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ON THE TOTAL WORKING TIME OF A REPAIRED UNIT

In the paper we define the breakdown process of a repaired unit as a regenerative process valued in [0, 1]. We analyze the probability distribution of the total working time of the unit, the estimation of this distribution, and its limiting properties. In the considerations, auxiliary breakdown processes, valued 0 or 1, are used with similar probabilistic characteristics, and dependence of working and repair times in a renewal cycle is allowed.

1. Introduction. In the probabilistic description of a repaired unit it is often assumed that the unit may be in one of the two states: work or breakdown. Assume for simplicity that the working and repair times of the unit are independent random variables with given probability distributions. The analysis of the reliability function of the unit under these assumptions is well known (see [1], p. 80, and [2], p. 283).

Examples justify the need for generalizations of the above description. It happens that switching between working and breakdown states of a unit is not immediate: before failure the unit is working for some time with reduced intensity; moreover, the length of the cycle between the renewals of a unit may depend upon the total working time in this cycle. We do not however question the assumption that the renewal moment is the start of a cycle which does not depend upon the past.

Assume that the breakdown process of the repaired unit is a regenerative process in Smith's sense and is valued in the interval [0, 1]. From the theory of regenerative processes we have the solution of some problems, e.g., concerning the probability of being in the working state. Here we consider the probability distribution of the total working time of the unit, the estimation of this distribution, and its limiting properties.

2. Example. Consider a unit with both one reserve and one renewal. In this system the moment of switching into work of the reserve unit is the start of both working and repair times. At the moment of breakdown the reserve unit may be unrepaired and the system fails. Assume that the working and repair times of the unit are independent random variables with given probability distributions. It is easy to see that the breakdown process is a regenerative process, valued 0 or 1, with dependent working and repair times.

Let ξ denote the working time and let η denote the repair time in a cycle. Cycles of the regenerative process which describes the unit are in this example defined by the distribution functions of the random variables: the cycle length $Z = \max(\xi, \eta)$, the working time $X = \xi$, and the repair time $Y = \max(0, \eta - \xi)$. The distribution of the total breakdown time in this system was considered by Parthasarathy and Ramanarayanan [3].

3. The probability distribution. Let $a(t)$, $t \geqslant 0$, be a regenerative process satisfying the additional condition $0 \leqslant a(t) \leqslant 1$, $t \geqslant 0$. Recall that in the definition of the regenerative process we introduce the space $\mathscr{C}$ of cycles $C = \big(Z, z(t),\ 0 \leqslant t \leqslant Z\big)$, where Z is a positive number called the *cycle length* and $z(t)$ is a real function defined during the cycle time, and the probabilistic measure P defined on $\mathscr{C}$ for which the probabilities $P(A) = \Pr\big(z(t) \in A,\ Z \geqslant t\big)$ are defined for some sets $A \in \mathscr{A}$. The stochastic process $a(t) = z_{N(t)+1}(t - S_{N(t)})$, $S_{N(t)} \leqslant t < S_{N(t)+1}$, where $N(t)$, $t \geqslant 0$, is the renewal process generated by the sequence $\{Z_n\}$ of independent cycle lengths and where $S_0 = 0$, $S_n = S_{n-1} + Z_n$, $n = 1, 2, \ldots$, is called a *breakdown process*.

Let $X = \int\limits_0^Z z(u)\,du$ be the total working time of the repaired unit and let $Y = Z - X$ be the total breakdown time of the unit during one cycle. Generally, in a regenerative process the random variables X and Y may be dependent. The joint distribution function of these random variables is denoted by

$$F(x, y) = \Pr(X < x,\ Y < y), \qquad x \geqslant 0,\ y \geqslant 0.$$

The stochastic process

$$A(t) = \int\limits_0^t a(u)\,du, \qquad t \geqslant 0,$$

is called a *process of total working time of the unit in the breakdown process* $a(t)$, $t \geqslant 0$.

THEOREM 1. *The probability distribution* $W(t, a) = \Pr\big(A(t) < a\big)$, $a \geqslant 0$, *of the total working time satisfies the renewal equation*

$$W(t, a) = \begin{cases} 1, & a > t, \\ V(t, a) + W^* F(t, a), & a \leqslant t, \end{cases} \tag{1}$$

where

$$V(t, a) = \Pr\left(\int_0^t z(u)\,du < a,\ Z \geqslant t\right)$$

and

$$W^*F(t, a) = \int_{u=0}^{a} \int_{v=0}^{t-u} W(t-v-u, a-u)\, F(du, dv), \qquad t \geqslant 0,\ a \geqslant 0,$$

is the convolution of W *and* F.

COROLLARY 1. *The solution of equation* (1) *is*

$$W(t, a) = \begin{cases} 1 & a > t, \\ \sum_{n=0}^{\infty} V^*F^{*n}(t, a), & a \leqslant t, \end{cases}$$

where $*$ *denotes the convolution defined in Theorem* 1 *and*

$$W^*F^{*(n+1)} = (W^*F^{*n})^*F, \qquad n = 1, 2, \ldots$$

Note that Theorem 1 and Corollary 1 are unsuitable for practical purposes.

4. The estimation. For every regenerative process $a(t)$, $t \geqslant 0$, we define two additional regenerative processes $a^*(t)$ and $a^{**}(t)$, $t \geqslant 0$, valued 0 or 1, with cycles of identical length and identical total working times in a cycle. To do this we define, for every cycle $C = \big(Z, z(t), 0 \leqslant t \leqslant Z\big) \in \mathscr{C}$, the cycles C^* and C^{**} of length Z and with functions

$$z^*(t) = \begin{cases} 1, & 0 \leqslant t \leqslant X, \\ 0, & X < t \leqslant Z, \end{cases} \qquad z^{**}(t) = \begin{cases} 1, & Z-X < t \leqslant Z, \\ 0, & 0 \leqslant t \leqslant Z-X, \end{cases}$$

and then we define the regenerative process as before using the same probabilistic measure P. Analogously as for the breakdown process $a(t)$, $t \geqslant 0$, we define the total working time in the processes $a^*(t)$ and $a^{**}(t)$, $t \geqslant 0$, and denote them by $A^*(t)$ and $A^{**}(t)$, $t \geqslant 0$, respectively. It is easy to see that the inequality $A^{**}(t) \leqslant A(t) \leqslant A^*(t)$, $t \geqslant 0$, holds. Thus we obtain

COROLLARY 2. *The probability distributions of the total working time of the processes* $A(t)$, $A^*(t)$, *and* $A^{**}(t)$, $t \geqslant 0$, *satisfy the inequalities*

$$\Pr\big(A^{**}(t) < a\big) \geqslant \Pr\big(A(t) < a\big) \geqslant \Pr\big(A^*(t) < a\big), \qquad t \geqslant 0,\ a \geqslant 0.$$

Now we find the left- and right-most terms in the inequality in Corollary 2. This is an extension of the results for the case of independent working and repair times in a cycle (see [5], and [2], p. 296).

For the n-th cycle, denote by X_n, Y_n, and $Z_n = X_n+Y_n$ the working time, repair time, and cycle length, respectively. For $n = 1, 2, \ldots$ these triples of random variables are independent and the random variables (X_n, Y_n) have the same probability distribution function F. Let $T_0 = 0$, $U_0 = 0$, $T_n = T_{n-1}+X_n$, $U_n = U_{n-1}+Y_n$, $n = 1, 2, \ldots$

THEOREM 2. *For $a < t$ we have*

$$\Pr\big(A^*(t) < a\big) = \sum_{n=1}^{\infty}[\Pr(T_n < a,\ U_{n-1} < t-a) - \Pr(T_n < a,\ U_n < t-a)].$$

Proof. It is easy to see that if $a < t$, then

$$P_0 = \Pr\big(A^*(t) < a,\ a^*(t) = 0\big) = \sum_{n=1}^{\infty}\int_0^a \Pr(T_n = du,\ U_{n-1} < t-u,$$

$$U_n \geqslant t-u) = \sum_{n=1}^{\infty}\int_0^a [\Pr(T_n = du,\ U_{n-1} < t-u) - \Pr(T_n = du,\ U_n < t-u)].$$

Since

$$\int_0^a \Pr(T_n = du,\ U_n < t-u) = \Pr(T_n < a,\ U_n < t-a) +$$

$$+\int_{t-a}^{t} \Pr(T_n < t-v,\ U_n = dv),$$

we have

$$P_0 = \sum_{n=1}^{\infty}[\Pr(T_n < a,\ U_{n-1} < t-a) - \Pr(T_n < a,\ U_n < t-a)] +$$

$$+\sum_{n=1}^{\infty}\int_{t-a}^{t}[\Pr(T_n < t-v,\ U_{n-1} = dv) - \Pr(T_n < t-v,\ U_n = dv)].$$

Similarly,

$$P_1 = \Pr\big(A^*(t) < a,\ a^*(t) = 1\big) = \sum_{n=1}^{\infty}\int_0^a\int_{t-a}^{t-u}\Pr(X_{n+1} \geqslant t-u-v)F_n(du, dv)$$

$$= \sum_{n=1}^{\infty}\int_{t-a}^{t}\int_0^{t-v}\big(1-\Pr(X_{n+1} < t-u-v)\big)\Pr(T_n = du \mid U_n = v)\Pr(U_n = dv)$$

$$= \sum_{n=1}^{\infty}\int_{t-a}^{t}[\Pr(T_n < t-v,\ U_n = dv) - \Pr(T_{n+1} < t-v,\ U_n = dv)].$$

Since $\Pr\big(A^*(t) < a\big) = P_0+P_1$, by substitution and reduction we obtain Theorem 2.

Let $N_1(t)$, $t \geqslant 0$, denote the renewal process generated by the sequence of random variables $\{X_n\}$ and let $N_2(t)$, $t \geqslant 0$, denote the renewal process generated by the sequence of random variables $\{Y_n\}$. The renewal processes $N_1(t)$ and $N_2(t)$, $t \geqslant 0$, are dependent iff F is a distribution function of dependent random variables. It follows from Theorem 2, similarly as in the case of independence, that the mentioned renewal processes may be used to express the probability distribution of the process $A(t)$, $t \geqslant 0$. The general expression is the same as in the independence case.

COROLLARY 3. *The distribution function of the process of total working time in the breakdown process $a^*(t)$, $t \geqslant 0$, is of the following form:*

$$\Pr\big(A^*(t) < a\big) = \Pr\big(N_2(t-a) < N_1(a)\big), \quad a < t.$$

Hence we get

COROLLARY 4. *The probability distribution function of the process of total working time in the breakdown process $a^{**}(t)$, $t \geqslant 0$, is of the form*

$$\Pr\big(A^{**}(t) > a\big) = \Pr\big(N_1(a) < N_2(t-a)\big), \quad a < t.$$

Note that the total unit breakdown time

$$B^{**}(t) = \int_0^t \big(1 - a^{**}(u)\big)\,du$$

has, according to Corollary 3, the probability distribution function

$$\Pr\big(B^{**}(t) < b\big) = \Pr\big(N_1(t-b) < N_2(b)\big), \quad 0 \leqslant b < t.$$

Hence $B^{**}(t) = t - A^{**}(t)$, $t \geqslant 0$, and, consequently, for $b = t-a$ we get Corollary 4.

5. Limiting properties of working time.

THEOREM 3. *If (X_n, Y_n), $n = 1, 2, \ldots$, are independent random vectors identically distributed with parameters*

$$\mathrm{E}X_n = \mu_1, \quad \mathrm{E}Y_n = \mu_2, \quad \mathrm{D}^2X_n = \sigma_1^2, \quad \mathrm{D}^2Y_n = \sigma_2^2,$$
$$\mathrm{Cov}(X_n, Y_n) = \varrho\sigma_1\sigma_2,$$

then the process $A(t)$ is, as $t \to \infty$, asymptotically normally distributed with parameters

$$\mathrm{E}A(t) = \mu_1 t/\mu + o(t), \quad \mathrm{D}^2A(t) = \sigma^2 t + o(t),$$

where

$$\mu = \mu_1 + \mu_2, \quad \sigma^2 = (\mu_1^2\sigma_2^2 - 2\mu_1\mu_2\sigma_1\sigma_2\varrho + \mu_2^2\sigma_1^2)/\mu^3,$$

and $o(t)$ is an expression which divided by t tends to 0 as $t \to \infty$.

Theorem 3 is a corollary to the general theory of regenerative processes (see [4]). Note that this follows from the Corollaries 2-4, from the central limit theorem for pairs of random variables and the limit theorem for the pair of renewal processes $(N_1(t), N_2(t))$, $t \geqslant 0$. The proof is analogous as in the independent case.

Remark. The processes $A^*(t)$ and $A^{**}(t)$, $t \geqslant 0$, have identical asymptotical distributions as in Theorem 3.

THEOREM 4. *Under the assumptions of Theorem* 3, *the random vector* (T_n, U_m) *is, for* $n\to\infty$, $m\to\infty$, $m/n\to\nu^2$, *asymptotically normally distributed with mean vector* $(n\mu_1, m\mu_2)$, *variance vector* $(n\sigma_1^2, m\sigma_2^2)$, *and correlation* $\varrho^* = \varrho\min(\nu, 1/\nu)$.

THEOREM 5. *Under the assumptions of Theorem* 3, *the random vector* $(N_1(t_1), N_2(t_2))$ *is, for* $t_1\to\infty$, $t_2\to\infty$, $t_2\mu_1/t_1\mu_2\to\nu^2$, *asymptotically normally distributed with mean vector* $(t_1/\mu_1, t_2/\mu_2)$, *variance vector* $(t_1\sigma_1^2/\mu_1^3, t_2\sigma_2^2/\mu_2^3)$, *and correlation* ϱ^*.

6. Asymptotical expansion of the mean. Theorem 3 gives the first term of the asymptotical expansion of the mean of the process of total working time in the breakdown process. Now we consider the next term of this expansion.

THEOREM 6. *Under the assumptions of Theorem* 3 *and if the distribution function of the cycle length is non-lattice, then*

$$\mathrm{E}A^*(t) = \frac{1}{\mu}\big(H(t)+1\big) - \frac{1}{2\mu}(\sigma_1^2+\mu_1^2)+o(1),$$

$$\mathrm{E}A^{**}(t) = \mathrm{E}A^*(t) - \frac{1}{\mu}(\sigma_1\sigma_2\varrho+\mu_1\mu_2)+o(1),$$

where

$$H(t)+1 = \frac{t}{\mu}+\frac{1}{2\mu}(\sigma^2+\mu^2)+o(1),$$

and $o(1)$ *is an expression which tends to* 0 *as* $t\to\infty$.

Proof. Let

$$R_1(t) = A^*(S_{N(t)+1}) - A^*(t), \tag{2}$$

$$R_2(t) = A^{**}(S_{N(t)+1}) - A^{**}(t), \quad t \geqslant 0, \tag{3}$$

be residual working times in the breakdown processes $a^*(t)$ and $a^{**}(t)$, $t\geqslant 0$, in the cycle which is actual at the moment t, where $N(t)$, $t\geqslant 0$, is the renewal process of finished cycles. Since

$$A^*(S_n) = A^{**}(S_n), \quad n = 1, 2, \ldots,$$

using the renewal function $H(t) = \mathrm{E}N(t)$, $t \geqslant 0$, we get

$$\Pr\big(R_i(t) \geqslant v\big) = g_i(t, v) + \int_0^t g_i(t-u, v)\, dH(u), \quad i = 1, 2,$$

where

$$g_1(t, v) = \Pr\big(\max(0, X-t) \geqslant v, Z \geqslant t\big),$$
$$g_2(t, v) = \Pr\big(\min(X, t) \geqslant v, Z \geqslant t\big), \quad t \geqslant 0,\ v \geqslant 0.$$

Hence we have

$$\mathrm{E}R_i(t) = g_i(t) + \int_0^t g_i(t-u)\, dH(u), \quad i = 1, 2,$$

where

$$(4) \qquad g_i(t) = \int_0^\infty g_i(t, v)\, dv, \quad i = 1, 2.$$

Note that (4) are nonnegative, nondecreasing, and integrable functions. By the fundamental renewal theorem, after suitable calculations we get

$$\mathrm{E}R_1(t) = \frac{1}{2\mu}\mathrm{E}X^2 + o(1), \quad \mathrm{E}R_2(t) = \frac{1}{2\mu}\mathrm{E}X^2 + \frac{1}{\mu}\mathrm{E}XY + o(1), \quad t \to \infty.$$

From (2) and (3) and from Wald's equality $\mathrm{E}A(S_{N(t)+1}) = \big(H(t)+1\big)\mu$, using the asymptotical expansion of the renewal function, we get the thesis of Theorem 6.

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