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# A NOTE ON THE $k$-TH DISTANCE RANDOM VARIABLES

**1. Introduction.** Let $\boldsymbol{X}_n = (X_{n1}, X_{n2}, \ldots, X_{nm})$ for $n = 0, 1, 2, \ldots$ be a sequence of independent random $m$-dimensional vectors with the common multivariate distribution function $F(\boldsymbol{x}) = F(x_1, x_2, \ldots, x_m)$. We write the sequence of random variables

$$Y_n = \Big(\sum_{i=1}^{m} (X_{0i} - X_{ni})^2\Big)^{1/2} \quad (n = 1, 2, \ldots).$$

Denote by $Z_1^{(n)} \leqslant Z_2^{(n)} \leqslant \ldots \leqslant Z_n^{(n)}$ the order statistics in the sequence $Y_1, Y_2, \ldots, Y_n$. The random variable $Z_k^{(n)}$ for $k = 1, 2, \ldots, n$ will be called the *k-th distance random variable.*

**2. Limiting distributions.** Denote by $F_k^{(n)}(x) = \mathrm{P}(Z_k^{(n)} < x)$, $x \geqslant 0$ $(n = 1, 2, \ldots)$, the distribution function of $Z_k^{(n)}$. The *limiting distribution* of the $k$-th distance random variable is defined by

$$F_k(x) = \lim_{n\to\infty} F_k^{(n)}(u_n) \quad (k = 1, 2, \ldots),$$

where $k$ is a fixed natural number and $u_n = x/n^{1/m}$, $x \geqslant 0$.

Kopociński in [3] and Trybuś in [5] have investigated the limiting properties of the distance random variable

$$Z_1^{(n)} = \min(Y_1, Y_2, \ldots, Y_n).$$

Note that $Y_n$ $(n = 1, 2, \ldots)$ is the sequence of exchangeable random variables, since

$$\mathrm{P}(Y_{i_1} < x_{i_1}, \ldots, Y_{i_r} < x_{i_r}) = \int_{R^m} \prod_{j=1}^{r} H(x_{i_j}; \boldsymbol{u})\, dF(\boldsymbol{u}), \qquad x_{i_j} \geqslant 0,$$

where

$$H(x; \boldsymbol{u}) = \mathrm{P}\left(\Big(\sum_{i=1}^{m} (u_i - X_{ni})^2\Big)^{1/2} < x\right) \quad (n = 1, 2, \ldots).$$

The limiting distributions of the order statistics for exchangeable random variables have been studied among others by Berman [1] and Chanda [2].

THEOREM. *If the random vectors $\boldsymbol{X}_n$ have the bounded probability density $f(\boldsymbol{x}) \leqslant M < \infty$, then*

$$F_k(x) = \frac{1}{(k-1)!} \int_{R^m} \Big( \int_0^{V_m(x) f(\boldsymbol{u})} t^{k-1} e^{-t} dt \Big) f(\boldsymbol{u}) d\boldsymbol{u}, \qquad x \geqslant 0,$$

*where*

$$V_m(x) = \frac{\pi^{m/2} x^m}{\Gamma\big((m/2)+1\big)}$$

*is the volume of an m-dimensional sphere with radius x.*

Proof. It is easy to show that

$$F_k^{(n)}(x) = \sum_{r=k}^{n} \sum_{1 \leqslant i_1 < \ldots < i_r \leqslant n} \mathrm{P}(Y_{i_1} < x, \ldots, Y_{i_r} < x, Y_{i_{r+1}} \geqslant x, \ldots, Y_{i_n} \geqslant x)$$

$$= \int_{R^m} \sum_{r=k}^{n} \binom{n}{r} H^r(x; \boldsymbol{u}) \big(1 - H(x; \boldsymbol{u})\big)^{n-r} dF(\boldsymbol{u}).$$

Thus

$$F_k^{(n)}(x) = \int_{R^m} F_k^{(n)}(x; \boldsymbol{u}) dF(\boldsymbol{u}),$$

where $F_k^{(n)}(x; \boldsymbol{u})$ is the distribution function of the $k$-th order statistics in the sequence of $n$ independent random variables with the common cumulative distribution function $H(x; \boldsymbol{u})$.

Let $\eta$ be an arbitrary positive constant and denote by $E_{\eta,n}$ the set of the points for which the oscillation of $f(\boldsymbol{x})$ in the sphere $K(\boldsymbol{x}, x/n^{1/m})$ is more than $\eta$. For $\boldsymbol{x} \in E_{\eta,n}^c$ we have

$$nH(u_n; \boldsymbol{x}) = n \int_{|\boldsymbol{u}| < x/n^{1/m}} f(\boldsymbol{x}+\boldsymbol{u}) d\boldsymbol{u} = V_m(x)[f(\boldsymbol{x}) + \theta(\boldsymbol{x})\eta],$$

where $|\theta(\boldsymbol{x})| \leqslant 1$.

Using Theorem 4 of [4], p. 119, we obtain

$$\lim_{n\to\infty} F_k^{(n)}(u_n; \boldsymbol{x}) = \frac{1}{(k-1)!} \int_0^{V_m(x)[f(\boldsymbol{x})+\theta(\boldsymbol{x})\eta]} t^{k-1} e^{-t} dt.$$

The rest of the proof is similar to the proof of Theorem 2 of [3].

COROLLARY ([3] and [5]). *The limiting distribution of the distance random variable $Z_1^{(n)}$ is of the form*

$$F_1(x) = 1 - \int_{R^m} \exp\big(-V_m(x) f(\boldsymbol{u})\big) f(\boldsymbol{u}) d\boldsymbol{u}.$$

**References**

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[2] K. C. Chanda, *Asymptotic distribution of sample quantiles for exchangeable random variables*, Cal. Stat. Assoc. Bull. 20 (1971), p. 135-142.
[3] B. Kopociński, *Rozkład zmiennej losowej dystansowej*, Prace Naukowe WSE Wrocław 6 (1967), p. 83-92.
[4] — *Zarys teorii odnowy i niezawodności*, Warszawa 1973.
[5] G. Trybuś, *On the distance random variable*, Zastosow. Matem. 14 (1974), p. 237-244.

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## UWAGA O $k$-TYCH DYSTANSOWYCH ZMIENNYCH LOSOWYCH

STRESZCZENIE

W pracy znaleziono rozkład graniczny $k$-tej dystansowej zmiennej losowej, korzystając z teorii rozkładów granicznych ekstremalnych statystyk pozycyjnych. Rezultat ten jest uogólnieniem wyników Kopocińskiego [3] i Trybusia [5].

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