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CONSTRUCTION AND FEATURES OF THE OPTIMUM RATIONAL FUNCTION USED IN THE ADI-METHOD

1. Introduction. The following approximation problem has been posed in connection with the ADI-method (see, e.g., Varga [11], p 212):

For given numbers, a natural m and a real k' $(0 < k' < 1)$, find parameters $r_{1m}, r_{2m}, \ldots, r_{mm}$ such that

$$\max_{k'\leqslant x\leqslant 1}\left|\prod_{j=1}^{m}\frac{x-r_{jm}}{x+r_{jm}}\right|$$

is the minimum.

This paper deals with the determination of the parameters r_{jm} and with their properties. In Section 2, the Wachspress algorithm determining the parameters r_{jm} for $m = 2^p$ (called, shortly, *WR-algorithm*) is recalled and it is shown how this algorithm can be applied to the calculation of the extremal points u_{jm} of the optimum function

$$\prod_{j=1}^{m}\frac{x-r_{jm}}{x+r_{jm}}.$$

It is known that the optimal parameters r_{jm} are expressed by elliptic Jacobi functions. Therefore, some monotonicity properties of these functions are proved in Section 3. In Section 6, it is shown that the extremal points u_{jm} are expressed by similar formulas as those for the parameters r_{jm}. In Section 7, using these and Jordan's formulas, we give algorithms JR and JU determining r_{jm} and u_{jm} for $m = 2^p$. This new algorithm JR determines the parameters r_{jm} with a smaller error (1) than the WR-algorithm. Section 6 is devoted to the examination of properties of r_{jm} and u_{jm}. The theorems of Section 3 are used here to prove the monotonicity and the limit properties of these parameters. The modification of the algorithm JR, given in Section 7, is characterized by a smaller number of operations but – as it follows from numerical examples – produces a greater error. In Section 8, known approximate methods of determining the parameters

(1) This is the machine rounding error arising during the computation.

r_{jm} are recalled and properties of a certain function connected with parameters r_{jm}, the existence of which has been experimentally verified by C. Boor and J. R. Rice, are proved. At last, Section 9 gives a hypothetical estimation of the norm of the optimum function, the proof of which has failed. Only examples confirming the conjecture are given.

Throughout this paper theorems and formulas are numbered independently in each section. If we refer to formulas (theorems) from other sections, two numbers, separated by a decimal point, are given — denoting the section and formula (theorem), respectively. A single number as reference means the formula (theorem) in the same section.

This paper is the first part of [15].

2. Algorithm WR of determining parameters $r_{j,2^p}$ and algorithm WU of determining parameters $u_{j,2^p}$. The alternating direction implicit method (ADI-method) for solving partial differential equations has been given by Peaceman and Rachford [8]. It is one of the most effective difference methods of solving the Poisson partial differential equation on a rectangle. It can also be applied to an iterative solution of other systems of linear equations with matrices satisfying some rather strong assumptions (see Varga [11], p. 212). An important role in this method is played by acceleration parameters.

Let S_m denote the set of parameters $s_1, s_2, \ldots, s_m$ ordered increasingly and belonging to the interval $[\alpha, \beta]$ with $\alpha > 0$. The corresponding rational function is

$$f_m(x;\ S_m) \stackrel{\text{df}}{=} \prod_{j=1}^{m} \frac{x-s_j}{x+s_j}.$$

As acceleration parameters in the ADI-method (see, e.g., Varga [11], p. 212) ought to be assumed elements of such a set $R_m \equiv R_m(\alpha, \beta)$ (α and β depend upon the matrix of the system) for which

$$\max_{\alpha\leqslant x\leqslant\beta} |f_m(x;\ R_m)| = L_m(\alpha, \beta), \tag{1}$$

where

$$L_m(\alpha, \beta) \stackrel{\text{df}}{=} \min_{S_m} \max_{\alpha\leqslant x\leqslant\beta} |f_m(x;\ S_m)|.$$

We assume the elements of $R_m(\alpha, \beta)$ to be ordered increasingly

$$r_{1m} < r_{2m} < \ldots < r_{mm}.$$

The following theorem of Wachspress (see Varga [11], p. 223, and also Wachspress [13], p. 183) asserts the existence and uniqueness of $R_m(\alpha, \beta)$:

THEOREM 1. *There exists exactly one optimum rational function $f_m(x; R_m)$ for which condition* (1) *is satisfied. The elements $r_{1m}, r_{2m}, \ldots, r_{mm}$ of R_m belong to the interval (a, β) and are pairwise different. The function $f_m(x; R_m)$ is uniquely characterized by the property that $f_m(x; R_m)$ attains alternatively the values $L_m(a, \beta)$ and $-L_m(a, \beta)$ in $m+1$ points u_{jm} of the interval $[a, \beta]$ such that $a = u_{0m} < u_{1m} < \ldots < u_{mm} = \beta$.*

The set of extremal points u_{jm} is denoted by $U_m(a, \beta)$. The elements of the optimal set R_m are consequently denoted by r_{jm}, and the extremal points of the optimum function by u_{jm}.

The determination of $R_m(a, \beta)$ for any m is not easy, since — as was shown by Jordan (see Wachspress [12]) — the optimal parameters r_{jm} are expressed by elliptic functions. In the case $m = 2^p$ only (see Varga [11], p. 225, and Wachspress [12] and [13]), the simple algorithm of Wachspress of determining r_{jm} is known. In all other cases, one uses approximations.

The WR-algorithm makes use of the following property of the optimal parameters r_{jm}:

LEMMA 1 (Varga [11], p. 224). *The optimal parameters r_{jm} satisfy the relation*

$$r_{jm} = \frac{a\beta}{r_{m-j+1,m}}.$$

From this lemma it follows that

$$\max_{a\leqslant x\leqslant\beta}\left|\prod_{j=1}^{2m}\frac{x-r_{j,2m}}{x+r_{j,2m}}\right| = \max_{\sqrt{a\beta}\leqslant y\leqslant(a+\beta)/2}\left|\prod_{j=1}^{m}\frac{y-\tilde{r}_{jm}}{y+\tilde{r}_{jm}}\right|,$$

where

$$y = \frac{\sqrt{a\beta}}{2}\left(\frac{\sqrt{a\beta}}{x}+\frac{x}{\sqrt{a\beta}}\right), \tag{2}$$

$$\tilde{r}_{jm} = \frac{1}{2}\left(r_{j,2m}+\frac{a\beta}{r_{j,2m}}\right) \quad (j = 1, 2, \ldots, m).$$

It can be shown (see Varga [11], p. 224) that the parameters $\tilde{r}_{jm}$ are elements of the optimal set $R_m\left(\sqrt{a\beta}, (a+\beta)/2\right)$. Thus, finding $R_{2m}(a, \beta)$ is reduced to determining $R_m\left(\sqrt{a\beta}, (a+\beta)/2\right)$. Hence the algorithm WR for $m = 2^p$ follows immediately.

Algorithm WR.

Step I. Construct the sequences $\{a_i\}$ and $\{\beta_i\}$ by the formulas

$$\begin{aligned} &a_0 = a, &&\beta_0 = \beta, \\ &a_{i+1} = \sqrt{a_i\beta_i}, &&\beta_{i+1} = (a_i+\beta_i)/2 \quad (i = 0, 1, \ldots, p-1). \end{aligned} \tag{3}$$

The elements of these sequences form the interval ends obtained consecutively after having applied p times transformation (2).

Step II. Form the elements s_{jn} ($j = 1, 2, \ldots, n$; $n = 1, 2, 4, \ldots, 2^p$) by the formulas

$$s_{11} = \sqrt{a_p \beta_p} = a_{p+1},$$

$$s_{n+j,2n} = s_{jn} + \sqrt{(s_{jn} - a_k)(s_{jn} + a_k)}, \tag{4}$$

$$s_{n-j+1,2n} = \frac{a_{k-1}\beta_{k-1}}{s_{n+j,2n}} \tag{5}$$

for $j = 1, 2, \ldots, n$; $k = p, p-1, \ldots, 1$; $n = 2^{p-k}$.

The elements s_{jm} are those of $R_m(a, \beta)$. Formulas (4) follow from the transformation inverse to transformation (2), and formulas (5) are a consequence of Lemma 1 (they can also be obtained in a similar way as formulas (4)).

The elements of $U_m(a, \beta)$ can be determined analogously. Let us prove now some auxiliary lemmas. The first lemma is a modification of a lemma given in [14] by the author.

LEMMA 2. *The element $u > 0$ belongs to the set $\{u_{1m}, u_{2m}, \ldots, u_{m-1,m}\}$ if and only if*

$$\sum_{j=1}^{m} \frac{r_{jm}}{u^2 - r_{jm}^2} = 0. \tag{6}$$

Proof. Since the elements u_{im} ($i = 1, 2, \ldots, m-1$) are extremal points of $f_m(x; R_m)$, they are zeros of the partial derivative of that function with respect to x. Let us calculate it. We obtain

$$\frac{\partial}{\partial x} f_m(x; R_m) = f_m(x; R_m) \sum_{j=1}^{m} \left(\frac{1}{x - r_{jm}} - \frac{1}{x + r_{jm}}\right)$$
$$= f_m(x; R_m) \sum_{j=1}^{m} \frac{2r_{jm}}{x^2 - r_{jm}^2}.$$

Thus, if u equals u_{im} ($i = 1, 2, \ldots, m-1$), it satisfies (6). The logarithmic partial derivative of $f_m(x; R_m)$ is a rational function with the numerator being a polynomial of (even) degree $2(m-1)$, having thus $m-1$ positive zeros. They must be the elements of $\{u_{1m}, u_{2m}, \ldots, u_{m-1,m}\}$. Hence the proof of the lemma is complete.

LEMMA 3. *If $u \in U_m(a, \beta)$, then $a\beta/u \in U_m(a, \beta)$.*

Proof. Let u belong to the set $\{u_{1m}, u_{2m}, \ldots, u_{m-1,m}\}$. It follows then from Lemma 2 that u satisfies (6). Using Lemma 1, we can transform the left-hand side of (6) in the following manner:

$$\sum_{j=1}^{m} \frac{r_{jm}}{u^2 - r_{jm}^2} = \sum_{j=1}^{m} \frac{\alpha\beta / r_{jm}}{u^2 - (\alpha\beta / r_{jm})^2} = -\frac{\alpha\beta}{u^2} \sum_{j=1}^{m} \frac{r_{jm}}{(\alpha\beta / u)^2 - r_{jm}^2}.$$

Since $\alpha\beta \neq 0$, we have

$$\sum_{j=1}^{m} \frac{r_{jm}}{(\alpha\beta / u)^2 - r_{jm}^2} = 0$$

and, by Lemma 2, $\alpha\beta/u$ belongs to the set $\{u_{1m}, u_{2m}, \ldots, u_{m-1,m}\}$. The thesis of the lemma is also true for the boundary points of $U_m(\alpha, \beta)$ equal to α and β. This completes the proof.

We give now the algorithm of determining u_{jm} for $m = 2^p$.

Algorithm WU.

Step I. Similarly as in the WR-algorithm, construct sequences (3).

Step II. Form the elements v_{jn} ($j = 0, 1, \ldots, n$; $n = 1, 2, 4, \ldots, 2^p$) using the formulas

$$v_{01} = \alpha_0, \quad v_{11} = \beta_0,$$

$$v_{0,2n} = \alpha_{k-1}, \quad v_{n,2n} = \alpha_k, \quad v_{2n,2n} = \beta_{k-1},$$

$$v_{n+j,2n} = v_{jn} + \sqrt{(v_{jn} - \alpha_k)(v_{jn} + \alpha_k)}, \quad v_{n-j,2n} = v_{jn} - \sqrt{(v_{jn} - \alpha_k)(v_{jn} + \alpha_k)}$$

for $j = 1, 2, \ldots, n-1$; $k = p, p-1, \ldots, 1$; $n = 2^{p-k}$.

The elements v_{jm} form the set $U_m(\alpha, \beta)$ which is sought. The last formula does not guarantee the numerical stability. By virtue of Lemma 3 it can be substituted by the formula

$$v_{n-j,2n} = \frac{\alpha_{k-1}\beta_{k-1}}{v_{n+j,2n}}$$

which guarantees the numerical stability of one step of the algorithm WU. For similar reasons, we use in the algorithm WR formula (5) instead of the formula

$$s_{n-j+1,2n} = s_{jn} - \sqrt{(s_{jn} - \alpha_k)(s_{jn} + \alpha_k)}. \tag{7}$$

The numerical stability of the algorithms is considered in [16], where we show that consecutive elements s_{jn} are zeros of certain polynomials of second degree from which the numerical superiority of (5) over (7) immediately follows.

We assume in what follows that $\alpha = k'$ and $\beta = 1$. This is an inessential limitation. Also, we write $L_m(k')$, $R_m(k')$ and $U_m(k')$ instead of $L_m(\alpha, \beta)$, $R_m(\alpha, \beta)$ and $U_m(\alpha, \beta)$, respectively. Thus, parameters r_{jm} and u_{jm} are now functions of variable k', i.e. $r_{jm} = r_{jm}(k')$ and $u_{jm} = u_{jm}(k')$.

3. Limit properties of the elliptic Jacobi functions. As we already mentioned, Jordan has expressed the elements r_{jm} by elliptic functions. Before presenting the formulas, we deal with the elliptic Jacobi functions sn, cn and dn. The necessary information about these functions can be found, e.g., in the book of Oberhettinger and Magnus [7]. In particular, it is known (see [9], formulas 8.151.1-8.151.3) that

$$\text{(1)} \qquad \mathrm{dn}(0;\, k) = 1, \quad \mathrm{dn}\big(\tfrac{1}{2}K(k);\, k\big) = \sqrt{k'}, \quad \mathrm{dn}\big(K(k);\, k\big) = k',$$

$$\text{(2)} \qquad \mathrm{dn}\big(u+K(k);\, k\big) = k'/\mathrm{dn}(u;\, k),$$

$$\text{(3)} \qquad \mathrm{dn}\big(u+2K(k);\, k\big) = \mathrm{dn}(u;\, k) = \mathrm{dn}(-u;\, k),$$

where $k' = \sqrt{1-k^2}$, and

$$K \equiv K(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \int_0^{\pi/2} \frac{dx}{\sqrt{1-k^2\sin^2 x}}.$$

In addition, we know ([7], p. 31) that

$$\text{(4)} \qquad \lim_{k\to 0} K(k) = \pi/2,$$

$$\text{(5)} \qquad \lim_{k\to 1} K(k) = \infty$$

and

$$\text{(6)} \qquad \lim_{k\to 0} \mathrm{dn}(u;\, k) = 1,$$

$$\lim_{k\to 1} \mathrm{dn}(u;\, k) = 1/\cosh u.$$

In what follows we are interested in the function $\mathrm{dn}(u;\, k)$ with a special form of the first argument:

$$u = \xi K(k) \qquad \text{for } \xi \in [0, 1],\ k \in [0, 1].$$

Under this assumption $K(k)$, $\mathrm{dn}\big(\xi K(k);\, k\big)$, $\mathrm{sn}\big(\xi K(k);\, k\big)$ and $\mathrm{cn}\big(\xi K(k);\, k\big)$ are real functions. Here is the first property of $\mathrm{dn}\big(\xi K(k);\, k\big)$ which – as will be shown in Section 6 – decides upon the monotonicity of the optimal parameters:

THEOREM 1. *If* $\xi \in (0, 1]$, *the function* $\mathrm{dn}\big(\xi K(k);\, k\big)$ *is decreasing with respect to* k $(0 < k < 1)$.

Proof. For $\xi = 1$, the assertion of the theorem follows immediately from the last of equations (1). Now we assume that

$$\text{(7)} \qquad \xi \in (0, 1).$$

This assumption is necessary because of the way the proof is performed. The function $\mathrm{dn}(u;\, k)$ is defined as the inverse to the elliptic integral of first kind (see [7], p. 19). We now recall this definition.

Let

$$u(x;\ k) = \int_0^x \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}. \tag{8}$$

The function $u(x;\ k)$ of the variables x and k is continuous. If, for a fixed k, the value of x changes from 0 to 1, the value of $u(x;\ k)$ changes from 0 to $K(k)$. From (8) we determine x as a function of u and k. Then

$$\mathrm{dn}(u;\ k) \stackrel{\mathrm{df}}{=} \sqrt{1-k^2x^2}, \tag{9}$$

where, as known ([7], p. 20), $x = \mathrm{sn}(u;\ k)$.

In the theorem we consider the case $u = \xi K(k)$. Hence

$$u(x;\ k) = \xi u(1;\ k). \tag{10}$$

Therefore, under (7) we get

$$0 < u(x;\ k) < u(1;\ k). \tag{11}$$

Hence it follows that

$$0 < x < 1 \quad (k \in [0, 1)). \tag{12}$$

Remark. From Theorem 3 and formula (9) we have

$$\lim_{k\to 1^-} \mathrm{sn}(\xi K;\ k) = 1.$$

Let

$$U(x;\ k) \stackrel{\mathrm{df}}{=} \frac{u(x;\ k)}{u(1;\ k)} \quad (0 < x < 1).$$

For fixed ξ, it follows from (10) that $U(x;\ k) = \xi = \mathrm{const}$. $U(x;\ k)$ is a continuous function of x and k ($u(1;\ k) \neq 0$). Now, let us determine its partial derivatives

$$U_x(x;\ k) = \frac{u_x(1;\ k)}{u(1;\ k)}, \tag{13}$$

$$U_k(x;\ k) = \frac{u(1;\ k)u_k(x;\ k) - u(x;\ k)u_k(1;\ k)}{u^2(1;\ k)}, \tag{14}$$

where

$$u_x(x;\ k) = \frac{1}{\sqrt{(1-x^2)(1-k^2x^2)}}, \tag{15}$$

$$u_k(x;\ k) = \int_0^x \frac{kt^2\,dt}{(1-k^2t^2)\sqrt{(1-t^2)(1-k^2t^2)}}. \tag{16}$$

Assumption (7), from which (12) follows, guarantees the continuity of derivatives (15) and (16). Hence, due to the continuity of $u(x;\ k)$, it follows that derivatives (13) and (14) are continuous. From (11) and (12) we also deduce that derivative (14) is different from zero. Thus the function $U(x;\ k) = \xi$ satisfies the assumptions of the theorem on involved functions. Therefore, we can determine x as a function of k and ξ, and calculate the derivative

$$(17)\qquad \frac{d}{dk}x(k;\ \xi) = \frac{-u(1;\ k)u_k(x;\ k)+u(x;\ k)u_k(1;\ k)}{u(1;\ k)u_x(x;\ k)} \quad \text{for } \xi\in(0,1).$$

Since $x(k;\ \xi) = \operatorname{sn}(\xi K(k);\ k)$, the local character of the thesis of the theorem on involved functions is no obstacle. Now, we prove that $x = x(k;\ \xi)$ is an increasing function with respect to k. To do this, equality (17) is transformed. From (11) it follows that $u(x;\ k) \neq 0$; also, we know that $u(1;\ k) \neq 0$. Hence

$$(18)\qquad \frac{dx}{dk} = \frac{u(x;\ k)}{u_x(x;\ k)}\left(\frac{u_k(1;\ k)}{u(1;\ k)} - \frac{u_k(x;\ k)}{u(x;\ k)}\right).$$

By (11) and (15), the quotient $u(x;\ k)/u_x(x;\ k)$ is positive. We write

$$g(t) = 1/\sqrt{(1-t^2)(1-k^2t^2)}, \qquad h(t) = kt^2/(1-k^2t^2),$$

$$a = \int_0^x g(t)\,dt, \qquad b = \int_x^1 g(t)\,dt,$$

$$c = \int_0^x g(t)h(t)\,dt, \qquad d = \int_x^1 g(t)h(t)\,dt.$$

Now the expression of (18) in parentheses can be written as

$$(19)\qquad \frac{c+d}{a+b} - \frac{c}{a} = \frac{ad-bc}{a(a+b)}.$$

The function $h(t)$ is continuous and increasing for $0 \leqslant t < 1/k^2$ ($k\in(0,1]$). The function $g(t)$ does not change its sign in the interval $[0,1)$. Applying the theorem on the mean value for integrals, we obtain

$$(20)\qquad \begin{aligned} c &= h(t_1)a \quad (0 < t_1 < x),\\ d &= h(t_2)b \quad (x < t_2 < 1). \end{aligned}$$

The integrals a, b, c and d are positive for $k\in(0,1)$. Hence from (20) we have

$$ad = abh(t_2) > abh(t_1) = bc \qquad (k\in(0,1)).$$

Thus we have proved that expression (19) is positive. Hence it follows that derivative (18) is positive, i.e. that the function $x = x(k;\ \xi)$ is increasing with respect to k for $\xi \in (0, 1)$ and $k \in (0, 1)$. By virtue of definition (9), we conclude that the function $\operatorname{dn}(\xi K(k);\ k)$ is decreasing with respect to k for $k \in (0, 1)$. This completes the proof of the theorem.

Remark. As it was mentioned in the proof, the function $x = x(k;\ \xi)$ is equal to $\operatorname{sn}(\xi K(k);\ k)$. It has thus been proved indirectly that, for $\xi \in (0, 1)$, the function $\operatorname{sn}(\xi K(k);\ k)$ is increasing with respect to k $(0 < k < 1)$. On the boundaries of the interval this function is constant $(\operatorname{sn}(0;\ k) = 0$, and $\operatorname{sn}(K(k);\ k) = 1)$. This fact, as well as the thesis of Theorem 1, were known earlier (see, e.g., the graphs in the book of Jahnke and Emde [4], p. 98). However, the author could not find a published proof of this property of elliptic functions in the literature.

Here is a generalization of Theorem 1:

THEOREM 2. *If* $\eta \in [0, 1]$, *the function*

$$t(k;\ \xi, \eta) \stackrel{\mathrm{df}}{=} \frac{\operatorname{dn}(\xi K(k);\ k)}{\operatorname{dn}(\eta K(k);\ k)}$$

of the variable k *is in the interval* $(0, 1)$ *strictly increasing for* $0 \leqslant \xi < \eta$ *and strictly decreasing for* $\eta < \xi \leqslant 1$.

Proof. Introduce the auxiliary notation

$$r(z) = \operatorname{dn}(zK(k);\ k).$$

Calculate the derivative of $t(k;\ \xi, \eta)$ with respect to k:

$$\frac{\partial}{\partial k} t(k;\ \xi, \eta) = \frac{r(\eta) r_k(\xi) - r(\xi) r_k(\eta)}{r^2(\eta)} = \frac{r(\xi)}{r(\eta)} \left(\frac{r_k(\xi)}{r(\xi)} - \frac{r_k(\eta)}{r(\eta)} \right). \tag{21}$$

Remark. The existence of the derivatives $r_k(\xi), r_{k\xi}(\xi)$, etc. follows from the formulas given by Cayley in the book [3], p. 102. The author did not succeed in proving the theorem directly from these formulas.

Now, consider the function

$$s(k;\ \xi) \stackrel{\mathrm{df}}{=} \frac{r_k(\xi)}{r(\xi)} \quad (\xi \in [0, 1],\ k \in (0, 1)).$$

The derivative of this function with respect to ξ equals

$$\frac{\partial}{\partial \xi} s(k;\ \xi) = \frac{r(\xi) r_{k\xi}(\xi) - r_k(\xi) r_\xi(\xi)}{r^2(\xi)}. \tag{22}$$

It is known (see [7], p. 21) that

$$\frac{d}{du} \operatorname{dn}(u;\ k) = -\sqrt{(1 - \operatorname{dn}^2(u;\ k))(\operatorname{dn}^2(u;\ k) - k'^2)}.$$

Whence for $u = \xi K(k)$ we have

$$(23) \qquad \frac{r_\xi(\xi)}{r(\xi)} = -K(k)\sqrt{\big(1-r^2(\xi)\big)\left(1-\frac{r^2(1)}{r^2(\xi)}\right)}.$$

We know from Theorem 1 that, for $k \in (0, 1)$, the function $r(\xi)$ for $\xi \in (0, 1]$ and the function $r(1)/r(\xi) = r(1-\xi)$ for $\xi \in [0, 1)$ are decreasing with respect to k. The function $K(k)$ is increasing. Due to this, expression (23) is a decreasing function of k for $\xi \in (0, 1)$. Hence

$$(24) \qquad \frac{r(\xi)r_{\xi k}(\xi) - r_\xi(\xi)r_k(\xi)}{r^2(\xi)} < 0 \quad \text{for } \xi, k \in (0, 1).$$

Calculate now the derivative $r_{\xi k}(\xi)$. We obtain

(25)

$$r_{\xi k}(\xi) = -\frac{d}{dk}K(k)\sqrt{\big(1-r^2(\xi)\big)\big(1-r^2(1-\xi)\big)} + \frac{K(k)\varphi(\xi)}{\sqrt{\big(1-r^2(\xi)\big)\big(1-r^2(1-\xi)\big)}},$$

where

$$\varphi(\xi) = r(\xi)r_k(\xi)\big(1-r^2(1-\xi)\big) + r(1-\xi)r_k(1-\xi)\big(1-r^2(\xi)\big).$$

We know from (9) that $r(\xi) = \sqrt{1-k^2x(k;\ \xi)}$. By virtue of (17), the derivative x_k is continuous and so does the derivative

$$(26) \qquad r_k(\xi) = \frac{-kx(k;\ \xi)\big(x(k;\ \xi)+kx_k(k;\ \xi)\big)}{\sqrt{1-k^2x^2(k;\ \xi)}} \quad (k,\ \xi \in (0, 1)).$$

Therefore, $\varphi(\xi)$ is a continuous function of ξ and k. For $k, \xi \in (0, 1)$, the denominator of the right-hand side of (25) is different from zero. From this it follows at last that derivative (25) is continuous for $\xi \in (0, 1)$.

Now we investigate the continuity of the partial derivative $r_{k\xi}(\xi)$. From (26) we obtain (for simplicity, the arguments of the functions are omitted)

$$(27) \qquad r_{k\xi} = -\frac{(1-k^2x^2)[kx_\xi(x+kx_k)+kx(x_\xi+kx_{k\xi})]+k^3x^2x_\xi(x+kx_k)}{(1-k^2x^2)\sqrt{1-k^2x^2}}.$$

From the proof of Theorem 1 we know that $x = x(k;\ \xi)$ is determined by the formula

$$\Phi(x;\ k,\ \xi) \equiv \xi K(k) - \int_0^x \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = 0.$$

The partial derivatives of the function $\Phi(x;\ k,\ \xi)$,

$$\Phi_\xi = K(k), \quad \Phi_x = -1/\sqrt{(1-x^2)(1-k^2x^2)},$$

$$\Phi_k = \xi\frac{dK}{dk} - \int_0^x \frac{kt^2\,dt}{(1-k^2t^2)\sqrt{(1-t^2)(1-k^2t^2)}},$$

are continuous for $\xi \in [0, 1)$ (see (12)) and so do the derivatives

$$x_\xi = K\sqrt{(1-x^2)(1-k^2x^2)},$$

$$x_k = \left[\xi\frac{dK}{dk} - \int_0^x \frac{kt^2\,dt}{(1-k^2t^2)\sqrt{(1-t^2)(1-k^2t^2)}}\right]\sqrt{(1-x^2)(1-k^2x^2)}. \tag{28}$$

From (28) we calculate the derivative $x_{k\xi}$:

$$x_{k\xi} = \left(\frac{dK}{dk} - x_\xi\frac{kx^2}{(1-k^2x^2)\sqrt{(1-x^2)(1-k^2x^2)}}\right)\sqrt{(1-x^2)(1-k^2x^2)} +$$
$$+\left(\xi\frac{dK}{dk} - \int_0^x \frac{kt^2\,dt}{(1-k^2t^2)\sqrt{(1-t^2)(1-k^2t^2)}}\right)\frac{-xx_\xi(1+k^2-2k^2x^2)}{\sqrt{(1-x^2)(1-k^2x^2)}}.$$

Hence the derivative $x_{k\xi}$ is also continuous for $\xi \in [0, 1)$.

In this way we have shown that derivative (27) is continuous for $\xi \in [0, 1)$. It equals, therefore, derivative (25). Hence the left-hand side of (24) is equal to the right-hand side of (22). Thus it follows that $s(k;\ \xi)$ is a decreasing function of ξ for $\xi \in (0, 1)$. Now, from (21) we have the following inequalities:

$$\begin{aligned} &t_k(k;\ \xi, \eta) > 0 \quad \text{for } 0 < \xi < \eta, \\ &t_k(k;\ \xi, \eta) < 0 \quad \text{for } \eta < \xi < 1. \end{aligned} \tag{29}$$

For $\xi = 0$ and $\xi = 1$, we get

$$\begin{aligned} &t(k;\ 0,\ \eta) = 1/\mathrm{dn}\big(\eta K(k);\ k\big) && \eta \in (0, 1], \\ &t(k;\ 1, \eta) = k'/\mathrm{dn}\big(\eta K(k);\ k\big) = \mathrm{dn}\big((1-\eta)K(k);\ k\big) && \eta \in [0, 1). \end{aligned}$$

So, from Theorem 1, the thesis of Theorem 2 follows for $\xi = 0$ and $\xi = 1$. For all other values of ξ, the thesis follows from (29). This completes the proof of the theorem.

Remark. For $\eta = 0$, Theorem 1 is a consequence of Theorem 2. We have, however, proved Theorem 1 first, because it is needed in the proof of Theorem 2.

In this paper we are dealing only with the limits $k \to 1^-$ and $k \to 0^+$. The investigation of the limits $k \to 0$ and $k \to 1$ is possible, however, for the investigation of the properties of the optimum rational function, this is not necessary. We prove now a theorem which enables the calculation of the limit $\lim\limits_{k'\to 0^+} r_{jm}$.

THEOREM 3. *For* $\xi \in (0,1]$, $\lim\limits_{k\to 1^-} \mathrm{dn}\big(\xi K(k);\ k\big) = 0$.

Proof. The majority of the known series expansions of the function $\mathrm{dn}(u;\ k)$ is fast convergent for $k \sim 0$. We are investigating the limit for $k \to 1^-$, therefore, it is more convenient to replace the modulus of k by its complementation $k' = \sqrt{1-k^2}$. To do this we apply the imaginary Jacobi transformation (see [7], p. 24) to the function $\mathrm{dn}(u;\ k)$. We obtain

$$\mathrm{dn}(u;k) = \frac{\mathrm{dn}(iu;k')}{\mathrm{cn}(iu;k')}. \tag{30}$$

It is known (see [9], formula 8.146.13) the expansion

$$\frac{\mathrm{dn}(u;\ k)}{\mathrm{cn}(u;\ k)} \tag{31}$$

$$= \frac{\pi}{2K(t)}\left[\frac{1}{\cos\big(u\pi/2K(k)\big)} - 4\sum_{n=1}^{\infty}(-1)^n \frac{q^{2n-1}}{1+q^{2n-1}}\cos\left((2n-1)\frac{u\pi}{2K(k)}\right)\right]$$

$$\left|\mathrm{im}\,\frac{u\pi}{2K(k)}\right| < \frac{\pi}{2}\,\mathrm{im}\,\tau,$$

where ([7], p. 20)

$$q = \exp\big(-\pi K(k')/K(k)\big), \qquad \tau = iK(k')/K(k). \tag{32}$$

Let

$$u = i\xi K(k), \qquad \xi \in (0,1). \tag{33}$$

Series (31) is convergent for arguments (33) and for the complementary modulus k', since

$$\left|\mathrm{im}\,\frac{\pi u}{2K(k')}\right| = \left|\mathrm{im}\,\frac{i\xi\pi K(k)}{2K(k')}\right| = \frac{\pi}{2}\,\xi\,\frac{K(k)}{K(k')} < \frac{\pi}{2}\,\mathrm{im}\,\tau' = \frac{\pi}{2}\,\frac{K(k)}{K(k')}.$$

Therefore, from (30) and (31), for arguments (33), we obtain

$$\mathrm{dn}\big(\xi K(k);\ k\big) = \frac{\mathrm{dn}\big(i\xi K(k);\ k'\big)}{\mathrm{cn}\big(i\xi K(k);\ k'\big)} = \frac{\pi}{2K(k')}\left[\frac{1}{\cos\big(i\xi\pi K(k)/2K(k')\big)} - \right. \tag{34}$$

$$\left. -4\sum_{n=1}^{\infty}(-1)^n \frac{q'^{2n-1}}{1+q'^{2n-1}}\cos\left((2n-1)\frac{i\xi\pi K(k)}{2K(k')}\right)\right],$$

where (see (32))

$$q' = \exp\big(-\pi K(k)/K(k')\big). \tag{35}$$

From (35) we obtain the relation $-\ln q' = \pi K(k)/K(k')$ which implies

$$\cos\left((2n-1)\frac{i\xi\pi K(k)}{2K(k')}\right) = \cos\left(-(2n-1)i\frac{\xi}{2}\ln q'\right) \tag{36}$$
$$= \cosh\left(-(2n-1)\frac{\xi}{2}\ln q'\right) = \frac{1}{2}\left(q'^{\xi(2n-1)/2} + q'^{-\xi(2n-1)/2}\right).$$

Therefore, formula (34) can now be presented in the form

$$\operatorname{dn}(\xi K(k);\, k) = \frac{\pi}{2K(k')}\left(\frac{2q'^{\xi/2}}{1+q'^{\xi}} - 4\sum_{n=1}^{\infty}(-1)^n b_n\right) \quad (\xi \in (0,1)), \tag{37}$$

where

$$b_n = \frac{1+q'^{\xi(2n-1)}}{1+q'^{(2n-1)}}\, q'^{(1-\xi/2)(2n-1)}.$$

It is known (see [7], p. 31) that $0 < q' < 1$ and

$$\lim_{k\to 1^-} q' = 0. \tag{38}$$

Thus

$$\lim_{k\to 1^-} b_n = 0.$$

The series $\sum_{n=1}^{\infty}(-1)^n b_n$ converges uniformly since

$$\sum_{n=1}^{\infty} b_n < 2\sum_{n=1}^{\infty} q'^{(1-\xi/2)(2n-1)} < \infty.$$

Now it follows that

$$\lim_{k\to 1^-}\sum_{n=1}^{\infty}(-1)^n b_n = \sum_{n=1}^{\infty}(-1)^n \lim_{k\to 1^-} b_n = 0.$$

In addition, from (4) and (38) we have

$$\lim_{k\to 1^-}\frac{q'^{\xi/2}}{1+q'^{\xi}} = 0 \quad \text{and} \quad \lim_{k\to 1^-} K(k') = \frac{\pi}{2}.$$

We have shown in this way that both components of the right-hand side of equality (36) tend to zero for $k \to 1^-$. For $\xi = 1$, the thesis of the theorem follows immediately from the last of equalities (1), which completes the proof.

Similarly as Theorem 1, the thesis of Theorem 3 was known earlier (see the graphs in [4], p. 98). The author did not, however, succeed in finding a published proof of it.

Let us consider now the limit properties of the function

$$\ln \operatorname{dn}(\xi K(k);\, k)/\ln \operatorname{dn}(\eta K(k);\, k).$$

In view of Theorem 3, for $k \to 1^-$, this function is an expression of type ∞/∞. Its limit will be stated in Theorem 4. In Section 8 we shall investigate the relationship between this theorem and the approximate formulas for r_{jm}.

THEOREM 4. *If* $\xi \in [0,1]$ *and* $\eta \in (0,1]$, *then*

$$\lim_{k\to 1^-} \frac{\ln \operatorname{dn}(\xi K(k);\, k)}{\ln \operatorname{dn}(\eta K(k);\, k)} = \frac{\xi}{\eta}.$$

Proof. Similarly as in the proof of Theorem 3, we use Jacobi's imaginary transformation (see (30)) for the function $\operatorname{dn}(u;\, k)$. The well known expansion of the logarithm of $\operatorname{cn}(u;\, k)$ (see [9], formula 8.146.21) is

$$\ln \operatorname{cn}(u;\, k) = \ln\cos\frac{\pi u}{2K(k)} - 4\sum_{n=1}^{\infty} \frac{1}{n}\, \frac{q^n}{1+(-1)^n q^n} \sin^2 \frac{n\pi u}{2K(k)}, \tag{39}$$

$$\left|\operatorname{im}\frac{u}{K(k)}\right| < \operatorname{im}\tau,$$

where q and τ are given by (32). It is known (see [7], p. 31) that

$$\lim_{k\to 0^+} \operatorname{cn}(u;\, k) = \cos u \quad \text{and} \quad \lim_{k\to 0^+} K(k) = \pi/2.$$

From (39) it follows, therefore, that the behaviour of $\ln\operatorname{cn}(u;\, k)$ for $k \to 0^+$ is determined by the first component of the right-hand side, because the second one tends to zero. Let

$$u = i\xi K(k), \qquad \xi \in [0,1). \tag{40}$$

Now

$$\left|\operatorname{im}\frac{u}{K(k')}\right| = \left|\operatorname{im}\frac{i\xi K(k)}{K(k')}\right| = \frac{\xi K(k)}{K(k')} < \operatorname{im}\tau' = \frac{K(k)}{K(k')}.$$

Thus expansion (39) holds for arguments (40) and for k'. Therefore, from (39) we have

$$\ln\operatorname{cn}\big(i\xi K(k);\, k'\big)$$
$$= \ln\cos\left(i\xi\frac{\pi}{2}\,\frac{K(k)}{K(k')}\right) - 4\sum_{n=1}^{\infty}\frac{1}{n}\,\frac{q'^n}{1+(-1)^n q'^n}\sin^2\left(i\xi n\frac{K(k)}{K(k')}\,\frac{\pi}{2}\right) \quad (\xi\in[0,1)).$$

This and (30) give, for arguments (40),

$$\begin{aligned}\ln\operatorname{dn}\big(\xi K(k);\, k\big) &= \ln\operatorname{dn}\big(i\xi K(k);\, k'\big) - \ln\operatorname{cn}\big(i\xi K(k);\, k'\big)\\ &= \ln\operatorname{dn}\big(i\xi K(k);\, k'\big) - \ln\cos\left(i\xi\frac{\pi}{2}\,\frac{K(k)}{K(k')}\right) +\\ &\quad + 4\sum_{n=1}^{\infty}\frac{1}{n}\,\frac{q'^n}{1+(-1)^n q'^n}\sin^2\left(i\xi n\frac{K(k)}{K(k')}\,\frac{\pi}{2}\right).\end{aligned} \tag{41}$$

From (6) it follows that

$$\lim_{k'\to 0+} \operatorname{dn}(u;\ k') = 1.$$

From what has been stated before we have

$$\lim_{k'\to 0+} \sum_{n=1}^{\infty} \frac{1}{n} \frac{q'^n}{1+(-1)^n q'^n} \sin^2\left(i\xi n \frac{K(k)}{K(k')} \frac{\pi}{2}\right) = 0.$$

Now, it can be deduced from (41) that

$$\lim_{k\to 1-} \frac{\ln \operatorname{dn}\big(\xi K(k);\ k\big)}{\ln \operatorname{dn}\big(\eta K(k);\ k\big)} = \lim_{k'\to 0+} \frac{-\ln\cos\big(i\xi\big(K(k)/K(k')\big)(\pi/2)\big)}{-\ln\cos\big(i\eta\big(K(k)/K(k')\big)(\pi/2)\big)}, \tag{42}$$

$$\xi \in [0, 1),\ \eta \in (0, 1).$$

It is known that, for sufficiently small k',

$$q' = (k'/4)^2\big(1+o(k')\big).$$

Hence from (36) we obtain, for $n = 1$,

$$\cos\left(i\xi \frac{\pi}{2} \frac{K(k)}{K(k')}\right) = \frac{1}{2} \frac{1+(k'/4)^{2\xi}\big(1+o(k')\big)}{(k'/4)^{\xi}\big(1+o(k')\big)}.$$

This and (42), for $\xi \in [0, 1)$ and $\eta \in (0, 1)$, give

$$\lim_{k\to 1-} \frac{\ln \operatorname{dn}\big(\xi K(k);\ k\big)}{\ln \operatorname{dn}\big(\eta K(k);\ k\big)} \tag{43}$$

$$= \lim_{k'\to 0+} \frac{\ln 2 - \ln\big(1+(k'/4)^{2\xi}\big(1+o(k')\big)\big) + \xi\ln(k'/4) + \ln\big(1+o(k')\big)}{\ln 2 - \ln\big(1+(k'/4)^{2\eta}\big(1+o(k')\big)\big) + \eta\ln(k'/4) + \ln\big(1+o(k')\big)} = \frac{\xi}{\eta}.$$

It remains to prove the theorem for

1° $\xi = 1$, $\eta \in (0, 1)$;

2° $\xi = 1$, $\eta = 1$;

3° $\xi \in (0, 1)$, $\eta = 1$.

The proof in cases 1° and 3° is similar. For $\xi = 1$, we have $\ln \operatorname{dn}\big(K(k);\ k\big) = \ln k'$. Hence, from (43) it follows

$$\lim_{k\to 1-} \frac{\ln \operatorname{dn}\big(K(k);\ k\big)}{\ln \operatorname{dn}\big(\eta K(k);\ k\big)}$$

$$= \lim_{k'\to 0+} \frac{\ln k'}{\ln 2 - \ln\big(1+(k'/4)^{2\eta}\big(1+o(k')\big)\big) + \eta\ln(k'/4) + \ln\big(1+o(k')\big)} = \frac{1}{\eta}.$$

The proof in case 3° is similar. On the other hand, for $\xi = \eta = 1$, we have

$$\lim_{k\to 1^-} \frac{\ln \mathrm{dn}(K(k);\, k)}{\ln \mathrm{dn}(K(k);\, k)} = 1,$$

which completes the proof of the theorem.

We investigate now the behaviour of

$$\ln \mathrm{dn}(\xi K(k);\, k)/\ln \mathrm{dn}(\eta K(k);\, k) \quad \text{for } k \to 0^+.$$

We know from (6) that this function is an expression of type $0/0$. The calculation of its limit will allow us the determination of

$$\lim_{k'\to 1^-} \frac{dr_{jm}}{dk'}.$$

THEOREM 5. *If* $\xi \in [0, 1]$ *and* $\eta \in (0, 1]$, *then*

$$\lim_{k\to 0^+} \frac{\ln \mathrm{dn}(\xi K(k);\, k)}{\ln \mathrm{dn}(\eta K(k);\, k)} = \frac{\sin^2(\xi\pi/2)}{\sin^2(\eta\pi/2)}.$$

Proof. It is known (see [9], formula 8.146.22) that

(44)

$$\ln \mathrm{dn}(u;\, k) = -8 \sum_{n=1}^{\infty} \frac{a_n}{2n-1} \sin^2\left((2n-1)\frac{u}{K(k)}\frac{\pi}{2}\right), \qquad \left|\mathrm{im}\frac{u}{K}\right| < \mathrm{im}\,\tau,$$

where

(45) $$a_n = \frac{q^{2n-1}}{1-q^{2(2n-1)}},$$

and q and τ are given by (32). The expansion of q with respect to k is also known (see [7], p. 18):

$$q^{1/4} = \left(\frac{k}{4}\right)^{1/2}\left[1 + 2\left(\frac{k}{4}\right)^2 + \ldots\right].$$

Hence, for sufficiently small k, we obtain

(46) $$q = \left(\frac{k}{4}\right)^2 \bigl(1 + o(k)\bigr).$$

We investigate now the behaviour of the coefficients of (45) for $k \to 0^+$. We have, for sufficiently small x,

$$\frac{x}{1-x^2} = x\bigl(1 + o(x)\bigr).$$

Thus, for sufficiently small q and for $x = q^{2n-1}$, from (45) and (46) we have

$$a_n = q^{2n-1}\big(1+o(q^{2n-1})\big) = \left(\frac{k}{4}\right)^{2(2n-1)}\big(1+o(k)\big) = o(k^{4n-3}). \tag{47}$$

Let

$$u = \xi K(k), \qquad \xi \in [0,1]. \tag{48}$$

Now

$$\left|\operatorname{im}\frac{u}{K}\right| = |\operatorname{im}\xi| = 0 < \operatorname{im}\tau = \frac{K(k')}{K(k)}.$$

It follows that series (44) converges for arguments (48). Thus from (47), for sufficiently small k and for arguments (48), one obtains

$$\ln\operatorname{dn}\big(\xi K(k);\ k\big) = -8\left(\frac{k}{4}\right)^2\big(1+o(k)\big)\sin^2\left(\xi\frac{\pi}{2}\right)+o(k^5).$$

Hence, for $\xi \in [0,1]$, $\eta \in (0,1]$, we have

$$\frac{\ln\operatorname{dn}\big(\xi K(k);\ k\big)}{\ln\operatorname{dn}\big(\eta K(k);\ k\big)}$$
$$= \frac{(-k^2/2)\big(1+o(k)\big)\sin^2(\xi\pi/2)+o(k^5)}{(-k^2/2)\big(1+o(k)\big)\sin^2(\eta\pi/2)+o(k^5)} = \frac{\big(1+o(k)\big)\sin^2(\xi\pi/2)\big)+o(k^3)}{\big(1+o(k)\big)\sin^2(\eta\pi/2)\big)+o(k^3)},$$

from which the thesis of the theorem follows immediately.

Theorem 5 will be used not only in the investigation of the limit behaviour of the elements of the sets $R_m(k')$ and $U_m(k')$ but also in the proof of the experimental research described in [2] (see Section 8).

4. Jordan's formulas and the determination of the elements of $U_m(k')$ for an arbitrary m. It is known [12] that $R_m(k')$ can be determined by Jordan's formula

$$r_{jm} = \operatorname{dn}\left[\left(1-\frac{2j-1}{2m}\right)K(k);\ k\right] \qquad (j = 1, 2, \ldots, m). \tag{1}$$

As usual, let $k' = \sqrt{1-k^2}$. Formula (1) guarantees that $R_m(k')$ is ordered increasingly. Wachspress (see [13], p. 192) has shown that the following problem of Cauer (see [7], p. 105) can be reduced to the problem of finding the optimum function satisfying (2.1) with

$$k' = \frac{1-t^2}{1+t^2} \tag{2}$$

by substituting

$$x = \frac{1-z^2}{1+z^2}. \tag{3}$$

CAUER'S PROBLEM. For a given real t $(0 \leqslant t < 1)$ and a natural m, find optimum parameters $a_1, a_2, \ldots, a_m$ such that

$$\max_{0 \leqslant z \leqslant t} \left| \prod_{j=1}^{m} \frac{a_j^2 - z^2}{1 - a_j^2 z^2} \right|$$

be the minimum.

Oberhettinger and Magnus ([7], p. 153) give the optimum a_j as

$$a_j = t\,\mathrm{sn}\left(\frac{2j-1}{2m} K(t^2);\ t^2\right) \quad (j = 1, 2, \ldots, m), \tag{4}$$

and the set of extremal points of the optimum function in Cauer's problem z_j as

$$z_j = t\,\mathrm{sn}\left(\frac{j}{m} K(t^2);\ t^2\right) \quad (j = 0, 1, \ldots, m). \tag{5}$$

Wachspress obtained (1) from (4) by using substitution (3) and Landen's transformation. The same leads to the following

THEOREM 1. *The elements of $U_m(k')$ are given by*

$$u_{jm} = \mathrm{dn}\left(\frac{m-j}{m} K(k);\ k\right) \quad (j = 0, 1, \ldots, m). \tag{6}$$

Proof. Parameters (5) form an increasing sequence. Now, let us apply substitution (3) to formula (5). Since the set $U_m(k')$ is increasingly ordered, we obtain

$$u_{m-j,m} = \frac{1 - z_j^2}{1 + z_j^2} = \frac{1 - t^2 \mathrm{sn}^2\big((j/m)K(t^2);\ t^2\big)}{1 + t^2 \mathrm{sn}^2\big((j/m)K(t^2);\ t^2\big)}.$$

Landen's transformation (see [7], p. 4 and 23) is recalled now:

$$\mathrm{dn}\left((1+k)u;\ \frac{2\sqrt{k}}{1+k}\right) = \frac{1 - k^2 \mathrm{sn}^2(u;\ k)}{1 + k^2 \mathrm{sn}^2(u;\ k)}, \qquad K\left(\frac{2\sqrt{k}}{1+k}\right) = (1+k)K(k).$$

Hence

$$u_{m-j,m} = \mathrm{dn}\left((1+t^2)\frac{j}{m} K(t^2);\ \frac{2t}{1+t^2}\right) = \mathrm{dn}\left[\frac{j}{m} K\left(\frac{2t}{1+t^2}\right);\ \frac{2t}{1+t^2}\right].$$

From (2) we have

$$\frac{2t}{1+t^2} = \sqrt{1-k'^2},$$

therefore,

$$u_{m-j,m} = \mathrm{dn}\left(\frac{j}{m} K\big(\sqrt{1-k'^2}\big);\ \sqrt{1-k'^2}\right).$$

The thesis of the theorem follows easily.

A new proof of Lemma 2.3 can be obtained from Theorem 1 (assuming $a = k'$ and $\beta = 1$). Namely, from (3.2) we have

$$u_{m-j,m} = \mathrm{dn}\left[\left(1-\frac{m-j}{m}\right)K(k);\ k\right] = \frac{k'}{\mathrm{dn}\big(-((m-j)/m)K(k);\ k\big)} = \frac{k'}{u_{jm}}.$$

The assumptions can easily be removed by an appropriate substitution of variables.

5. Algorithms JR and JU of determining sets $R_{2^p}(k')$ and $U_{2^p}(k')$. The determination of the exact values of r_{jm} and u_{jm} for an arbitrary m from (4.1) and (4.6) is troublesome. In practice, one usually replaces these formulas by some approximate expressions derived from the series expansion of $\mathrm{dn}\big(\xi K(k);\ k\big)$ with respect to powers of k'. Simple analytic formulas can be obtained in the case $m = 2^p$ only. This fact is a consequence of the following

THEOREM 1. *The elements of the sets $R_m(k')$ and $R_{2m}(k')$ are related as follows:*

$$(1)\qquad r_{m+j,2m} = \left(\frac{r_{jm}+k'^2+\sqrt{(1-k'^2)(r_{jm}^2-k'^2)}}{1+r_{jm}}\right)^{1/2} \qquad (j = 1, 2, \ldots, m),$$

$$(2)\qquad r_{m-j+1,2m} = \left(\frac{r_{jm}+k'^2-\sqrt{(1-k'^2)(r_{jm}^2-k'^2)}}{1+r_{jm}}\right)^{1/2} \qquad (j = 1, 2, \ldots, m).$$

Proof. It is known (see [7], p. 21) that

$$(3)\qquad \mathrm{dn}^2(u;\ k) = \frac{\mathrm{dn}(2u;\ k)+k^2\mathrm{cn}(2u;\ k)+k'^2}{1+\mathrm{dn}(2u;\ k)}.$$

From the identities (see [7], p. 19)

$$\mathrm{sn}^2(u;\ k)+\mathrm{cn}^2(u;\ k) = 1 \quad \text{and} \quad \mathrm{dn}^2(u;\ k)+k^2\mathrm{sn}^2(u;\ k) = 1,$$

for double u, we obtain the following

$$(4)\qquad \mathrm{cn}(2u;\ k) = \pm\sqrt{\frac{k^2-1+\mathrm{dn}^2(2u;\ k)}{k^2}}.$$

As it is known (see [9], p. 325),

$$(5)\qquad \mathrm{dn}(2K-u;\ k) = \mathrm{dn}(u;\ k) \quad \text{and} \quad \mathrm{cn}(2K-u;\ k) = -\mathrm{cn}(u;\ k).$$

Let

$$(6)\qquad v = \frac{1}{2}\left(1-\frac{2j-1}{2m}\right)K(k) \qquad (j = 1, 2, \ldots, m).$$

We have $\operatorname{cn}(2v;\ k) > 0$ (see [7], p. 22). From (5) for (6) it follows that formula (4) is equivalent to the following two ones:

$$\operatorname{cn}(2v;\ k) = \frac{1}{k}\sqrt{k^2-1+\operatorname{dn}^2(2v;\ k)}\,, \tag{7}$$

$$\begin{aligned}\operatorname{cn}\bigl(2(K-v);\ k\bigr) &= -\frac{1}{k}\sqrt{k^2-1+\operatorname{dn}^2\bigl(2(K-v);\ k\bigr)} \\ &= -\frac{1}{k}\sqrt{k^2-1+\operatorname{dn}^2(2v;\ k)}\,.\end{aligned} \tag{8}$$

It follows from (4.1) that $\operatorname{dn}(2v;\ k) = r_{jm}$ for arguments (6); thus the substitution of (7) into (3) for $u = v$ leads to the expression

$$\frac{r_{jm}+\sqrt{k^2-1+r_{jm}^2}+k^2}{1+r_{jm}}$$

on the right-hand side, and to $r^2_{m+j,2m}$ on the left-hand side, since

$$\frac{1}{2}\left(1-\frac{2j-1}{2m}\right) = 1-\frac{2(m+j)-1}{4m}\,,$$

which immediately implies formula (1), because $k^2-1 = -k'^2$. For $u = K-v$, the function $\operatorname{cn}(2u;\ k)$ in formula (3) can be replaced by the right-hand side of equality (8). The right-hand side of (3) is then equal to

$$\frac{r_{jm}-\sqrt{k^2-1+r_{jm}^2}+k^2}{1+r_{jm}}\,,$$

and the left-hand side equals $r^2_{m-j+1,2m}$ since

$$K-v = \left(1-\frac{2(m-j+1)-1}{4m}\right)K\,.$$

From this we are lead to formula (2), and the proof is complete.

Knowing the function $f_m(x;\ R_m)$, due to Theorem 1, it is very easy to construct the function $f_{2m}(x;\ R_{2m})$. Since k', $\sqrt{k'}$ and 1 belong always to the set of extremal points, it is easy to compare the norms of $f_m(x;\ R_m)$ and $f_{2m}(x;\ R_{2m})$.

Similarly as formula (2.7) in the *WR*-algorithm, formula (2) does not guarantee the numerical stability of one step of the algorithm *JR*, and, therefore, the elements $r_{1,2m}, r_{2,2m}, \ldots, r_{m,2m}$ ought to be determined by

$$r_{j,2m} = \frac{k'}{r_{2m-j+1,2m}} \qquad (j = 1, 2, \ldots, m)\,. \tag{9}$$

In a forthcoming paper [16] we show that the determination of the elements of R_{2m} is equivalent to the determination of the zeros of a certain trinomial. This fact exhibits the known numerical defects of formulas of type (2).

Algorithm JR. It follows from formula (4.1) and from property (3.1) that

$$R_1(k') = \{\mathrm{dn}(\tfrac{1}{2}K(k);\ k)\} = \{\sqrt{k'}\}.$$

If r_{11} is known, the elements of $R_2(k')$ are determined by (1) and (9). To obtain thus the elements of $R_m(k')$ for $m = 2^p$, it is necessary — with the aid of (1) and (9) — to find in sequence the elements of $R_2(k')$, $R_4(k')$, $\ldots$, $R_{m/2}(k')$ and $R_m(k')$. This method is used in the following ALGOL-60 procedure:

```
procedure JR(p, k, r);
    value p, k;
    integer p;
    real k;
    array r;
    comment procedure JR places in the array r[1 : 2↑p] the elements
        of the set R_m(k) for m = 2↑p in increasing order;
    begin
        integer i, j, m;
        real rj, k1, k2;
        k1 := (1.0 − k) × (1.0 + k);
        k2 := k × k;
        m := 1;
        r[1] := sqrt(k);
        for i := 1 step 1 until p do
            begin
                for j := 1 step 1 until m do
                    begin
                        rj := r[j];
                        r[m + j] := sqrt((rj + k2 + sqrt(k1 × (rj − k) ×
                            (rj + k)))/(1.0 + rj))
                    end j;
                m := m + m;
                for j := m ÷ 2 step −1 until 1 do
                    r[j] := k/r[m + 1 − j]
            end i
    end JR
```

The elements of the sets $R_m(k')$ and $U_m(k')$ are expressed by similar formulas (see (4.1) and (4.6)). These similarities are discussed in details in Section 6. Here we give only formulas similar to (1) and (2).

COROLLARY. *The elements of the sets $U_m(k')$ and $U_{2m}(k')$ are related as follows:*

$$u_{m+j,2m} = \left(\frac{u_{jm}+k'^2+\sqrt{(1-k'^2)(u_{jm}^2-k'^2)}}{1+u_{jm}}\right)^{1/2} \tag{10}$$

$$u_{m-j,2m} = \left(\frac{u_{jm}+k'^2-\sqrt{(1-k'^2)(u_{jm}^2-k'^2)}}{1+u_{jm}}\right)^{1/2} \tag{11}$$

$$(j = 0, 1, \ldots, m).$$

For the same reasons as before, formula (11) is replaced by

$$u_{m-j,2m} = \frac{k'}{u_{m+j,2m}} \quad (j = 0, 1, \ldots, m). \tag{12}$$

For $j = 0$, formulas (10), (11) and (12) are identical, since $u_{0m} = k'$ and $u_{m,2m} = \sqrt{k'}$. The derivation of these formulas is similar to the proof of Theorem 1. Arguments (6) are to be replaced by arguments

$$v = \frac{1}{2}\left(1-\frac{j}{m}\right)K(k) \quad (j = 0, 1, \ldots, m).$$

Therefore, the proof of the corollary is omitted. For $m = 2^p$, formulas (10) and (12) lead to the algorithm of constructing $U_m(k')$.

Algorithm JU. It follows from (4.6) and (3.1) that $U_1(k') = \{k', 1\}$. The elements of $U_2(k')$ can now be easily determined by the use of (10) and (12). To obtain $U_m(k')$ for $m = 2^p$, use of (10) and (12) allows the sequential construction of $U_2(k'), U_4(k'), \ldots, U_{m/2}(k')$ and $U_m(k')$. The procedure description in ALGOL-60 is the following:

```
procedure JU(p, k, u);
   value p, k;
   integer p;
   real k;
   array u;
   comment procedure JU places in the array u[0 : 2↑p] the elements
      of the set U_m(k) for m = 2↑p in increasing order;
   begin
      integer m, i, j;
      real k1, k2, uk, uj;
      m := 1;
      if p ≠ 0 then
```

```
begin
  k1 := (1.0 - k) × (1.0 + k);
  k2 := k × k;
  uk := u[1] := sqrt(k);
  m := 2;
  for i := 2 step 1 until p do
      begin
        for j := m-1 step -1 until 1 do
          begin
            uj := u[j];
            u[m+j] := sqrt ((uj + k2 + sqrt (k1 ×
                (uj - k) × (uj + k)))/(1.0 + uj))
          end j;
        u[m] := uk;
        m := m + m;
        for j := m ÷ 2-1 step -1 until 1 do
          u[j] := k/u[m-j]
      end i
  end p ≠ 0;
u[0] := k;
u[m] := 1.0
end JU
```

Remark. The elements $u_{0,2n}$, $u_{n,2n}$ and $u_{2n,2n}$ are not determined by (10) and (12) but directly by $u_{0,2n} = k'$, $u_{n,2n} = \sqrt{k'}$ and $u_{2n,2n} = 1$.

In a forthcoming paper [16] a detailed analysis of the errors of the algorithms *WR* and *JR* is performed. This analysis concerns also the algorithm *JU*. The algorithm calculates the alternans of the optimum function with great accuracy. This allows a practical verification whether the algorithms *WR* and *JR* give good approximations to the optimum function. The algorithms *JR* and *JU* require a greater number of calculations than the algorithms *WR* and *WU*, however, one obtains indirectly all sets $R_n(k')$ and $U_n(k')$ for n being a power of 2 smaller than m (see [16]).

6. Relations between $R_m(k')$ and $U_m(k')$. Monotonicity of parameters r_{jm} and u_{jm}. Paper [14] contains remarks about some analogy between the optimum rational function $f_m(x; R_m)$ in the limit case $k' \to 1^-$ and the Chebyshev polynomials $T_m(x)$. Now we deal with another property of the sets $R_m(k')$ and $U_m(k')$ which also is characteristic for the sets of zeros and extremal points of Chebyshev polynomials $T_m(x)$ (the parameters r_{jm} correspond to the zeros of the polynomial $T_m(x)$, and u_{jm} to the extremal points).

THEOREM 1. *For an arbitrary natural m and a real k' $(0 < k' < 1)$, we have*

$$U_{2m}(k') = R_m(k') \cup U_m(k').$$

Proof. Introduce the auxiliary notation

$$d_{jm} = \operatorname{dn}\left(\frac{j}{2m} K(k);\ k\right) \quad (j = 0, 1, \ldots, 2m). \tag{1}$$

From (4.1) and (4.6) it follows that

$$r_{jm} = d_{2m-2j+1,m} \quad (j = 1, 2, \ldots, m),$$
$$u_{jm} = d_{2m-2j,m} \quad (j = 0, 1, \ldots, m).$$

Now, we have

$$u_{2j,2m} = d_{4m-4j,2m} = d_{2m-2j,m} = u_{jm},$$
$$r_{2j-1,2m} = d_{4m-4j+2,2m} = d_{2m-2j+1,m} = r_{jm},$$

from which the thesis of the theorem follows.

COROLLARY. *For $m = 2^p$, the set $U_m(k')$ is the sum of all sets $R_n(k')$, where n is a natural power of 2 smaller than m, i.e.*

$$U_m(k') = U_1(k') \cup R_1(k') \cup R_2(k') \cup R_4(k') \cup \ldots \cup R_{2^{p-1}}(k').$$

This formula follows immediately from Theorem 1. This fact can be illustrated graphically as follows:

$$\begin{array}{ccccccccccccccc}
 & & & & & & & r_{11} & & & & & & & \\
 & & & r_{12} & & & & u_{12} & & & & r_{22} & & & \\
 & r_{14} & & u_{14} & & r_{24} & & u_{24} & & r_{34} & & u_{34} & & r_{44} & \\
r_{18} & u_{18} & r_{28} & u_{28} & r_{38} & u_{38} & r_{48} & u_{48} & r_{58} & u_{58} & r_{68} & u_{68} & r_{78} & u_{78} & r_{88}
\end{array}$$

The quantities occurring in the same column are identical, e.g. $r_{12}(k') = u_{14}(k') = u_{28}(k')$.

Let us state now a property of the auxiliary sequence d_{jm} defined by (1).

THEOREM 2. *For an arbitrary natural m, the elements d_{jm} satisfy the recurrent relations*

(2)
$$d_{j+2,m} = \frac{2(1-k'^2)d_{2m}d_{jm}}{1-k'^2-(1-d_{2m}^2)(1-d_{jm}^2)} - d_{j-2,m} \quad (j = 1, 2, \ldots, 2m-2),$$

where

$$d_{0m} = 1, \quad d_{-1m} = d_{1m}, \quad d_{2m} = \frac{2(1-k'^2)d_{1m}^2}{1-k'^2-(1-d_{1m}^2)^2} - 1.$$

Proof. From the formulas given in [7], p. 19 and 21, it is easy to derive the following equality:

$$(3)\qquad \mathrm{dn}(u+v;\, k)+\mathrm{dn}(u-v;\, k) = \frac{2k^2\,\mathrm{dn}(u;\, k)\,\mathrm{dn}(v;\, k)}{k^2-\big(1-\mathrm{dn}^2(u;\, k)\big)\big(1-\mathrm{dn}^2(v;\, k)\big)}.$$

Let $u = jK(k)/2m$ and $v = K(k)/m$. In accordance with (1) we obtain now formula (2) from formula (3). The expression for d_{2m} follows from (3) for $u = v = K(k)/2m$. All other equalities follow from the properties of the elliptic function $\mathrm{dn}(u;\, k)$ (see Section 3), which completes the proof.

The following simple corollary can be obtained from Theorem 2:

Given $d_{1m} = r_{mm}$, all remaining elements of the sets $R_m(k')$ and $U_m(k')$ can be obtained directly from (2). Thus, for any m, it suffices to know only one element (r_{mm} or $r_{1m} = k'/r_{mm}$) to determine the sets $R_m(k')$ and $U_m(k')$.

This fact will be used in Section 7 to modify the algorithm JR.

It follows from formulas (4.1) and (4.6) that, for fixed j and m, the parameters r_{jm} and u_{jm} are functions of the variable k'. The following theorem states the properties of this function:

THEOREM 3. 1° *The parameters $r_{jm} = r_{jm}(k')$ $(j = 1, 2, \ldots, m)$ and $u_{jm} = u_{jm}(k')$ $(j = 0, 1, \ldots, m-1)$ are increasing functions of the variable k'.*

In addition, they have the following limits:

$$2^\circ\ \lim_{k'\to 0+} r_{jm}(k') = \lim_{k'\to 0+} u_{j-1,m}(k') = 0 \qquad (j = 1, 2, \ldots, m);$$

$$3^\circ\ \lim_{k'\to 1-} \frac{d}{dk'}\, r_{jm}(k') = \frac{1}{2}\left(1+\cos\frac{2j-1}{2m}\pi\right) \qquad (j = 1, 2, \ldots, m);$$

$$4^\circ\ \lim_{k'\to 1-} \frac{d}{dk'}\, u_{jm}(k') = \frac{1}{2}\left(1+\cos\frac{j}{m}\pi\right) \qquad (j = 0, 1, \ldots, m).$$

Thus limits 3° and 4° are equal to the zeros and extremal points of the m-th Chebysmhev polynomial in the interval $[0, 1]$.

Proof. Part 1° of the theorem is a simple corollary to Theorem 3.1. Now $\mathrm{dn}(\xi K(k);\, k)$ for $\xi\in(0,1]$ is a decreasing function of the variable k. In our case,

$$(4)\qquad \left\{\begin{array}{ll} \xi = 1-\dfrac{2j-1}{2m} & (j = 1, 2, \ldots, m) \\ \text{or} & \\ \xi = 1-\dfrac{j}{m} & (j = 0, 1, \ldots, m-1); \end{array}\right.$$

thus ξ belongs to the interval $(0, 1]$. Part 1° follows immediately since $k = \sqrt{1-k'^2}$.

Let $r(\xi) = \mathrm{dn}\left(\xi K(\sqrt{1-k'^2}); \sqrt{1-k'^2}\right)$. We know from (3.6) that

$$\lim_{k'\to 1^-} r(\xi) = 1.$$

Since $r(1) = k'$, the application of de l'Hospital's rule gives

$$\lim_{k'\to 1^-} \frac{\ln r(\xi)}{\ln r(1)} = \lim_{k'\to 1^-} \frac{r(1)}{r(\xi)} \frac{dr(\xi)}{dk'} \frac{1}{dr(1)/dk'} = \lim_{k'\to 1^-} \frac{dr(\xi)}{dk'}. \tag{5}$$

In view of Theorem 3.5, we have

$$\lim_{k'\to 1^-} \frac{\ln r(\xi)}{\ln r(1)} = \lim_{k\to 0^+} \frac{\ln r(\xi)}{\ln r(1)} = \sin^2 \xi \frac{\pi}{2} = \frac{1}{2}(1-\cos\xi\pi).$$

Since in our case ξ is of form (4), parts 3° and 4° of the theorem follow immediately from the above. Part 2° is a simple conclusion from Theorem 3.3. Thus the proof is complete.

Parts 3° and 4° of the theorem have been published without the proof by the present author in [14]. The truth of these formulas was motivated there with the limit properties of the optimum function. Now we have an exact proof.

We know that the parameters r_{jm} increase with increasing k'; let us now investigate how behaves the quotient r_{jm}/r_{im}.

Let

$$\xi = 1 - \frac{2j-1}{2m}, \qquad \eta = 1 - \frac{2i-1}{2m} \qquad (i < j).$$

Now, the quotient

$$\mathrm{dn}\left(\xi K(k); k\right)/\mathrm{dn}\left(\eta K(k); k\right) \tag{6}$$

is equal to the function $t\left(\sqrt{1-k'^2}; \xi, \eta\right)$ from Theorem 3.2. Since $\xi < \eta$ and $k = \sqrt{1-k'^2}$, quotient (6) is a decreasing function of the variable k', and, therefore, r_{im} must increase more rapidly than r_{jm} for $k' \to 1^-$.

7. Modification of the algorithm ***JR*****.** For $m = 2^p$, it is easy to determine $d_{1m} = r_{mm}$. As it is known, $r_{11} = \sqrt{k'}$. Generally, from (5.1) we have

$$r_{2j,2j}^2 = \frac{r_{jj} + k'^2 + \sqrt{(1-k'^2)(r_{jj}^2 - k'^2)}}{1 + r_{jj}} \qquad (j = 1, 2, 4, \ldots, 2^{p-1}).$$

If r_{mm} are determined, Theorem 6.2 can be used to evaluate all remaining elements of $R_m(k')$ (for $j = 1, 3, 5, \ldots$) and $U_m(k')$ (for $j = 2, 4, 6, \ldots$). In reality, it suffices to calculate one half of the elements from (6.2), and the remaining ones from (5.9) and (5.12).

The following procedure *MJR* is a realization of this modified algorithm *JR*:

```
procedure MJR(p, k, r);
   value p, k;
   integer p;
   real k;
   array r;
   comment procedure MJR places in the array r[1 : 2↑p] the
      elements of the set R_m(k) for m = 2↑p in increasing order;
   begin
      integer j, m, m2;
      real k1, k2, d1, d2, d3, d4;
      m := 2↑p;
      m2 := m÷2+1;
      k1 := (1.0−k) × (1.0+k);
      k2 := k × k;
      d1 := sqrt(k);
      for j := 1 step 1 until p do
        d1 := sqrt((d1+k2+sqrt(k1 × (d1−k) × (d1+k)))/
           (1.0+d1));
      r[1] := k/d1;
      d3 := r[m] := d1;
      d2 := (1.0−d1) × (1.0+d1);
      k2 := k1+k1;
      d2 := k2 × d1 × d1/(k1−d2× d2)−1.0;
      k2 := k2 × d2;
      d2 := (1.0−d2) × (1.0+d2);
      for j := m−1 step −1 until m2 do
         begin
            d4 := r[j] := k2 × d3/(k1−(1.0−d3) × (1.0+d3) ×
               d2)−d1;
            d1 := d3;
            d3 := d4;
            r[m+1−j] := k/d4
         end j
   end MJR
```

The elements of the set $U_m(k')$ for $m = 2^p$ can be determined in a similar way. The algorithm *MJR* requires a smaller number of calculations than the algorithm *JR*; it gives, however, a greater error.

8. Approximate methods of determining the elements of $R_m(k')$. Peaceman and Rachford have proposed to calculate the parameters r_{jm} from the approximate formulas (see [11], p. 226)

$$r_{jm}^P = k'^{(2j-1)/2m} \quad (j = 1, 2, \ldots, m).$$

Wachspress ([1], p. 212) has given the formulas

$$r_{jm}^W = k'^{(j-1)/(m-1)} \quad (j = 1, 2, \ldots, m).$$

It is easy to verify that

$$r_{j,m-1}^P > r_{jm}^W > r_{j+1,m-1}^P.$$

The parameters r_{jm}^P and r_{jm}^W are decreasingly ordered. The optimum parameter $r_{m-j+1,m}$ can be represented in the form $(k')^f$, where

$$f = \frac{\ln r_{m-j+1,m}}{\ln k'}.$$

Hence, it follows from Theorem 3.4 for $k' \sim 0$ that $f \sim (2j-1)/2m$; therefore, r_{jm}^P is an approximate value of $r_{m-j+1,m}$. On the other hand, the parameter r_{jm}^W is equal to an approximate value of the extremal point $u_{m-1-j,m-1}$.

It has been shown in [1] that Wachspress parameters result in a better convergence of the ADI-method than Peaceman's parameters. These results have been confirmed by Lynch and Rice in [5]; the authors have observed, in addition, that the ordering of the elements of $R_m(k')$ plays here some role. It appears that the smallest elements of $R_m(k')$ have the greatest influence on the reduction of the initial error of the solution of the system of linear equations.

In practice, instead of constructing an optimum function of high order, one applies, cyclically, a smaller set of optimal parameters. It is then possible, as was shown by Samarskiĭ in [10], p. 461, to determine an approximate rational function which, used cyclically, gives (with accuracy up to some factor), for $k' \to 0$, the same rapidity of convergence of the ADI-method as that of the optimum function. From the point of view of the ADI-method, this simple algorithm, proposed by Samarskiĭ, is very useful though the norm of this function is not minimal.

Before Jordan gave formula (4.1), Boor and Rice [2] tried to find such a formula in a experimental way. Namely, they posed the hypothesis that there exists a function $f(y;\, k)$ such that the elements r_{jm}^* and u_{jm}^* belong to $R_m(k')$ and $U_m(k')$, respectively, and

$$(1) \qquad \begin{aligned} r_{jm}^* &= \sqrt{k'} \exp\left[-f\left(\frac{2j-m-1}{m};\ k'\right) \ln\sqrt{k'}\right], \\ u_{jm}^* &= \sqrt{k'} \exp\left[-f\left(\frac{2j-m-2}{m};\ k'\right) \ln\sqrt{k'}\right], \end{aligned}$$

where

1° $f(y;\ k')$ is a decreasing function of the variable k' for $y \in (0,1)$,

2° $f(y;\ k') = -f(-y;\ k')$,

3° $f(0;\ k') = 0$, $f(1;\ k') = 1$,

4° $f(y;\ 1) = \sin(y\pi/2)$, $\lim\limits_{k'\to 0+} f(y;\ k') = y$.

Jordan's formulas confirm the hypothesis of Boor and Rice. The following theorem holds:

THEOREM 1. *The function*

$$(2)\quad f(\xi;\ k') \stackrel{\mathrm{df}}{=} 1-2\frac{\ln\mathrm{dn}\big(((1-\xi)/2)K(k);\ k\big)}{\ln\mathrm{dn}\big(K(k);\ k\big)} \qquad (-1 \leqslant \xi \leqslant 1,\ 0 \leqslant k \leqslant 1),$$

where $k' = \sqrt{1-k^2}$, *has the following properties*:

(i) $f(0;\ k') = 0$, $f(1;\ k') = 1$,

(ii) $f(\xi;\ k') = -f(-\xi;\ k')$,

(iii) $\lim\limits_{k'\to 0+} f(\xi;\ k') = \xi$,

(iv) $\lim\limits_{k'\to 1-} f(\xi;\ k') = \sin(\xi\pi/2)$.

Proof. Let $\xi = 0$. From (3.1) we have then

$$f(0;\ k') = 1-2\frac{\ln\mathrm{dn}\big(K(k)/2;\ k\big)}{\ln\mathrm{dn}\big(K(k);\ k\big)} = 1-2\frac{\ln\sqrt{k'}}{\ln k'} = 0.$$

For $\xi = 1$, we have

$$f(1;\ k') = 1-2\frac{\ln\mathrm{dn}(0;\ k)}{\ln\mathrm{dn}(K;\ k)} = 1.$$

Property (i) is thus proved.

From the definition of $f(\xi;\ k')$ and from (3.2) we obtain

$$\begin{aligned} f(-\xi; k) &= 1-2\frac{\ln\mathrm{dn}\big((1+\xi)/2)K(k);\ k\big)}{\ln\mathrm{dn}\big(K(k);\ k\big)} = 1-2\frac{\ln\mathrm{dn}\big((1-(1-\xi)/2)K;\ k\big)}{\ln\mathrm{dn}(K;\ k)} \\ &= 1-2\frac{\ln k' - \ln\mathrm{dn}\big(((1-\xi)/2)K;\ k\big)}{\ln k'} = -f(\xi;\ k'), \end{aligned}$$

which completes the proof of property (ii).

It follows from Theorem 3.4 that, for $\xi \in [0,1]$,

(3)

$$\lim_{k'\to 0+} f(\xi;\ k') = 1-2\lim_{k\to 1-}\frac{\ln\mathrm{dn}\big(((1-\xi)/2)K(k);\ k\big)}{\ln\mathrm{dn}\big(K(k);\ k\big)} = 1-2\,\frac{1-\xi}{2} = \xi.$$

For negative values of ξ, using (2) and (ii), we have

$$\lim_{k'\to 0^+} f(\xi, k') = -\lim_{k'\to 0^+} f(-\xi; k') = \xi \qquad (\xi \in [-1, 0]).$$

From the above and from (2) property (iii) follows.

Property (iv) is a simple conclusion from Theorem 3.5. Namely, for $\xi \in [0, 1]$, we have

$$\lim_{k'\to 1^-} f(\xi; k') = 1 - 2\lim_{k\to 0^+} \frac{\ln \mathrm{dn}\big(((1-\xi)/2)K(k); k\big)}{\ln \mathrm{dn}\big(K(k); k\big)} = 1 - 2\sin^2\left(\frac{1-\xi}{2}\,\frac{\pi}{2}\right)$$
$$= \cos\left((1-\xi)\frac{\pi}{2}\right) = \sin \xi \frac{\pi}{2}.$$

For $\xi \in [-1, 0]$, the reasoning is similar, which completes the proof of the theorem.

We have proved in this way that function (2) satisfies conditions 2°-4°. The author did not succeed in proving the monotonicity of function (2), i.e. in proving property 1°. Numerical experiments have confirmed the conjection that

$$g(\xi; k) \stackrel{\mathrm{df}}{=} \frac{\ln \mathrm{dn}\big(\xi K(k); k\big)}{\ln \mathrm{dn}\big(K(k); k\big)} \qquad (\xi \in [0, 1])$$

is a strictly decreasing function of k' for $0 < \xi < 1/2$ and that it is strictly increasing for $1/2 < \xi < 1$. It follows from this that

$$f(\xi; k') = 1 - 2g\left(\frac{1-\xi}{2}; k'\right) \qquad (\xi \in (0, 1))$$

is an increasing function of k', which contradicts the hypothesis of Boor and Rice (probably, there is a misprint in their paper). Our conjection is confirmed by the fact that $\xi < \sin(\xi\pi/2)$ for $\xi \in (0, 1)$; thus, it follows from (iii) and (iv) that the limit of $f(\xi; k')$ for $k' \to 1^-$ is greater than that for $k' \to 0^+$.

It remains to test the relation of parameters (1) with r_{jm} and u_{jm}. It appears that

$$r^*_{jm} = r_{m-j+1,m} \quad \text{and} \quad u^*_{jm} = u_{m-j+1,m}.$$

A verification of these formulas is simple. We have

$$r^*_{jm} = \sqrt{k'}\exp\left[-f\left(\frac{2j-1-m}{m}; k'\right)\ln\sqrt{k'}\right] = \mathrm{dn}\left(-\frac{2j-1}{2m}K; k\right)$$
$$= \frac{k'}{r_{jm}} = r_{m-j+1,m}.$$

Similar transformations have to be performed for the second formula. Thus we see that parameters (1) are decreasingly ordered and belong to the sets $R_m(k')$ and $U_m(k')$.

9. Estimation of the norm of the optimum function. Let $L_m(k')$ denote the norm of the optimum function

$$L_m(k') \overset{\mathrm{df}}{=} \max_{k' \leqslant x \leqslant 1} |f_m(x;\ R_m)|.$$

This norm can be estimated by using the approximations of elliptic functions (4.1). If we have at our disposal only certain approximations $\tilde{r}_{jm}$ of the optimal parameters, it is interesting to know how much the norm of $f_m(x;\ \tilde{R}_m)$ differs from that of the optimum function, and how to estimate the norm $L_m(k')$ when only the values of the approximate function are known. These questions are answered by the following theorem:

THEOREM 1 (Wachspress [13], p. 182). *If the function $f_m(x;\ R^m)$ is continuous in the interval $[k', 1]$ and assumes the non-zero values $w_0, w_1, w_2, \ldots, w_m$ with alternating signs at the points $k' \leqslant x_0 < x_1 < \ldots < x_m \leqslant 1$, then*

$$L_m(k') \geqslant \min_{0 \leqslant i \leqslant m} |w_i|.$$

Not only the functions $f_m(x;\ R_m)$ have such a property. Theorem 1 holds for a wider class of rational functions used in uniform approximation (see [6], p. 138).

Numerous numerical experiments lead to the following hypothesis:

If the function $f_m(x;\ \tilde{R}_m)$ is continuous in the interval $[k', 1]$ and assumes the non-zero values $w_0, w_1, w_2, \ldots, w_m$ with alternating signs at the points $k' \leqslant x_0 < x_1 < \ldots < x_m \leqslant 1$, then

$$L_m(k') \geqslant \frac{1}{2} \min_{1 \leqslant i \leqslant m} (|w_{i-1}| + |w_i|). \tag{1}$$

In the case of uniform polynomial approximation such a theorem has been proved by Remez (see [6], p. 80). The truth of the above-given hypothesis is shown by numerous examples and the properties of the optimum rational function $f_m(x;\ R_m)$ approximating zero, similar to those of Chebyshev polynomials. Inequality (1) can be proved only for $m = 1$. The details are omitted here. We give only some examples. Approximations of the optimum rational function are obtained by algorithms JR, MJR and WR. The values of all these functions have been evaluated at the points of the sets $U_m(k')$ obtained by algorithms JU and WU. Algorithms JR and JU give good approximations of both the function and its alternans. We can thus safely assume in the majority of cases that $E(JR) \approx L_m(k')$, where $E(JR)$ denotes the value of the right-hand side of inequality (1) for the function $f_m(x;\ RJ_m)$ at the points of UJ_m. Let us use similarly the symbols $E(MJR)$ and $E(WR)$. The examples are given in Table 1.

TABLE 1

m	k	$E(JR)$	$E(MJR)$	$E(WR)$
2	.999	$.312\ 812\ 769\ 42_{10}-7$	$.312\ 812\ 769\ 41_{10}-7$	$.312\ 812\ 771\ 21_{10}-7$
2	.8	$.155\ 281\ 375\ 17_{10}-2$	$.155\ 281\ 375\ 17_{10}-2$	$.155\ 281\ 375\ 17_{10}-2$
4	.8	$.120\ 561\ 527\ 37_{10}-5$	$.120\ 561\ 527\ 32_{10}-5$	$.120\ 561\ 526\ 85_{10}-5$
8	.8	$.726\ 754\ 093\ 71_{10}-12$	$.726\ 754\ 088\ 55_{10}-12$	$.726\ 750\ 769\ 60_{10}-12$
8	.01	$.446\ 726\ 834\ 67_{10}-4$	$.446\ 726\ 834\ 26_{10}-4$	$.446\ 726\ 834\ 32_{10}-4$
16	.01	$.997\ 824\ 323\ 79_{10}-9$	$.997\ 824\ 272\ 29_{10}-9$	$.997\ 824\ 201\ 54_{10}-9$
16	.5	$.172\ 377\ 697\ 71_{10}-16$	$.172\ 377\ 682\ 21_{10}-16$	$.172\ 253\ 790\ 39_{10}-16$

In all but the first examples given in Table 1 the values of $E(JR)$ are greater or equal to the values of $E(MJR)$ and $E(WR)$. This fact indicates that inequality (1) is true. In the first example neither $f_m(x;\ RJ_m)$ nor $f_m(x;\ RW_m)$ represent a good approximation to the optimum function. Therefore, the inequality $E(JR) < E(WR)$ is not in contradiction with the hypothesis. The calculations for the first example have been repeated with double accuracy. They lead to an optimum function with the norm

$$E = .312\ 812\ 773\ 672_{10}-7$$

which is greater than $E(JR)$, $E(MJR)$ and $E(WR)$; this is in accordance with the conjecture.

From the point of view of the ADI-method, the use of the rational function determined by the algorithm WR is not much worse because its norm is at most two times greater than that of the function obtained by the algorithm JR. The greatest defect of the algorithm WR is the appearance of negative expressions under the square root, which does not enable the construction of the rational function.

In the forthcoming paper [16] we are dealing with the analysis of the errors obtained in algorithms JR and Wachspress' algorithm. In a theoretical and experimental way, the numerical superiority of the algorithm JR over the Wachspress algorithm is proved.

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KONSTRUKCJA I WŁASNOŚCI OPTYMALNEJ FUNKCJI WYMIERNEJ STOSOWANEJ W METODZIE ADI

STRESZCZENIE

W związku z metodą naprzemiennych kierunków (metoda ADI) postawiono następujące zadanie:

Dla danej liczby naturalnej m i rzeczywistej k' ($0 < k' < 1$) znaleźć takie parametry $r_{1m}, r_{2m}, \ldots, r_{mm}$, dla których

$$\max_{k' \leqslant x \leqslant 1} \left| \prod_{j=1}^{m} \frac{x - r_{jm}}{x + r_{jm}} \right|$$

jest najmniejsze. Z twierdzenia Wachspressa wynika, że parametry r_{jm} są określone jednoznacznie i są parami różne. Niech $R_m(k')$ i $U_m(k')$ oznaczają odpowiednio zbiory uporządkowanych rosnąco parametrów $r_{jm} = r_{jm}(k')$ i $u_{jm} = u_{jm}(k')$. W pracy zajmujemy się własnościami i konstrukcją optymalnej funkcji wymiernej

$$f_m(x;\, R_m) = \prod_{j=1}^{m} \frac{x-r_{jm}}{x+r_{jm}}.$$

Funkcja ta ma $m+1$ punktów ekstremalnych u_{jm} $(k' = u_{0m} < u_{1m} < \ldots < u_{mm} = 1)$, w których przyjmuje, z naprzemiennymi znakami, maksymalną co do modułu wartość. Ponieważ we wzorze Jordana (4.1) na parametry r_{jm} występuje funkcja eliptyczna Jacobiego $\operatorname{dn}(u;\, k)$, w praktyce wyznacza się parametry r_{jm} dla dowolnego m z wzorów przybliżonych (patrz § 8). Jedynie dla $m = 2^p$ znany jest prosty algorytm Wachspressa (WR) wyznaczania parametrów r_{jm}. Przypominamy go w § 2. Ponadto pokazujemy, jak ten algorytm można zastosować do wyznaczania parametrów u_{jm} dla $m = 2^p$. Aby zbadać np. graniczne własności parametrów r_{jm} i u_{jm}, w § 3 zajmujemy się funkcjami eliptycznymi. Dowodzimy m. in. (patrz twierdzenia 3.3, 3.4 i 3.5), że

$$\lim_{k\to 1^-} \operatorname{dn}(\xi K(k);\, k) = 0 \qquad (\xi \in (0,\, 1]),$$

$$\lim_{k\to 1^-} \frac{\ln \operatorname{dn}(\xi K(k);\, k)}{\ln \operatorname{dn}(\eta K(k);\, k)} = \frac{\xi}{\eta} \qquad (\xi \in [0,\, 1),\, \eta \in (0,\, 1]),$$

$$\lim_{k\to 0^+} \frac{\ln \operatorname{dn}(\xi K(k);\, k)}{\ln \operatorname{dn}(\eta K(k);\, k)} = \frac{\sin^2(\xi\pi/2)}{\sin^2(\eta\pi/2)} \qquad (\xi \in [0,\, 1],\ \eta \in (0,\, 1]).$$

Wykazujemy również, że funkcja $\operatorname{dn}(\xi K(k);\, k)/\operatorname{dn}(\eta K(k);\, k)$ jest monotoniczna (patrz twierdzenie 3.2).

W § 4 dajemy wzór na parametry u_{jm} (patrz twierdzenie 4.1), otrzymany przez transformację Landena z wzoru (4.5) na ekstremalne punkty optymalnej funkcji wymiernej z zadania Cauera.

Algorytm WR ma złe własności numeryczne — zawodzi dla $k' \sim 1$ lub dla m dużego. Dlatego jednym z celów pracy jest opisanie innego, numerycznie stabilnego algorytmu (§ 5). Algorytm ten (JR) jest prostym wnioskiem z twierdzenia 5.1, w którym opisana jest zależność między elementami zbiorów $R_m(k')$ i $R_{2m}(k')$ (patrz wzory (5.1) i (5.2)). Analogiczne związki spełniają elementy zbiorów $U_m(k')$ i $U_{2m}(k')$ (patrz wzory (5.10) i (5.11)). Odpowiedni algorytm oznaczamy przez JU. Dla zapewnienia numerycznej stabilności, wzory (5.2) i (5.11) zastępujemy wzorami (5.9) i (5.12). Przedstawione w § 5 dwie procedury w Algolu 60 realizują stabilną wersję algorytmów JR i JU.

W § 6 zajmujemy się własnościami zbiorów $R_m(k')$ i $U_m(k')$, analogicznymi do własności zbiorów zer i punktów ekstremalnych m-tego wielomianu Czebyszewa $T_m(x)$ (patrz twierdzenie 6.1). W twierdzeniu 6.3 dowodzimy monotoniczności parametrów r_{jm} i u_{jm} oraz wyznaczamy granice parametrów r_{jm} i u_{jm} dla $k'\to 0^+$, a także granice ich pochodnych dla $k'\to 1^-$.

Algorytm JR wymaga prawie dwukrotnie więcej działań niż algorytm WR. W § 7 dajemy modyfikację algorytmu JR, która wymaga mniej działań. Numeryczny eksperyment wykazał jednak, że ten zmodyfikowany algorytm wyznacza parametry r_{jm} z nieco większym błędem.

Jak już wspomnieliśmy, parametry r_{jm} dla dowolnego m oblicza się z wzorów przybliżonych. Różnymi wersjami tych wzorów zajmujemy się w § 8. Dajemy również dowód istnienia i własności funkcji $f(y;\ k')$, takiej, że parametry wyrażone wzorami (8.1) należą do zbiorów $R_m(k')$ i $U_m(k')$. Istnienie tej funkcji i jej własności zostały doświadczalnie stwierdzone przez Boora i Rice'a. Ich spostrzeżenia są przez nas potwierdzone (patrz twierdzenie 8.1).

W końcu § 9 dajemy hipotetyczne oszacowanie normy optymalnej funkcji wymiernej (patrz wzór (9.1)). O słuszności tej hipotezy świadczą liczne doświadczenia numeryczne.