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ON THE NON-EQUIVALENCE OF TWO CRITERIA OF COMPARABILITY OF STATIONARY POINT PROCESSES

1. Introduction. This note is concerned with the comparability of stochastic point processes defined on the real line R. It seems to be natural to define the comparability of such processes by the comparability of the density of their points. Basic to any point process are, on the one hand, the counting and, on the other hand, the interval properties.

Now, the density of points can be expressed in terms either of counting or interval properties. Two stochastic point processes are said to be *comparable* if (i) the number of points in corresponding bounded Borel subsets of R, or (ii) the intervals between corresponding events are in relation with respect to a certain stochastic semi-ordering. These two possibilities of definition were proposed by D. Stoyan.

In the special case of stationary point processes one may expect, intuitively, that these two criteria of comparability are equivalent. However, it turns out, even simple examples of stationary renewal processes do not have this equivalence property.

2. Definitions. The basic definitions concerning stationary point processes defined on the real line and the notation are according to [3].

Let M be the system of all local finite subsets of R. The points of $\varphi \in M$ will be numbered, as usually,

$$\ldots < x_{-1}(\varphi) \leqslant 0 < x_0(\varphi) < x_1(\varphi) < \ldots$$

Further, $\mathfrak{M}$ denotes the smallest σ-algebra of subsets of M with the property that, for all bounded Borel subsets B of R, the mapping $h_B\colon M \to N$,

$$h_B(\varphi) = \sum_k \delta_{x_k(\varphi)}(B),$$

is $\mathfrak{M}$-measurable.

By a *stochastic point process* defined on R one understands a probability space of the form $[M, \mathfrak{M}, P]$.

For every $t \in R$, the *shifting operator* T_t will be defined on M by the formula $x \in T_t\varphi$ if and only if $x+t \in \varphi$.

A stochastic point process $[M, \mathfrak{M}, P]$ is said to be *stationary* if the probability measure P is invariant with respect to the shift T_t for every $t \in R$.

In the sequel only stationary point processes will be considered.

Definition 1. Two stationary point processes $[M, \mathfrak{M}, P_i]_{i=1,2}$ defined on the real line are said to be *comparable* ($P_1 \subseteq_1$ or $\supseteq_1 P_2$) if the distribution functions

$$F_{t,i}(n) = P_i\big(\varphi\colon x_n(\varphi) \geqslant t\big), \quad i = 1, 2,$$

satisfy the relation $F_{t,1}(n) \geqslant$ or $\leqslant F_{t,2}(n)$ for every $t > 0$ and for $n = 1, 2 \ldots$

Definition 2. Two stationary point processes $[M, \mathfrak{M}, P_i]_{i=1,2}$ defined on the real line are said to be *comparable* ($P_1 \subseteq_2$ or $\supseteq_2 P_2$) if the n-dimensional distribution functions

$$F_{n,i}(t_0, \ldots, t_{n-1}) \\ = P_i^{(0)}\big(\varphi\colon x_0(\varphi) < t_0, x_1(\varphi) - x_0(\varphi) < t_1, \ldots, x_{n-1}(\varphi) - x_{n-2}(\varphi) < t_{n-1}\big), \\ i = 1, 2,$$

satisfy the relation $F_{n,1} \overset{(1)}{\leqslant}$ or $\overset{(1)}{\geqslant} F_{n,2}$ for $n = 1, 2, \ldots$, where $P_i^{(0)}$ denotes the Palm measure corresponding to the stationary point process $[M, \mathfrak{M}, P_i]$.

Two n-dimensional distribution functions F and G are said to satisfy the relation $F \overset{(1)}{\leqslant} G$ if, for every measurable functional f of R^n having the property

$$[(x_i \leqslant y_i\colon i = 1, 2, \ldots, n) \Rightarrow f(x) \leqslant f(y)],$$

the inequality

$$\int_{R^n} f(x)\,dF(x) \leqslant \int_{R^n} f(x)\,dG(x)$$

holds (assuming the existence of the integrals (see [4])).

Therefore, definition 1 contains the comparability of the number of points in an interval of length t for every $t > 0$, but definition 2 contains the comparability of intervals between $n+1$ successive events for $n = 1, 2, \ldots$

If the random variables $x_i(\varphi) - x_{i-1}(\varphi)$ are independent and identically distributed with the distribution function A, then one obtains a recurrent stationary point process (renewal process) $[M, \mathfrak{M}, P_{[A]}]$, where the probability measure $P_{[A]}$ is induced by A (see [3]). In this case definitions 1 and 2 can be written in the following forms (a) and (b), respectively:

(a) $P_{[F]} \subseteq_1 P_{[G]}$ if, for $n = 1, 2, \ldots$ and for every $t > 0$,

$$F_1(t) \overset{\text{def}}{=} \frac{1}{m_F}\int_0^t \big(1 - F(u)\big)\,du \leqslant \frac{1}{m_G}\int_0^t \big(1 - G(u)\big)\,du \overset{\text{def}}{=} G_1(t) \tag{1}$$

and

(2) $$F_1 * F^{*n}(t) \leqslant G * G^{*n}(t),$$

where

$$m_A \stackrel{\text{def}}{=} \int_0^\infty \big(1 - A(u)\big) du,$$

and $*$ denotes the convolution operator.

(b) $P_{[F]} \subseteq_2 P_{[G]}$ if, for every $t > 0$,

(3) $$F(t) \geqslant G(t).$$

3. Counterexample. In this section some connections between counting and interval properties are investigated. As mentioned above, one might expect that $P_1 \subseteq_1 P_2$ if and only if $P_1 \supseteq_2 P_2$. Jacobs, however, gave an example in [2] that the implication $P_1 \supseteq_2 P_2 \Rightarrow P_1 \subseteq_1 P_2$ does not in general follow. It will be shown here that also the inverse implication ($P_1 \subseteq_1 P_2 \Rightarrow P_1 \supseteq_2 P_2$) is not always true.

PROPOSITION. *For the distribution functions F and G defined on R^+ by*

(4) $$F(t) = 1 - e^{-t} \quad \text{and} \quad G(t) = \begin{cases} \frac{1}{2} & \text{if } t \leqslant 1, \\ 1 & \text{if } t > 1, \end{cases}$$

the relation $P_{[F]} \subseteq_1 P_{[G]}$ holds, although $P_{[F]}$ and $P_{[G]}$ are not comparable with respect to relation $\subseteq_2$.

Proof. Since $F(1) > G(1)$ and $F(2) < G(2)$, the second assertion follows immediately from (3). It is also easy to see that inequality (1) is satisfied for every $t > 0$. Therefore, it remains to show inequality (2) for $n = 1, 2, \ldots$ and for every $t > 0$.

From (4), by simple calculations, we have

$$F_1 * F^{*n}(t) = e^{-t}\left[\frac{t^{n+1}}{(n+1)!} + \frac{t^{n+2}}{(n+2)!} + \ldots\right]$$

and

$$G_1 * G^{*n}(t)$$
$$= \begin{cases} \dfrac{1}{2^n}\left[\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{k}(t-k)\right] & \text{if } k \leqslant t < k+1,\ k = 0, 1, \ldots, n, \\ 1 & \text{if } n+1 \leqslant t. \end{cases}$$

The function $F_1 * F^{*n}$ is convex in the interval $(0, n)$. Now, since $G_1 * G^{*n}$ is linear in $(i, i+1)$, $i = 0, 1, \ldots, n-1$, it is sufficient to prove inequality (2) for $t = 1, 2, \ldots, n$ and for $t \in (n, n+1)$.

From (2), setting $t = n$, one obtains the inequality

(5) $$e^n \leqslant 2^n\left(1 + n + \ldots + \frac{n^n}{n!}\right),$$

which can be proved by complete induction. Therefore, inequality (2) holds for $t = n$. Moreover, this leads to

$$F_1 * F^{*n}(t) \leqslant G_1 * G^{*n}(t) \quad \text{for } t \in [n, n+1).$$

Since $F_1 * F^{*n}$ is convex and $G_1 * G^{*n}$ concave in $(n/2, n)$, it remains now to prove inequality (2) for $t = 1, 2, \ldots, k \leqslant n/2$. It turns out, by an elementary reasoning, that even

$$(F_1 * F^{*n})'(t) \leqslant (G_1 * G^{*n})'(t-0) \quad \text{for } t = 1, 2, \ldots, k \leqslant n/2.$$

Thus, since n is arbitrary, the proof is complete.

4. Concluding remarks. The same proposition can be obtained, varying slightly the proof, for

$$\tilde{F} \stackrel{\text{def}}{=} \chi_{(1,+\infty)} * F \quad \text{and} \quad \tilde{G} \stackrel{\text{def}}{=} \chi_{(1,+\infty)} * G,$$

for which $\tilde{F}(0+0) = \tilde{G}(0+0) = 0$, or for absolute continuous distribution functions being sufficiently close to $\tilde{F}$ and $\tilde{G}$, where F and G are given by (4), and $\chi_{(1,+\infty)}$ denotes the indicator of $(1, +\infty)$.

This statement simultaneously shows the difficulty of the problem (investigated, for instance, in [1]) of determining the interval distributions of a stationary renewal process from its counting distributions.

For the case considered by Jacobs in [2] (where $P_{[F]} \supseteq_2 P_{[G]} \Rightarrow P_{[F]} \subseteq_1 P_{[G]}$ is not true) further examples can be easily found. Namely, let F and G be distribution functions defined on R^+ and assume that there exists an $x_0 \in R^+$ with $F(x_0) < 1$ and $F(x) = G(x)$ for every $x \geqslant x_0$. Then from $F(t) \leqslant G(t)$ for every $t > 0$ inequality (1) follows if and only if $F \equiv G$.

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O NIERÓWNOWAŻNOŚCI DWU KRYTERIÓW PORÓWNYWALNOŚCI STACJONARNYCH PROCESÓW PUNKTOWYCH

STRESZCZENIE

W pracy rozważono dwa kryteria porównywalności stochastycznych procesów punktowych. Dwa stochastyczne procesy punktowe nazywają się *porównywalnymi*, jeżeli (i) liczby punktów w odpowiednich ograniczonych zbiorach borelowskich prostej rzeczywistej lub (ii) odstępy między odpowiednimi punktami są w relacji względem pewnego stochastycznego uporządkowania częściowego.

W przypadku stacjonarnych procesów punktowych intuicyjnie można by oczekiwać, że te dwa kryteria są równoważne. Na prostym przykładzie pokazano jednak, że nawet nie wszystkie pary stacjonarnych procesów odnowienia mają tę własność.