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ON THE DISTRIBUTION OF A NONNEGATIVE DIFFERENCE BETWEEN TWO χ^2-DISTRIBUTED SECOND DEGREE POLYNOMIAL STATISTICS

Motivation of this note is similar to that of the papers by Banerjee and Nagase [3] and Nagase and Banerjee [5], in which some basic results concerning quadratic expressions in normal variables have been commented and alternatively proved. Here an alternative proof is provided for the theorem stating that a nonnegative definite difference between two χ^2-distributed second degree polynomial statistics is also χ^2-distributed. As a preliminary result, a necessary and sufficient condition for the nonnegative definiteness of a second degree polynomial is established.

1. Introduction. Let $\mathcal{M}_{m,n}$ denote the linear space of $(m \times n)$-matrices over the real field. We write $\boldsymbol{A} \in \mathcal{M}_m^s$ if $\boldsymbol{A} \in \mathcal{M}_{m,m}$ and $\boldsymbol{A}$ is symmetric, and $\boldsymbol{A} \in \mathcal{M}_m^{\geqslant}$ if $\boldsymbol{A} \in \mathcal{M}_m^s$ and $\boldsymbol{A}$ is nonnegative definite. Moreover, $\boldsymbol{I}$ stands for an identity matrix and, given $\boldsymbol{A} \in \mathcal{M}_{m,n}$, the symbols $\mathscr{C}(\boldsymbol{A})$ and $\operatorname{tr}(\boldsymbol{A})$ denote the column space and trace of $\boldsymbol{A}$, while $\boldsymbol{A}'$ and $\boldsymbol{A}^-$ stand for the transpose and a generalized inverse of $\boldsymbol{A}$, respectively, $\boldsymbol{A}^-$ being understood as any solution to the equation $\boldsymbol{A}\boldsymbol{A}^-\boldsymbol{A} = \boldsymbol{A}$.

Rao [6], p. 187, gave the result which is quoted here as

LEMMA 1. *Let* $\boldsymbol{y} \sim N_p(\boldsymbol{0}, \boldsymbol{I})$ *and let* $Q_0 = Q_1 - Q_2$, *where, for* $i = 1, 2$, $Q_i = \boldsymbol{y}'\boldsymbol{A}_i\boldsymbol{y}$ *with* $\boldsymbol{A}_i \in \mathcal{M}_p^s$. *If* $Q_1 \sim \chi^2(k_1)$, $Q_2 \sim \chi^2(k_2)$, *and* Q_0 *is nonnegative definite, then* $Q_0 \sim \chi^2(k_1 - k_2)$.

Commenting this result Rao writes that its proof would be immediate if the idempotency of $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$, along with the nonnegative definiteness of $\boldsymbol{A}_1$, $\boldsymbol{A}_2$ and $\boldsymbol{A}_1 - \boldsymbol{A}_2$, were known to entail the idempotency of $\boldsymbol{A}_1 - \boldsymbol{A}_2$. Now, it appears that this can easily be established when using an observation of Milliken and Akdeniz [4], which is restated here as

LEMMA 2. *If* $\boldsymbol{A}_1, \boldsymbol{A}_2 \in \mathcal{M}_p^{\geqslant}$ *are such that* $\boldsymbol{A}_1 - \boldsymbol{A}_2 \in \mathcal{M}_p^{\geqslant}$, *then* $\mathscr{C}(\boldsymbol{A}_2) \subset \mathscr{C}(\boldsymbol{A}_1)$.

In fact, by Lemma 2, there exists $\boldsymbol{L} \in \mathcal{M}_{p,p}$ such that $\boldsymbol{A}_2 = \boldsymbol{A}_1\boldsymbol{L} = \boldsymbol{L}'\boldsymbol{A}_1$. The equality $\boldsymbol{A}_1^2 = \boldsymbol{A}_1$ then implies that

$$\boldsymbol{A}_1\boldsymbol{A}_2 = \boldsymbol{A}_1^2\boldsymbol{L} = \boldsymbol{A}_1\boldsymbol{L} = \boldsymbol{A}_2$$

and, similarly, that $A_2A_1 = A_2$, thus leading to the required equality

$$(A_1-A_2)^2 = A_1-A_2.$$

The purpose of the present note is to show that such a direct argumentation can also be applied to prove an extension of Lemma 1, as given by Rao and Mitra [7], p. 177, in which general second degree polynomials in y appear in place of quadratic forms and, in addition, y is admitted to follow a noncentral singular normal distribution. The proof proposed here uses a criterion for the nonnegative definiteness of a second degree polynomial, which is established as a preliminary result.

2. Nonnegative definiteness of a second degree polynomial. In view of the natural interest in the problem of nonnegative definiteness of a second degree polynomial, it is likely that a solution to it is somewhere available in the literature, but no relevant reference is known to the authors.

LEMMA 3. *A polynomial*

$$T = y'Ay+2b'y+c, \tag{1}$$

with $A \in \mathscr{M}_p^s$ *and* $b \in \mathscr{M}_{p,1}$, *is nonnegative definite if and only if*

$$A \in \mathscr{M}_p^{\geqslant}, \tag{2}$$

$$b \in \mathscr{C}(A), \tag{3}$$

and

$$c-b'A^-b \geqslant 0. \tag{4}$$

Proof. It is obvious that T can equivalently be written as

$$(y' \quad 1)\begin{pmatrix} A & b \\ b' & c \end{pmatrix}\begin{pmatrix} y \\ 1 \end{pmatrix}.$$

By Albert's [1] Theorem 1, this establishes the sufficiency of conditions (2)-(4). To prove their necessity observe that the nonnegative definiteness of T implies that, for every $y \in \mathscr{M}_{p,1}$ and every scalar a,

$$(y'Ay)a^2+(2b'y)a+c \geqslant 0,$$

which gives (2) and

$$cA-bb' \in \mathscr{M}_p^{\geqslant}. \tag{5}$$

But from (5) it follows that $cA \in \mathscr{M}_p^{\geqslant}$ and $b \in \mathscr{C}(cA)$. Therefore, if $c = 0$, then $b = 0$, and conditions (3) and (4) are trivially fulfilled. On the other hand, if $c \neq 0$, then, in view of (2), the relation $cA \in \mathscr{M}_p^{\geqslant}$ implies $c > 0$. Consequently, the condition $b \in \mathscr{C}(cA)$ reduces to (3), and then (4) follows from Proposition 2 in [2] and the fact that under (3) the product $b'A^-b$ is invariant with respect to the choice of a generalized inverse of A.

3. An alternative proof of Rao and Mitra's result. To make the note self-contained we begin with the quotation of Rayner and Livingstone's [8] result on the distribution of a second degree polynomial statistic.

LEMMA 4. *Let* $\boldsymbol{y} \sim N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, *where* $\boldsymbol{\mu} \in \mathscr{M}_{p,1}$ *and* $\boldsymbol{\Sigma} \in \mathscr{M}_p^{\geqslant}$. *Then the polynomial* T *defined in* (1) *is* χ^2*-distributed if and only if*

(6)
$$\boldsymbol{\Sigma A \Sigma A \Sigma} = \boldsymbol{\Sigma A \Sigma},$$

(7)
$$\boldsymbol{\Sigma A \Sigma}(\boldsymbol{A\mu}+\boldsymbol{b}) = \boldsymbol{\Sigma}(\boldsymbol{A\mu}+\boldsymbol{b}),$$

and

(8)
$$(\boldsymbol{A\mu}+\boldsymbol{b})'\boldsymbol{\Sigma}(\boldsymbol{A\mu}+\boldsymbol{b}) = \boldsymbol{\mu}'\boldsymbol{A\mu}+2\boldsymbol{b}'\boldsymbol{\mu}+c,$$

in which case the number of degrees of freedom is $k = \operatorname{tr}(\boldsymbol{A\Sigma})$ *and the noncentrality parameter is* $\delta = \boldsymbol{\mu}'\boldsymbol{A\mu}+2\boldsymbol{b}'\boldsymbol{\mu}+c$.

It was noted by Rao and Mitra [7], p. 171, that equality (7) can be replaced by the relation

(9)
$$\boldsymbol{\Sigma}(\boldsymbol{A\mu}+\boldsymbol{b}) \in \mathscr{C}(\boldsymbol{\Sigma A \Sigma}).$$

On the other hand, however, it should be observed that (9) is trivially fulfilled whenever the polynomial (1) is known in advance to be nonnegative definite, for then Lemma 3 implies that $\boldsymbol{b} \in \mathscr{C}(\boldsymbol{A})$ and $\boldsymbol{A} \in \mathscr{M}_p^{\geqslant}$, which results in $\boldsymbol{\Sigma}(\boldsymbol{A\mu}+\boldsymbol{b}) \in \mathscr{C}(\boldsymbol{\Sigma A})$ and $\mathscr{C}(\boldsymbol{\Sigma A}) = \mathscr{C}(\boldsymbol{\Sigma A \Sigma})$, respectively.

Now we are in a position to provide an alternative proof of the result given by Rao and Mitra [7], p. 177, as an extension of Lemma 1.

THEOREM. *Let* $\boldsymbol{y} \sim N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, *where* $\boldsymbol{\mu} \in \mathscr{M}_{p,1}$ *and* $\boldsymbol{\Sigma} \in \mathscr{M}_p^{\geqslant}$, *and let* $T_0 = T_1 - T_2$, *where, for* $i = 1, 2$, $T_i = \boldsymbol{y}'\boldsymbol{A}_i\boldsymbol{y}+2\boldsymbol{b}_i'\boldsymbol{y}+c_i$ *with* $\boldsymbol{A}_i \in \mathscr{M}_p^s$ *and* $\boldsymbol{b}_i \in \mathscr{M}_{p,1}$. *If* $T_1 \sim \chi^2(k_1, \delta_1)$, $T_2 \sim \chi^2(k_2, \delta_2)$, *and* T_0 *is nonnegative definite, then* $T_0 \sim \chi^2(k_0, \delta_0)$ *with* $k_0 = k_1 - k_2$ *and* $\delta_0 = \delta_1 - \delta_2$.

Proof. In view of Lemma 4, the proof consists in showing that conditions (6)-(8) are satisfied for the polynomial

$$T_0 = \boldsymbol{y}'(\boldsymbol{A}_1-\boldsymbol{A}_2)\boldsymbol{y}+2(\boldsymbol{b}_1-\boldsymbol{b}_2)'\boldsymbol{y}+(c_1-c_2)$$

if they are satisfied for each of the polynomials T_1 and T_2 and if, in addition, T_0 fulfills the conditions specified in Lemma 3. First observe that from condition (2) written for $\boldsymbol{A} = \boldsymbol{A}_1-\boldsymbol{A}_2$ it follows immediately that

$$\boldsymbol{\Sigma A}_1\boldsymbol{\Sigma} - \boldsymbol{\Sigma A}_2\boldsymbol{\Sigma} \in \mathscr{M}_p^{\geqslant}.$$

Hence, by Lemma 2, $\mathscr{C}(\boldsymbol{\Sigma A}_2\boldsymbol{\Sigma}) \subset \mathscr{C}(\boldsymbol{\Sigma A}_1\boldsymbol{\Sigma})$ or, equivalently, $\boldsymbol{\Sigma A}_2\boldsymbol{\Sigma} = \boldsymbol{\Sigma A}_1\boldsymbol{\Sigma L}$ for some $\boldsymbol{L} \in \mathscr{M}_{p,p}$. Using now (6) with $\boldsymbol{A} = \boldsymbol{A}_1$ we obtain the equality

(10)
$$\boldsymbol{\Sigma A}_1\boldsymbol{\Sigma A}_2\boldsymbol{\Sigma} = \boldsymbol{\Sigma A}_2\boldsymbol{\Sigma},$$

from which it follows that if (6) is satisfied for $\boldsymbol{A} = \boldsymbol{A}_1$ and $\boldsymbol{A} = \boldsymbol{A}_2$, then it is also satisfied for $\boldsymbol{A} = \boldsymbol{A}_1-\boldsymbol{A}_2$. Further, note that, in view of the remark following Lemma 4, the polynomial T_0 obviously fulfills (7) due to the assumption of its nonnegative definiteness. Finally, observe that applying (7) to T_0 and T_2 we obtain

$$[(\boldsymbol{A}_1-\boldsymbol{A}_2)\boldsymbol{\mu}+(\boldsymbol{b}_1-\boldsymbol{b}_2)]'\boldsymbol{\Sigma}(\boldsymbol{A}_2\boldsymbol{\mu}+\boldsymbol{b}_2)$$
$$= [(\boldsymbol{A}_1-\boldsymbol{A}_2)\boldsymbol{\mu}+(\boldsymbol{b}_1-\boldsymbol{b}_2)]'\boldsymbol{\Sigma}(\boldsymbol{A}_1-\boldsymbol{A}_2)\boldsymbol{\Sigma}\boldsymbol{A}_2\boldsymbol{\Sigma}(\boldsymbol{A}_2\boldsymbol{\mu}+\boldsymbol{b}_2),$$

which is zero by (10) and by (6) written for $\boldsymbol{A} = \boldsymbol{A}_2$. Consequently,

$$(\boldsymbol{A}_1\boldsymbol{\mu}+\boldsymbol{b}_1)'\boldsymbol{\Sigma}(\boldsymbol{A}_2\boldsymbol{\mu}+\boldsymbol{b}_2) = (\boldsymbol{A}_2\boldsymbol{\mu}+\boldsymbol{b}_2)'\boldsymbol{\Sigma}(\boldsymbol{A}_2\boldsymbol{\mu}+\boldsymbol{b}_2),$$

thus implying that (8) holds for T_0. Since the relations $k_0 = k_1-k_2$ and $\delta_0 = \delta_1-\delta_2$ follow directly, the proof is complete.

Added in proof. The considerations of the present note appeared stimulating for further research. Some months later, the authors [9] established a generalization of Rao and Mitra's [7] result, quoted above as the theorem. In the notation of this note, the generalization states that if $T_1 \sim \chi^2(k_1, \delta_1)$ and $T_2 \sim \chi^2(k_2, \delta_2)$, then a necessary and sufficient condition for T_0 to be distributed as a χ^2-variable is that it is nonnegative definite with probability 1 or, alternatively, that

$$\mathscr{C}(\boldsymbol{\Sigma}\boldsymbol{A}_2\boldsymbol{\Sigma}) \subset \mathscr{C}(\boldsymbol{\Sigma}\boldsymbol{A}_1\boldsymbol{\Sigma})$$

and

$$\boldsymbol{\Sigma}\boldsymbol{A}_2\boldsymbol{\Sigma}(\boldsymbol{A}_1\boldsymbol{\mu}+\boldsymbol{b}_1) = \boldsymbol{\Sigma}(\boldsymbol{A}_2\boldsymbol{\mu}+\boldsymbol{b}_2).$$

References

[1] A. Albert, *Conditions for positive and nonnegative definiteness in terms of pseudoinverses*, SIAM J. Appl. Math. 17 (1969), p. 434-440.

[2] J. K. Baksalary and R. Kala, *Two properties of a nonnegative definite matrix*, Bull. Acad. Polon. Sci., Sér. Sci. Math., 28 (1980), p. 233-235.

[3] K. S. Banerjee and G. Nagase, *A note on the generalization of Cochran's theorem*, Comm. Statist. A — Theory Methods 5 (1976), p. 837-842.

[4] G. A. Milliken and F. Akdeniz, *A theorem on the difference of the generalized inverses of two nonnegative matrices*, ibidem 6 (1977), p. 73-79.

[5] G. Nagase and K. S. Banerjee, *On the distribution of a second degree polynomial statistic in singular normal variates*, ibidem 5 (1976), p. 407-412.

[6] C. R. Rao, *Linear statistical inference and its applications* (2nd ed.), J. Wiley, New York 1973.

[7] — and S. K. Mitra, *Generalized inverse of matrices and its applications*, J. Wiley, New York 1971.

[8] A. A. Rayner and D. Livingstone, *On the distribution of quadratic forms in singular normal variates*, S. Afr. J. Agric. Sci. 8 (1965), p. 357-369.

[9] J. K. Baksalary and R. Kala, *On the difference between two second degree polynomials, each following a chi-square distribution*, Sankhyā, Ser. A, 42 (1980), p. 123-127.

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