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ON THE SOLUTION OF LINEAR INHOMOGENEOUS MATRIX DIFFERENCE EQUATIONS OF ORDER n WITH VARIABLE COEFFICIENTS

Abstract. A new method for finding the solution in a closed form of linear inhomogeneous matrix difference equations of order n with variable coefficients is presented.

1. Introduction and problem formulation. It is well known that many problems of linear discrete-time systems can be reduced to solving inhomogeneous matrix difference equations of order n with variable coefficients. In [1] Cholewicki has shown that a linear inhomogeneous difference equation of order 2 with variable coefficients is easily solvable in a closed form by the use of continuants [7]. Cholewicki's method has been extended to linear inhomogeneous difference equations of order n ($n > 2$) with variable coefficients in [5].

The purpose of this short paper is to present a new method for finding the solution in a closed form of linear inhomogeneous matrix difference equations of order n with variable coefficients.

Consider a linear inhomogeneous matrix difference equation of order n of the form

$$y_{i+n} = \sum_{j=1}^{n} a_{j,i+n-j} y_{i+n-j} + f_i \quad (i = 1, 2, \ldots), \tag{1}$$

where y_i is the m-dimensional vector of unknown variables, a_{kl} ($k = 1, 2, \ldots, n$) are ($m \times m$)-matrices of known coefficients depending on the index $l = 1, 2, \ldots$, and f_i is the m-dimensional vector of known functions.

The problem can be formulated as follows:

Given a_{kl}, f_i, and the initial conditions $y_1, y_2, \ldots, y_n$, find the solution of (1) in a closed form.

2. Problem solution. Let us define for (1) a sequence of ($m \times m$)-matrices D_j^k ($j = 1, 2, \ldots, n$; $k = n+1, n+2, \ldots$) by the recurrence equation

$$D_l^{i+n} = \sum_{j=1}^{n} a_{j,i+n-j} D_l^{i+n-j} \quad (i = 1, 2, \ldots; \ l = 1, 2, \ldots, n) \tag{2}$$

with the initial conditions

(3) $$D_j^k = \begin{cases} I_m & \text{for } j = k \\ 0 & \text{for } j \neq k \end{cases} \quad (j = 1, 2, \ldots, n),$$

where I_m is the identity matrix of order m. We also define a sequence of $(m \times m)$-matrices $D^{i,k}$ ($i = 1, 2, \ldots$; $k = n+1, n+2, \ldots$) by the recurrence equation

(4) $$D^{i+n,k} = \sum_{j=1}^{n} a_{j,i+n-j} D^{i+n-j,k}$$

with the initial conditions

(5) $$D^{i,k} = \begin{cases} I_m & \text{for } k = i+1, \\ 0 & \text{for } k > i+1. \end{cases}$$

THEOREM. *The solution of* (1) *is of the form*

(6) $$y_i = \sum_{j=1}^{n} D_j^i y_j + \sum_{j=1}^{i-n} D^{i,j+n+1} f_j \quad (i = n+1, n+2, \ldots).$$

Proof. From (6) we have

(7) $$y_{i+n-k} = \sum_{j=1}^{n} D_j^{i+n-k} y_j + \sum_{j=1}^{i-k} D^{i+n-k,j+n+1} f_j.$$

Premultiplying (7) by $a_{k,i+n-k}$, summing k from 1 to n and taking into account (2)–(5) we obtain

$$\begin{aligned} y_{i+n} &= \sum_{k=1}^{n} a_{k,i+n-k} y_{i+n-k} + f_i \\ &= \sum_{j=1}^{n} \Big(\sum_{k=1}^{n} a_{k,i+n-k} D_j^{i+n-k}\Big) y_j + \sum_{j=1}^{i-1} \Big(\sum_{k=1}^{n} a_{k,i+n-k} D^{i+n-k,j+n+1}\Big) f_j + f_i \\ &= \sum_{j=1}^{n} D_j^{i+n} y_j + \sum_{j=1}^{i} D^{i+n,j+n+1} f_j. \end{aligned}$$

This completes the proof.

3. Concluding remarks. The solution (6) in a closed form of (1) is presented. The results given in [1] for $n = 2$ and given in [5] for $n > 2$ are particular cases of those presented in this paper for $m = 1$. It can be easily shown that D_j^k and $D^{i,k}$ for $m = 1$ are equal to suitable continuants for $n = 2$ (see [1]) and to generalized continuants for $n > 2$ (see [5]). The presented method can be used, for example, for the analysis of a cascade of linear, various and active $2n$-ports (see [2]–[4]).

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