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AN INTERVAL METHOD FOR NONLINEAR SYSTEMS OF EQUATIONS

1. Introduction. This paper gives a new method with some modifications of the method of [1]. This process can be used for finding solutions of a nonlinear system of equations in a finite n-dimensional interval.

The fixed symbols are as follows:

$f_i(x) = 0$, $i = 1, 2, \ldots, n$, $a \leqslant x \leqslant b$, $a, b, x \in R^n$ is such a nonlinear system of equations, where all f_i: $R^n \to R$ are twice continuously differentiable on $[a, b]$;

$$e_i(u,x) = u + |f_i(x)|, \quad \text{where } u \in R \text{ and } u \geqslant 0;$$

$$g_i(x) = 1 + \|f_i'(x)\|_1 = 1 + \sum_j |\partial_j f_i(x)|;$$

$$h_i > \frac{1}{2} \sup_{x \in [a,b]} \|f_i''(x)\|_\infty = \frac{1}{2} \sup_{x \in [a,b]} \max_k \sum_j |\partial_j \partial_k f_i(x)|;$$

$$r(u,x) = \max_i \frac{1}{2nh_i}\left((4nh_i e_i(u,x) + g_i^2(x))^{1/2} - g_i(x)\right);$$

$$I(c,r) = \{x \in R^n: |x_i - c_i| \leqslant r;\ i = 1,2, \ldots, n;\ c \in R^n,\ r \in R\}$$

is an interval (cube) in R^n with centre c and radius r (R^n is canonically ordered).

2. Two theorems for characterization of the solutions.

THEOREM 1. *If $u_0 \geqslant 0$, $c \in [a, b]$ and $r(u_0, c) \geqslant u_0$, then there are no solutions of $f_i(x) = 0$, $i = 1, 2, \ldots, n$, in $I(c, r) \cap [a,b]$.*

Proof. Let $\alpha = \{\alpha_1, \ldots, \alpha_n\}$ be a solution of our problem. If

$$\hat{\alpha} = \{0, \alpha_1, \ldots, \alpha_n\}, \quad c = \{c_1, \ldots, c_n\}, \quad \hat{c} = \{u_0, c_1, \ldots, c_n\},$$

then

$$f_i(x) = f_i(c) + f_i'(c)(x-c) + \frac{1}{2}(x-c) f_i''(\xi)(x-c),$$

where $x, \xi \in [a, b]$ imply

$$u_0 + |f_i(c)| = 1\cdot(u_0 - 0) + \left|f_i'(c)(\alpha - c) + \frac{1}{2}(\alpha - c) f_i''(\xi)(\alpha - c)\right|$$

$$\Leftrightarrow e_i(u_0, c) < g_i(c)\,\|\hat{\alpha} - \hat{c}\|_\infty + h_i n \|\hat{\alpha} - \hat{c}\|_\infty^2 .$$

From this quadratic inequality we obtain

$$\|\hat{\alpha} - \hat{c}\|_\infty > \frac{1}{2nh_i}\Big((4nh_i e_i(u_0, c) + g_i^2(c))^{1/2} - g_i(c)\Big),$$

i.e.,

$$\|\hat{\alpha} - \hat{c}\|_\infty > r(u_0, c).$$

And if $r(u_0, c) \geqslant u_0$, then

$$\|\hat{\alpha} - \hat{c}\|_\infty > r(u_0, c) \Rightarrow \|\alpha - c\|_\infty > r(u_0, c) \Rightarrow \alpha \notin I(c, r) \cap [a, b].$$

Remark. If $u_0 > 0$, then there are $c_1, c_2, \ldots, c_M \in [a, b]$ such that

$$\bigcup_{k=1}^{M} I(c_k, r_k) \supset [a, b].$$

(Further on we shall show a simple construction of such a "cover-system".) If u_0 is a "sufficiently small" positive number and c_k is an element of $c_1, \ldots, c_M$ such that $r_k = \min\{r_1, \ldots, r_M\}$, then c_k is a "sufficiently short distance" to a solution of our problem (because $r(u_0, c_k) < u_0$ can only be realized in a "small neighbourhood" of the solutions).

THEOREM 2. *Let α be the unique solution of our problem. Let $u_0 > 0$ and*

$$\bigcup I(c_k, r_k) \supset [a, b] \quad \textit{for } c_1, \ldots, c_M \in [a, b].$$

If $c_k (= c)$ is an element of $c_1, \ldots, c_M$ such that

$$r_k (= r) = \min\{r_1, \ldots, r_M\}$$

and r has been obtained with f_i and $f_i(c) \neq 0$, then

$$\alpha \in I(c, \lambda r) \cap [a, b],$$

where

$$\lambda \geqslant (nh_i r + \|f_i'(c)\|_1)/|f_i'(\xi) s_{c\alpha}|$$

and $s_{c\alpha}$ is a unit vector (to $\|\cdot\|_\infty$) in the direction $c - \alpha$ and $\xi \in [c, \alpha]$.

Proof. The vector c and the number r satisfy the equation

$$r = \frac{1}{2nh_i}\Big((4nh_i(u_0 + |f_i(c)|) + (1 + \|f_i'(c)\|_1)^2)^{1/2} - (1 + \|f_i'(c)\|_1)\Big).$$

Hence

$$u_0+|f_i(c)| = nh_ir^2+r+r\|f_i'(c)\|_1.$$

Since $u_0 > r$ and $f_i(\alpha) = 0$, we have

$$|f_i(c)-f_i(\alpha)| < nh_ir^2+r\|f_i'(c)\|_1.$$

If now we use

$$|f_i(c)-f_i(\alpha)| = |f_i'(\xi)(c-\alpha)| = |f_i'(\xi)s_{c\alpha}|\cdot\|c-\alpha\|_\infty,$$

then ($f_i'(\xi)s_{c\alpha} \neq 0$ since $f_i(c) \neq 0$)

$$\|c-\alpha\|_\infty < r\big(nh_ir+\|f_i'(c)\|_1\big)/|f_i'(\xi)s_{c\alpha}|.$$

Remark. If $u_0 > \bar{u}_0$ (where u_0 and $\bar{u}_0$ are "small" positive numbers), then

$$s_{c\alpha} \sim (\bar{c}-c)/\|\bar{c}-c\|_\infty \quad \text{and} \quad f_i'(\xi) \sim f_i'(\bar{c}).$$

Thus the formula of Theorem 2 can be used in practice as well.

3. The method. Henceforth the function

$$h_i(x,\delta) = 0.5\|f_i''(x)\|_\infty+\delta, \qquad \delta \geqslant 0,$$

will be used in place of the constant h_i (in the formula for $r(u, x)$). If δ is sufficiently large, then $h_i(x,\delta) \geqslant h_i$ for every $x\in[a, b]$ and the former remarks are useful henceforward.

The idea of our method is as follows. First we choose an initial value u_0 and make a cover-system to u_0 (with centres $c_1, \ldots, c_M$ and radii $r_1, \ldots, r_M$). Then we define an interval

$$[\bar{a}, \bar{b}] = I(c, \lambda r)\cap[a, b] \qquad (r = r_k = \min\{r_1, \ldots, r_M\} \text{ and } c = c_k)$$

and make a new cover-system to $\bar{u}_0(< u_0)$ and $[\bar{a}, \bar{b}]$. At this time we can define an interval

$$[\bar{\bar{a}}, \bar{\bar{b}}] = I(\bar{c}, \bar{\lambda}\bar{r})\cap[a, b], \ldots$$

In this way we determine a sequence $c, \bar{c}, \bar{\bar{c}}, \ldots$ The description of our method will be complete when we answer the following questions:

How are the cover-systems made to $u_0, \bar{u}_0, \ldots$?

How are $\delta, u_0, \bar{u}_0, \ldots, \lambda, \bar{\lambda}, \ldots$ chosen?

We made the cover-system to u_0 and $[a, b]$ (to $\bar{u}_0$ and $[\bar{a}, \bar{b}], \ldots$) as follows. (The construction is illustrated for $n = 2$ by the first numerical example.) Let $c_1 = (a+b)/2$. If

$$r_1 = r(u_0, c_1) \geqslant 0.5\|b-a\|_\infty,$$

then the construction of the cover-system is finished ($M = 1$). Otherwise, the set $[a, b] - I(c_1, r_1)$ is divided into intervals as follows: If $r_1 < (b_k-a_k)/2$ for a fix k ($k = 1, 2, \ldots, n$), then we define (store up) two new intervals:

$$[\{c_{11}-h_1, \ldots, c_{1,k-1}-h_{k-1}, a_k, a_{k+1}, \ldots, a_n\},$$
$$\{c_{11}+h_1, \ldots, c_{1,k-1}+h_{k-1}, c_{1k}-r_1, b_{k+1}, \ldots, b_n\}]$$

and

$$[\{c_{11}-h_1, \ldots, c_{1,k-1}-h_{k-1}, c_{1k}+r_1, a_{k+1}, \ldots, a_n\},$$
$$\{c_{11}+h_1, \ldots, c_{1,k-1}+h_{k-1}, b_k, b_{k+1}, \ldots, b_n\}],$$

where

$$c_{1i} = (a_i+b_i)/2 \quad \text{and} \quad h_i = \min\{r_1, (b_i-a_i)/2\}.$$

(The number of the new intervals is between 2 and $2n$.) Then we compute $r_2 = r(u_0, c_2)$, where c_2 is the centre of the first issued interval. If this interval needs dividing ($I(c_2, r_2)$ does not cover the interval), then we store up the new intervals after the former ones. The construction of the cover-system is finished if an interval does not need dividing and if it was the last to be stored up. (Our computer program stored up $[a, b]$ and the issued intervals with their centres and radii. The centres and the radii were stored up in the columns of two two-dimensional blocks.)

The parameters of our method were chosen as follows:

(a) We used "correct" δ ($h_i(x, \delta) \geqslant h_i$ for every $x \in [a, b]$) if we had striven for (full) safety. Otherwise, we used $\delta = 0$ (this is only correct for $f_i''(x) = \text{const}$, $i = 1, 2, \ldots, n$) or a "mean" δ. (We think δ is often not an important parameter of our method.)

(b) We used $0 < u_0 < \|b-a\|_\infty$ "in the case of safety" and $\|b-a\|_\infty \leqslant u_0 \leqslant 10\,\|b-a\|_\infty$ otherwise. (The safety of our method is full for $u_0 \sim 0$ (λ and $\bar{u}_0$ are not needed in this case), but the number of examined points (M) can become too large. If $u_0 \gg \|b-a\|_\infty$, then $M = 1$ "on the first level", so λ and $\bar{u}_0$ become very important parameters in this case. (λ of Theorem 2 is a theoretical choice for every $\bar{u}_0$.))

(c) The formulas $\lambda = \min\{2, 1+100/M\}$ and $\bar{u}_0 = u_0 - r$ supplied two good properties in practice. On the one hand the method kept its "mobility", on the other hand M could not become too large on the new levels.

4. Numerical examples. Two examples are shown in this part. At first some possibilities of the use of the method are illustrated by an elementary example. The second example points to the quantity of computations with our method.

(a) In the system of equations

$$x_1x_2-8 = 0, \quad x_1^2-5x_1+x_2+2 = 0$$

we have

$$e_1(u,x) = u+|x_1x_2-8|, \quad e_2(u,x) = u+|x_1^2-5x_1+x_2+2|;$$
$$g_1(x) = 1+|x_1|+|x_2|, \quad g_2(x) = 2+|2x_1-5|;$$

$$h_1(x,\delta) = 0.5, \quad h_2(x,\delta) = 1;$$

and

$$r(u, x) = \max_i \frac{1}{4h_i}(\sqrt{8h_i e_i + g_i^2} - g_i).$$

If u_0 is sufficiently small, then our method can be used to prove that some $[a, b]$ do not include solutions. For instance, let

$$[a, b] = [\{-5, -1\}, \{1, 1\}] \quad \text{and} \quad u_0 = 0.1.$$

Then $r(0.1, \{-2, 0\}) \sim 1.71$ and we must define two new intervals:

$$[\{-5, 1\}, \{-3.71, 1\}] \quad \text{and} \quad [\{-0.29, -1\}, \{1, 1\}].$$

Since

$$r(0.1, \{-4.355, 0\}) \sim 2.14, \quad r(0.1, \{0.355, 0\}) \sim 2.24,$$

we need not define further intervals. Here $M = 3$, $r = 1.71$ on the first level and $[a, b]$ cannot include solutions because of $r > u_0$.

Now, let

$$[a, b] = [\{4, -1\}, \{8, 3\}] \quad \text{and} \quad u_0 = 2.$$

Then the cover-system of the first level is defined as follows:

$$r(2, \{6, 1\}) = 1$$

and new intervals are

$$[\{4, -1\}, \{5, 3\}], \quad [\{7, -1\}, \{8, 3\}], \quad [\{5, -1\}, \{7, 0\}], \quad [\{5, 2\}, \{7, 3\}];$$

$$r(2, \{4.5, 1\}) \sim 0.75$$

and new intervals are

$$[\{4, -1\}, \{5, 0.25\}] \quad [\{4, 1.75\}, \{5, 3\}];$$

$$r(2, \{7.5, 1\}) \sim 1.56$$

and new intervals are

$$[\{7, -1\}, \{8, -0.56\}], \quad [\{7, 2.56\}, \{8, 3\}];$$

$$r(2, \{6, -0.5\}) \sim 1.45;$$

$$r(2, \{6, 2.5\}) \sim 1.11;$$

$$r(2, \{4.5, -0.375\}) \sim 1.56;$$

$$r(2, \{4.5, 2.375\}) \sim 0.57$$

and new intervals are

$$[\{4,\ 1.75\},\ \{5,\ 1.805\}],\quad [\{4,\ 2.945\},\ \{5,\ 3\}];$$

$$r(2, \{7.5,\ -0.78\}) \sim 1.47;$$

$$r(2, \{7.5,\ 2.78\}) \sim 1.66;$$

$$r(2, \{4.5,\ 1.7775\}) \sim 0.503;$$

$$r(2, \{4.5,\ 2.9725\}) \sim 0.79.$$

Here $M = 11$, $r = 0.503$, $c = \{4.5,\ 1.7775\}$ on the first level. If c is a good initial approximation for a fast finishing method (see [2]), then our method is stopped. Otherwise, it continues with

$$[\bar{a},\ \bar{b}] = [\{4,\ 0.7715\},\ \{5.506,\ 2.7835\}] \quad \text{and} \quad \bar{u}_0 = 1.497.$$

(The problem has three solutions for $[a, b] = R^2$: $\{-1,\ -8\}$, $\{2, 4\}$ and $\{4, 2\}$.)

(b) In the system of equations

$$x_1x_2 + x_3^2 - 6x_3 + 13 = 0, \quad x_1^2 + x_2^2 + x_1x_2 - 4 = 0,$$

$$x_1x_3 + x_2x_3 + \sqrt{x_4} - 2 = 0, \quad 2x_1^2 - x_2x_3x_4 + 16 = 0,$$

we have

$$h_1(x, \delta) = 1, \quad h_2(x, \delta) = 1.5, \quad h_3(x, \delta) = 0.5\max\{2,\ 0.25/\sqrt{x_4^3}\} + \delta,$$

$$h_4(x, \delta) = 0.5\max\{4, |x_4| + |x_3|,\ |x_4| + |x_2|,\ |x_3| + |x_2|\} + \delta.$$

We worked with $\delta = 0$ and $u_0 = 4$ in four intervals $[a, b]$. They were

$$[\{-2, 0, 1, 2\},\ \{0, 2, 3, 4\}],\quad [\{-3, 1, 2, 1\},\ \{1, 3, 4, 5\}],$$

$$[\{-2,\ -1, 1, 4\},\ \{2, 3, 3, 6\}],\quad [\{-4, 0, 1, 2\},\ \{0, 4, 5, 6\}].$$

Then we worked with $\delta = 3$ and $u_0 = 10$ in four intervals $[a, b]$. They were

$$[\{-4,\ -3,\ -1, 4\},\ \{4, 5, 5, 6\}],\quad [\{-3,\ -2, 0, 2\},\ \{3, 4, 4, 6\}],$$

$$[\{-5,\ -2, 2, 1\},\ \{3, 2, 6, 5\}],\quad [\{-4, 0, 1, 2\},\ \{2, 6, 5, 4\}].$$

When we stopped the process by $u < 0.01$, the errors of our approximations were between 0.001 and 0.07. (The problem has a unique solution for $x \in R^4$ and $x_4 \geqslant 0$. It is $\{-2, 2, 3, 4\}$.) The working time of a TPA1148 minicomputer was between 15 secs and 50 secs for our examples. The number of used points was between 1022 and 3539. (If we cover these intervals with cubes having edges of 0.1, then the number of cubes is between $1.6 \cdot 10^5$ and $7.68 \cdot 10^6$.)

References

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