ZASTOSOWANIA MATEMATYKI
APPLICATIONES MATHEMATICAE
XII, 1 (1971)

A. SZUSTALEWICZ (Wrocław)

ON A VARIANT OF THE MATRIX TRANSFORMATION METHOD FOR APPROXIMATE SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS OF SECOND DEGREE

This paper presents a variant of the matrix transformation method [1], which allows a transformation of a partial differential equation of second degree (4) into n independent ordinary differential equations.

In the presented variant of the matrix transformation method $(n\times n)$-matrices of the form

$$T = \begin{pmatrix} 0 & r & 0 & 0 & \dots & 0 \\ s & 0 & r & 0 & \dots & 0 \\ 0 & s & 0 & r & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \dots & 0 & 0 & s & 0 \end{pmatrix}, \quad r, s > 0, \tag{1}$$

are used.

Eigenvalues of the matrix (1) are of the form

$$\lambda_i = 2\sqrt{rs}\cos\frac{i\pi}{n+1} \quad (i = 1, 2, \dots, n). \tag{2}$$

The eigenvector corresponding to the i-th $(i = 1, 2, \dots, n)$ eigenvalue [1] λ_i is the i-th column of matrix $P_{n\times n} = (p_{ij})$, where

$$p_{ij} = (\sqrt{r})^{n-i}(\sqrt{s})^{i-1}\sin\frac{ij\pi}{n+1}.$$

---

[1] Eigenvalues η_i $(i = 1, 2, \dots, n)$ of the matrix

$$T_1 = \begin{pmatrix} t & r & 0 & 0 & \dots & 0 \\ s & t & r & 0 & \dots & 0 \\ 0 & s & t & r & \dots & 0 \\ & \cdot & \cdot & \cdot & \cdot & \\ 0 & \dots & 0 & 0 & s & t \end{pmatrix}$$

can be expressed by $\eta_i = t+\lambda_i$, where λ_i $(i = 1, 2, \dots, n)$ are defined by (2). These eigenvectors are equal to those of (1).

The inverse matrix of P is $P^{-1} = P^T E_1$, where P^T is the transposed matrix of P, and $E_1 = [d_1, d_2, \ldots, d_n]$ is the diagonal matrix in which $d_i = 2r^{i-n}s^{1-i}/(n+1)$. Thus

$$T = P\Lambda P^T E_1, \tag{3}$$

where $\Lambda = [\lambda_1, \lambda_2, \ldots, \lambda_n]$ is the diagonal matrix of eigenvalues.

The partial differential equation of the function $u = u(x, y)$, which may be reduced to n independent ordinary differential equations, is of the form (²)

$$a_0(x)\frac{\partial^2 u}{\partial x^2} + a_1(x)\left[c_1\frac{\partial^2 u}{\partial y^2} + c_2\frac{\partial u}{\partial y}\right] + a_2(x)\frac{\partial u}{\partial x} + a_3(x)u = f(x, y) \tag{4}$$

with boundary conditions on the lines $y = c$, $y = \mathrm{d}$, and boundary or initial conditions on the lines $x = a$, $x = b$.

Let us divide the sector $\langle c, d\rangle$ into $n+1$ equal parts determined by the points $y_k = y_0 + kh$ $(k = 1, 2, \ldots, n)$, where

$$h = (d-c)/(n+1), \quad y_0 = c, \quad y_{n+1} = d.$$

We approximate the partial derivatives $\partial u/\partial y$ and $\partial^2 u/\partial y^2$ by the difference quotients

$$\left.\frac{\partial u}{\partial y}\right|_{y=y_k} \approx \frac{u_{k+1} - u_{k-1}}{2h} \quad \text{or} \quad \frac{u_{k+1} - u_k}{h},$$

$$\left.\frac{\partial^2 u}{\partial y^2}\right|_{y=y_k} \approx \frac{u_{k+1} - 2u_k + u_{k-1}}{h^2},$$

where $u_k = u_k(x) = u(x, y_k)$.

Now, approximating equation (4) on the straight lines $y = y_k$ $(k = 1, 2, \ldots, n)$ by the equation

$$a_0(x)u_k'' + a_1(x)\left[c_1\frac{u_{k+1} - 2u_k + u_{k-1}}{h^2} + c_2\frac{u_{k+1} - u_{k-1}}{2h}\right] + \tag{5}$$
$$+ a_2(x)u_k' + a_3(x)u_k = f_k(x)$$

we obtain a system of n difference equations. Putting

$$2c_1 + hc_2 = r, \quad 2c_1 - hc_2 = s \tag{6}$$

we write equation (5) as

(7)
$$a_0(x)u_k'' + \frac{a_1(x)}{2h^2}[ru_{k+1} + su_{k-1}] + a_2(x)u_k' + \left[a_3(x) - \frac{2a_1(x)}{h^2}c_1\right]u_k = f_k(x).$$

(²) The equation may have an analogous form with respect to the variable y.

Let us define linear operators for the function $v(x)$ as follows:

$$Av = a_0(x)v'' + a_2(x)v' + \left[a_3(x) - \frac{2c_1 a_1(x)}{h^2}\right] v,$$

$$Bv = \frac{1}{2h^2}\, a_1(x)v.$$

Then equation (7) takes the form

$$Au_k + B[ru_{k+1} + su_{k-1}] = f_k(x). \tag{8}$$

Now, writing equation (8) explicitly we obtain the system

$$\begin{cases} Au_1 + B[ru_2] = f_1(x) - B[su_0], \\ Au_2 + B[su_1 + ru_3] = f_2(x), \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ Au_{n-1} + B[su_{n-2} + ru_n] = f_{n-1}(x), \\ Au_n + B[su_{n-1}] = f_n(x) - B[ru_{n+1}]. \end{cases} \tag{9}$$

The boundary conditions on the straight lines $y = c$ and $y = d$ should be such that it were possible to calculate approximate values of the right--hand sides of the first and last equations.

Let us denote the right-hand side of the i-th equation in (9) by $w_i(x)$ and let us define the vectors

$$\boldsymbol{u}(x) = [u_1(x), u_2(x), \dots, u_n(x)],$$

$$\boldsymbol{w}(x) = [w_1(x), w_2(x), \dots, w_n(x)].$$

Now we are able to write the system (9) in vector form

$$A\boldsymbol{u}(x) + B[T\boldsymbol{u}(x)] = \boldsymbol{w}(x),$$

where T is defined by (1). Applying (3), we have

$$A\boldsymbol{u}(x) + B[P\Lambda P^T E_1 \boldsymbol{u}(x)] = \boldsymbol{w}(x),$$

and hence, multiplying on the left-hand side by $P^T E_1$, we have

$$A[P^T E_1 \boldsymbol{u}(x)] + B[\Lambda P^T E_1 \boldsymbol{u}(x)] = P^T E_1 \boldsymbol{w}(x).$$

Introducing now vectors

$$\boldsymbol{U}(x) = P^T E_1 \boldsymbol{u}(x), \qquad \boldsymbol{W}(x) = P^T E_1 \boldsymbol{w}(x), \tag{10}$$

we obtain the system of n independent ordinary differential equations for the functions $U_i(x)$.

After solving the new system and making use of (10) we determine the vector $\boldsymbol{u}(x)$ from the formula $\boldsymbol{u}(x) = P\boldsymbol{U}(x)$. The number n cannot be chosen arbitrarily. The numbers r and s in the matrix T should be positive to guarantee real eigenvalues. Thus, on the basis of (6), the following conditions are to be staisfied:

$$2c_1+hc_2 > 0, \quad 2c_1-hc_2 > 0.$$

If $c_2 \neq 0$, then assuming that $c_1 > 0$ (if $c_1 < 0$, then we may multiply equation (4) by (-1)) we have $h < 2c_1/|c_2|$.

Since $h = (d-c)/(n+1)$, we have

$$n > \frac{(d-c)\,|c_2|}{2c_1} - 1.$$

If we approximate $\partial u/\partial y$ by the quotient $(u_{k+1}-u_k)/h$, the conditions are as follows:

$$c_1+hc_2 > 0$$
$$c_1 > 0.$$

If $c_2 \geqslant 0$, then n is arbitrary, and if $c_2 < 0$, then $h < c_1/|c_2|$, and hence

$$n > \frac{(d-c)\,|c_2|}{c_1} - 1.$$

Reference

[1] R. Zuber, *Matrix transformation method of approximate solutions of partial differential equations*, Bull. Acad. Polon. Sci. 16 (1968), no. 7, p. 587-592.

DEPT. OF NUMERICAL METHODS
UNIVERSITY OF WROCŁAW

Received on 13. 3. 1970

A. SZUSTALEWICZ (Wrocław)

WARIANT METODY TRANSFORMACJI MACIERZOWEJ DLA PRZYBLIŻONEGO ROZWIĄZYWANIA CZĄSTKOWYCH RÓWNAŃ RÓŻNICZKOWYCH DRUGIEGO RZĘDU

STRESZCZENIE

Przedstawiony został wariant metody transformacji macierzowej, podanej w [1], który pozwala przekształcić cząstkowe równanie różniczkowe postaci (7) do układu n niezależnych zwyczajnych równań różniczkowych. W wariancie tym wykorzystywane są macierze kwadratowe postaci (1) oraz ich wartości i wektory własne.
