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ON AN INTEGRAL INEQUALITY CONNECTED WITH HARDY'S INEQUALITY

Let F denote the class of functions $f = f(t)$, defined and absolutely continuous on the interval $[-1, 1]$, satisfying the conditions $f(-1) = f(1) = 0$ and such that

$$\int_{-1}^{1} p^a \dot{f}^2 dt < \infty \tag{1}$$

where $p = (1-t^2)^{-1/2}$, $\dot{f} = df/dt$ and a is a positive number. We shall prove the following

THEOREM *If $f \in F$ and*

$$\int_{-1}^{1} p^{a+4} f f_0 dt = 0, \tag{2}$$

where $f_0 = p^{-1-a}$, then the following inequality holds true

$$\lambda_a \int_{-1}^{1} p^{a+4} f^2 dt \leqslant \int_{-1}^{1} p^a \dot{f}^2 dt \tag{3}$$

with $\lambda_a = (1+a/2)^2$ for $0 < a < 4$ and $\lambda_a = 3a-3$ for $a \geqslant 4$. The coefficient λ_a in (3) *cannot be enlarged. Moreover, for $0 < a < 4$, there is equality in* (3) *for $f = 0$ only.*

If condition (2) is rejected, then we have

$$(1+a) \int_{-1}^{1} p^{a+4} f^2 dt \leqslant \int_{-1}^{1} p^a \dot{f}^2 dt \tag{4}$$

for any $f \in F$ and there is equality in (4) for $f = \text{const}\, f_0$ only (see [2]). The reason for considering the additional restriction (2) is motivated by some problems of approximation in the theory of differential equations.

Proof of the theorem. We introduce a new independent variable x and a new function $u = u(x)$ which are connected with t and f by the relations

$$t = \operatorname{th} x/2, \quad f = (\operatorname{ch} x/2)^{-\alpha} u, \quad \alpha = 1 + a/2. \tag{5}$$

As a result of simple calculations we get the identities

$$J_1 \equiv \int_{-1}^{1} p^{a+4} f^2 dt = 1/2 \int_{-\infty}^{\infty} u^2 dx = 1/2\|u\|^2 \tag{6}$$

and

$$J_2 = \int_{-1}^{1} p^a \dot{f}^2 dt = 2\|u'\|^2 + \alpha^2 \|u\|^2 + \alpha(1-\alpha)/2 \int_{-\infty}^{\infty} (\operatorname{ch} x/2)^{-2} u^2 dx, \tag{7}$$

where $'$ denotes differentiation with respect to the variable x and $\|\ \|$ is the standard $L_2(-\infty,\infty)$ -norm. When deriving (7) the relation $(\operatorname{th} x/2) u_2 \to 0$ as $x \to \pm\infty$ is to be used which, on the other hand, is equivalent to the relation $p^{a+2} f^2 \to 0$ as $t \to \pm 1$, established in [2]. The function u is absolutely continuous on any finite interval of the real line and, due to (7), the norms $\|u\|$ and $\|u'\|$ are finite. The class of all such functions will be called U.

Let $u \in C_2(-\infty,\infty)$, i.e. u is twice continuously differentiable and vanishing outside of a segment of the x-axis. For such u the integral J_2 may be expressed in the form

$$J_2 = 2(Lu, u) + a^2/2\|u\|^2, \tag{8}$$

where

$$Lu = -u'' + \alpha(1-\alpha)/4 (\operatorname{ch} x/2)^{-2} u \tag{9}$$

and (,) denotes, as usually, the inner product in L_2. The second order differential operator L, if considered on C_2, is symmetric and it may be extended to a selfadjoint operator L_0 with a domain $D(L_0) \subset U$. Its spectrum has been studied by Titchmarsh [3]: it consists of a continuous part which is identical with the positive semiaxis $\varkappa \geqslant 0$ and of a point spectrum whose points $\varkappa_r$ lying on the negative semiaxis $\varkappa \leqslant 0$ are given by

$$\varkappa_r = -(\alpha/2 - 1/2 - r)^2, \quad r = 0, 1, \ldots, [\alpha/2 - 1/2] \tag{10}$$

and $[\beta]$ denotes the integer part of β.

Consider the smallest eigenvalue $\varkappa_0 = -a^2/16$ and corresponding eigenfunction $\tilde{u}$. They satisfy the relation

$$(L_0\tilde{u}, \tilde{u}) = \varkappa_0 \|\tilde{u}\|^2$$

which is equivalent to $J_2 = (2\varkappa_0 + a^2/2)\|\tilde{u}\|^2 = (1+a)J_1$. The last equality is identical with (4), where the $\leqslant$ sign has been replaced by the $=$ sign and where $f = \tilde{f} = (\mathrm{ch}\,x/2)^{-a}\tilde{u}$. However, the only function of class F for which there is equality in (4) is the function const f_0, therefore, due to relation $\tilde{u} \in U$, we have $f = \text{const } f_0$, and $\tilde{u} = \text{const } u_0$, where $u_0 = (\mathrm{ch}\,x/2)^{-a}$, is the only eigenfunction of L_0, corresponding to the eigenvalue $\varkappa_0$.

Let us now remark that it suffices to prove our theorem for functions $f \in C_2(-1, 1)$ only and then apply the standard limiting procedure. When working with corresponding functions $u \in U$, we can restrict ourselves to the class $C_2\ (-\infty, \infty)$.

Assume $u \in C_2(-\infty, \infty)$. The orthogonality condition (2), when expressed in terms of x and u's, takes the form

$$(u, u_0) = 0. \tag{11}$$

We make now use of spectral representation of the operator L_0. Due to (11), we have

$$(L_0 u, u) \geqslant \mu\|u\|^2 \tag{12}$$

for any $u \in C_2(-\infty, \infty)$ where μ is the point of the spectrum of L_0 which is different from and nearest to $\varkappa_0$, i.e. $\mu = 0$ for $0 < a < 4$ and $\mu = \varkappa_0 = -1/4(a-3)^2$ for $a \geqslant 4$. Due to (8) and the identity $L_0 u = Lu$ for $u \in C_2(-\infty, \infty)$, the inequality (12) may be written as $J_2 \geqslant \lambda_a J_1$, $\lambda_a = 4\mu + a^2$, $f \in C_2(-1, 1)$, which is the desired inequality (3).

Now for $0 < a < 4$, $\mu = 0$ is a point of the continuous part of the spectrum and there is no function other than zero for which there would be equality in (3) with both integrals convergent. If $a \geqslant 4$, then $\mu = \varkappa_1$ and an eigenfunction corresponding to this eigenvalue turns (3) to equality. This completes the proof of the theorem.

A particular case, $a = 1$, will be considered now under the additional assumption that $f \in F$ is an even function. In this case inequality (3) has the form

$$9/4 \int_{-1}^{1} p^5 f^2\,dt \leqslant \int_{-1}^{1} p\dot{f}^2\,dt \tag{13}$$

and the orthogonality condition becomes

$$\int_{-1}^{1} p^3 f\,dt = 0. \tag{14}$$

We shall show that in this particular case our theorem is equivalent to Hardy's theorem [1].

We shall make use of the identity

$$\int_{-1}^{1} p\dot{f}^2 dt - 2\int_{-1}^{1} p^5 f^5 dt = \int_{-1}^{1} p^{-3}\dot{h}^2 dt, \tag{15}$$

(see [2], formula (11)), where $h = p^2 f$ with $f \epsilon F$. Let $\{T_k\}$, $k = 0, 1, 2, \ldots$, denote the system of Chebyshev's polynomials. Denoting by a_i and b_i coefficients with respect to $\{T_k\}$ of functions h and $p^{-2}\dot{h}$ resp., i.e.

$$\begin{gathered} \pi a_0 = \int_{-1}^{1} phT_0 dt, \quad \pi a_k = 2\int_{-1}^{1} phT_k dt, \\ \pi b_0 = \int_{-1}^{1} p^{-1}\dot{h}T_0 dt, \quad \pi b_k = 2\int_{-1}^{1} p^{-1}\dot{h}T_k dt, \end{gathered} \tag{16}$$

we have, due to Parseval's relation,

$$I_1 = 2/\pi \int_{-1}^{1} ph^2 dt = 2a_0^2 + \sum_{k\geqslant 1} a_k^2,$$

$$I_2 = 2/\pi \int_{-1}^{1} p^{-3}\dot{h}^2 dt = 2b_0^2 + \sum_{k\geqslant 1} b_k^2.$$

We have also the following identities

$$a_1 = 2b_0, \quad 2b_k = (k+1)a_{k+1} - (k-1)a_{k-1} \tag{18}$$

for $k = 1, 2, \ldots$ These identities follow simply when performing integration by parts in (16). Due to relations (18) we can express the coefficients a_k in terms of b_k.

The orthogonality condition (14) is now equivalent to $a_0 = 0$. We have also relations $a_1 = a_3 = \ldots = 0$ and $b_0 = b_2 = \ldots = 0$ implied by the assumption h to be an even function. If we now put $\beta_k = b_{2k-1}$, $B_k = \beta_1 + \ldots + \beta_k$ we can then rewrite I_1 and I_2 in the form

$$I_1 = \sum_{k\geqslant 1} (B_k/k)^2, \quad I_2 = \sum_{k\geqslant 1} \beta_k^2. \tag{19}$$

According to Hardy's theorem we have $I_2 < 4I_1$ if only not all β_k are zero; here the coefficient 4 is the best one. The inequality (13) is now an immediate consequence of the identity (15). Conversely, Hardy's theorem follows simply from inequality (13) and identity (15) expressed in terms of β_k.

The above arguments may be modified so as to apply to the general case considered in our theorem, however, a generalization of Hardy's theorem is then needed. We hope to return to this problem elsewhere.

References

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[3] E. C. Titchmarsh, *Eigenfunction expansions associated with second-order differential equations*, Oxford 1946.

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