iterative matrix A are used to define the following properties of a four-terminal network:

1. Primary image impedance of a four-terminal network

$$Z_{OPA} = \sqrt{\frac{A_{11}A_{12}}{A_{22}A_{21}}}.$$

2. Secondary image impedance of a four-terminal network

$$Z_{OSA} = \sqrt{\frac{A_{22}A_{12}}{A_{11}A_{21}}}.$$

3. Mean image impedance of a four-terminal network

$$\sqrt{Z_{OPA}Z_{OSA}} = \sqrt{\frac{A_{12}}{A_{21}}}.$$

4. Voltage transfer function of a four-terminal network with secondary terminals open-circuited

$$K_A = \frac{1}{A_{11}}.$$

5. Voltage transfer function of a four-terminal network with terminals which are matched properly on the primary and secondary image impedances

$$K_{mA} = \frac{1}{A_{11} + A_{12}/Z_{OSA}}.$$

In order to calculate numerical values of the impedances which can be located in serial and parallel branches of a four-terminal network containing only series-parallel connections and no reciprocal coupling the following formulas are used:

a. for a T-section

$$Z_{1A}^{T} = \frac{A_{11}-1}{A_{21}}, \quad Z_{2A}^{T} = \frac{A_{22}-1}{A_{21}}, \quad Z_{3A}^{T} = \frac{1}{A_{21}};$$

b. for a Π-section

$$Z_{1A}^{\pi} = \frac{A_{21}}{A_{22}-1}, \quad Z_{2A}^{\pi} = \frac{A_{21}}{A_{11}-1}, \quad Z_{3A}^{\pi} = A_{21}.$$

Let us mention the measures of asymmetry of the four-terminal network described by the iterative matrix $\boldsymbol{A}$,

$$\sigma_{A} = \frac{A_{11}}{A_{22}} \quad \text{and} \quad \gamma = \frac{A_{11}-1}{A_{22}-1},$$

where the symbol γ_A represents the measure of asymmetry of the impedances which are located in serial branches of the T-section or in parallel branches of the Π-section.

We now introduce two matrices $\boldsymbol{B}$ and $\boldsymbol{C}$ of order 2 such that their non-commutative product is equal to the well-known iterative matrix $\boldsymbol{A}$ of order 2,

$$\boldsymbol{A} = \boldsymbol{B}\boldsymbol{C}, \tag{2}$$

where

$$\boldsymbol{B} = \begin{Vmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{Vmatrix} \quad \text{and} \quad \boldsymbol{C} = \begin{Vmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{Vmatrix}.$$

In general, two matrices $\boldsymbol{B}$ and $\boldsymbol{C}$ describe two various asymmetrical four-terminal networks.

From equation (2) we obtain a system of algebraic equations

$$\begin{aligned} B_{11}C_{11}+B_{12}C_{21} &= A_{11}, \\ B_{11}C_{12}+B_{12}C_{12} &= A_{12}, \\ B_{21}C_{11}+B_{22}C_{21} &= A_{21}, \\ B_{21}C_{12}+B_{22}C_{22} &= A_{22}, \end{aligned} \tag{3}$$

where the elements of the two matrices $\boldsymbol{B}$ and $\boldsymbol{C}$ are unknows.

To solve system (3) we introduce the following assumptions:

1° The determinants of the matrices $\boldsymbol{B}$ and $\boldsymbol{C}$ are equal,

$$\det \boldsymbol{B} = \det \boldsymbol{C} = |\sqrt{\det \boldsymbol{A}}\,|.$$

2° The primary image impedance of the four-terminal network described by the matrix $\boldsymbol{B}$ is equal to the primary image impedance of the four-terminal network described by the matrix $\boldsymbol{A}$,

$$Z_{OPB} = Z_{OPA}.$$

3° The secondary image impedance of the four-terminal network described by the matrix $\boldsymbol{B}$ is equal to the primary image impedance of the four-terminal network described by the matrix $\boldsymbol{C}$,

$$Z_{OSB} = Z_{OPC} = \sqrt{Z_{OPA} Z_{OSA}}.$$

4° The secondary image impedance of the four-terminal network described by the matrix $\boldsymbol{C}$ is equal to the secondary image impedance of the four-terminal network described by the matrix $\boldsymbol{A}$,

$$Z_{OSC} = Z_{OSA}.$$

5° The two four-terminal networks described by the two matrices $\boldsymbol{B}$ and $\boldsymbol{C}$ have identical voltage transfer functions of the four-terminal network with secondary terminals open-circuited,

$$K_{\boldsymbol{B}} = K_{\boldsymbol{C}}.$$

6° The two four-terminal networks described by the two matrices $\boldsymbol{B}$ and $\boldsymbol{C}$ have identical voltage transfer functions of the four-terminal network with terminals which are matched properly on the primary and secondary image impedances of a four-terminal network,

$$K_{m\boldsymbol{B}} = K_{m\boldsymbol{C}}.$$

On the basis of assumptions 1°-6° it is easy to establish relations between the elements of the matrices $\boldsymbol{B}$ and $\boldsymbol{C}$:

$$(4) \quad C_{11} = B_{11}, \quad C_{12} = B_{12}\sqrt{\frac{A_{22}}{A_{11}}}, \quad C_{21} = B_{21}\sqrt{\frac{A_{11}}{A_{22}}}, \quad C_{22} = B_{22}.$$

Next, we obtain the measures of asymmetry of the four-terminal networks $\sigma_{\boldsymbol{B}} = \sigma_{\boldsymbol{C}} = \sqrt{A_{11}/A_{22}}$, hence $\sigma_{\boldsymbol{A}} = \sigma_{\boldsymbol{B}}\sigma_{\boldsymbol{C}}$.

From (3), using relations (4), we obtain a new system of equations:

$$B_{11}^2 + B_{12}B_{21}\sqrt{A_{11}/A_{22}} = A_{11},$$

$$B_{12}(B_{11}\sqrt{A_{22}/A_{11}} + B_{22}) = A_{12},$$

$$B_{21}(B_{11} + B_{22}\sqrt{A_{11}/A_{22}}) = A_{21},$$

$$B_{22}^2 + B_{12}B_{21}\sqrt{A_{22}/A_{11}} = A_{22}.$$

The special solution of this system, which gives the description of a physically realizable four-terminal network, is expressed as

$$B_{11} = \sqrt{\tfrac{1}{2}\sqrt{A_{11}/A_{22}}(\sqrt{A_{11}A_{22}} + \sqrt{\det \boldsymbol{A}})},$$
$$B_{12} = A_{12}/\sqrt{2\sqrt{A_{22}/A_{11}}(\sqrt{A_{11}A_{22}} + \sqrt{\det \boldsymbol{A}})},$$
$$B_{21} = A_{21}/\sqrt{2\sqrt{A_{11}/A_{22}}(\sqrt{A_{11}A_{22}} + \sqrt{\det \boldsymbol{A}})},$$
$$B_{22} = \sqrt{\tfrac{1}{2}\sqrt{A_{22}/A_{11}}(\sqrt{A_{11}A_{22}} + \sqrt{\det \boldsymbol{A}})}.$$

In this case we obtain at once from realtions (4)

$$C_{11} = \sqrt{\tfrac{1}{2}\sqrt{A_{22}/A_{11}}(\sqrt{A_{11}A_{22}} + \sqrt{\det \boldsymbol{A}})},$$
$$C_{12} = A_{12}/\sqrt{2\sqrt{A_{11}/A_{22}}(\sqrt{A_{11}A_{22}} + \sqrt{\det \boldsymbol{A}})},$$
$$C_{21} = A_{21}/\sqrt{2\sqrt{A_{22}/A_{11}}(\sqrt{A_{11}A_{22}} + \sqrt{\det \boldsymbol{A}})},$$
$$C_{22} = \sqrt{\tfrac{1}{2}\sqrt{A_{22}/A_{11}}(\sqrt{A_{11}A_{22}} + \sqrt{\det \boldsymbol{A}})}.$$

The product of matrics (2), whose elements satisfy system (3) and assumptions 1°-6°, is called a *two-element polata.*

If the iterative matrix (1) describes a passive symmetrical four-terminal network, i.e., if $A_{11} = A_{22}$, the elements of a polata have to satisfy $\boldsymbol{B} = \boldsymbol{C} = \boldsymbol{a}_p$, where $\boldsymbol{a}_p$ is the primogenal quadratic root of the iterative matrix $\boldsymbol{A}$.

The relations between the corresponding impedances of the four-terminal network, described by two matrices $\boldsymbol{B}$ and $\boldsymbol{C}$, are as follows:

a. for a T-section

$$Z_{1B}^{T}/Z_{2B}^{T} = \gamma_B, \quad Z_{1C}^{T}/Z_{2C}^{T} = \gamma_C,$$

where $\gamma_B = \gamma_C$ and $Z_{1B}^{T}/Z_{1C}^{T} = Z_{2B}^{T}/Z_{2C}^{T} = Z_{3B}^{T}/Z_{3C}^{T} = \sqrt{\sigma_A}$;

b. for a Π-section

$$Z_{1B}^{\pi}/Z_{2B}^{\pi} = 1/\gamma_B, \quad Z_{1C}^{\pi}/Z_{2C}^{\pi} = 1/\gamma_C,$$

where $\gamma_B = \gamma_C$ and $Z_{1B}^{\pi}/Z_{1C}^{\pi} = Z_{2B}^{\pi}/Z_{2C}^{\pi} = Z_{3B}^{\pi}/Z_{3C}^{\pi} = \sqrt{1/\sigma_A}$.

Reference

[1] J. J. Włodek, *Rozwinięcie w łańcuch pierwiastkowy pewnych macierzy łańcuchowych,* Zastosow. Matem. 2 (1967), 187-195.

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ROZWINIĘCIE W POLATĘ DWUELEMENTOWĄ MACIERZY ŁAŃCUCHOWEJ STOPNIA 2

STRESZCZENIE

W pracy rozpatruje się metodę rozwinięcia danej z góry macierzy kwadratowej stopnia 2 w iloczyn dwu macierzy kwadratowych. Rozwiązanie tego problemu sprowadza się do rozwiązania układu 4 równań algebraicznych, w których występuje 8 niewiadomych. Szczególne rozwiązanie tego układu równań nazywa się *rozwinięciem w polatę dwuelementową*.