ZASTOSOWANIA MATEMATYKI
APPLICATIONES MATHEMATICAE
XIX, 3–4 (1987), pp. 519–529

ZOFIA MAJCHER (Opole) and J. PŁONKA (Wrocław)

ON SOME PARTITIONS OF A GRAPH. I

Abstract. Let G be a connected graph not necessarily finite and let $w \in V(G)$. An induced subgraph Q_w of G is called a *quasi-component* of G with the control point w if the following two conditions are satisfied:

1° $w \in V(Q_w)$;

2° $\forall w_1 \in V(Q_w)\ \forall x \in V(G) \setminus V(Q_w)\ (\{w_1, x\} \in E(G) \Rightarrow w_1 = w)$.

In this paper we find several properties of quasi-components.

A partition P of $V(G)$ is a *q-partition* of the graph G if every class of P induces a quasi-component of G. We prove that the set $\mathscr{P}_q(G)$ of all q-partitions of G is a complete lattice with respect to the relation $\leqslant$ defined similarly as for all partitions. Moreover, the supremum in $\mathscr{P}_q(G)$ coincides with that in the set $\mathscr{P}(G)$ of all partitions of G but the infimum may differ from that in $\mathscr{P}(G)$. We give an explicit form of the infimum.

1. Quasi-components of a graph. We consider only simple graphs not necessarily finite. If G is a graph, we denote by $V(G)$ the vertex set of G, and by $E(G)$ the set of all edges of G. If $S \subset V(G)$, we denote by $\langle S \rangle$ the subgraph induced by S in G.

Let G be a connected graph. An induced subgraph Q of G will be called a *quasi-component* of G if there exists $w \in V(Q)$ satisfying the following condition:

$$\forall w_1 \in V(Q)\ \forall x \in V(G) \setminus V(Q)\ (\{w_1, x\} \in E(G) \Rightarrow w_1 = w).$$

The vertex w will be called a *control point* of Q. We have:

(i) Every connected graph G is a quasi-component of G with an arbitrary vertex as its control point.

(ii) If $v \in V(G)$, then the subgraph $\langle \{v\} \rangle$ is a quasi-component of G.

(iii) Let Q be a quasi-component of G and $Q \neq G$. Then there is exactly one control point of Q.

If $Q \neq G$, then Q will be called a *proper quasi-component* of G, and Q will be called *improper* otherwise. If Q is a proper quasi-component of G and w is its control point, then we write $w = c(Q)$.

The term "control point" can be clarified as follows. Suppose we have some area Q and we want to control it. Then the best situation will be if there is a fixed point w such that every path out of Q goes through w and every path into this area goes through w; e.g., we have a counter showing how much electric energy comes into a factory or into a flat.

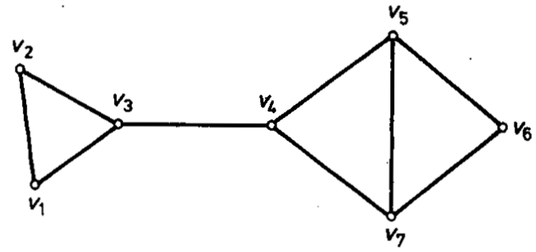

Fig. 1

EXAMPLE 1. In the graph in Fig. 1 we have the following quasi-components:

$\langle\{v_i\}\rangle$ for $i \in \{1, 2, \ldots, 7\}$,

$\langle\{v_1, v_2, v_3\}\rangle$, $\langle\{v_1, v_2, v_3, v_4\}\rangle$, $\langle\{v_4, v_5, v_6, v_7\}\rangle$,

$\langle\{v_3, v_4, \ldots, v_7\}\rangle$, $\langle\{v_1, v_2, \ldots, v_7\}\rangle$.

Quasi-components have some further properties. Namely, we get

LEMMA 1. *Let Q be a proper quasi-component of G, $w = c(Q)$ and let $u \in V(Q)$, $v \in V(G)\setminus V(Q)$. If $u = v_0, v_1, \ldots, v_{n-1}, v_n = v$ is a simple path from u to v, then there exists $i_0 \in \{0, 1, \ldots, n-1\}$ such that $v_{i_0} = w$, $v_j \in V(Q)$ for $j = 0, 1, \ldots, i_0$ and $v_j \in V(G)\setminus V(Q)$ for $j = i_0+1, \ldots, n$.*

Proof. The lemma follows from the assumption that w is the control point of Q and from the fact that the path $u, v_1, \ldots, v$ is simple, so it goes through w only once.

LEMMA 2. *Every quasi-component Q of a connected graph G is a connected subgraph of G.*

Proof. If $Q = G$ or $|V(Q)| = 1$, then the lemma holds trivially. Let Q be a proper non one-element quasi-component of G, $w = c(Q)$, and let $u_1, u_2 \in V(Q)$, $u_1 \neq u_2$. Hence there exists $v \in V(G)\setminus V(Q)$. Since G is connected, by Lemma 1 there exist two simple paths

$$u_1 = v_0, \ldots, v_{i_0-1}, w, v_{i_0+1}, \ldots, v_n = v,$$

$$u_2 = v'_0, \ldots, v'_{j_0-1}, w, v'_{j_0+1}, \ldots, v'_m = v$$

such that

$$v_i, v'_j \in V(Q) \quad \text{for } i = 0, 1, \ldots, i_0;\ j = 0, 1, \ldots, j_0,$$

$$v_i, v'_j \in V(G)\setminus V(Q) \quad \text{for } i = i_0+1, \ldots, n;\ j = j_0+1, \ldots, m.$$

Thus $u_1 = v_0, \ldots, v_{i_0-1}, , v'_{j_0-1}, \ldots, v'_0 = u_2$ is a path connecting u_1 and u_2 whose all vertices belong to $V(Q)$.

Let us recall that a vertex v of a connected graph G is an *articulation point* if the subgraph $\langle V(G)\setminus\{v\}\rangle$ is not connected. A subgraph B of a graph G is called a *block* if B is a maximal connected subgraph of G and contains no articulation point of B. If $|V(B)| > 2$, then B is a block iff every two vertices of B belong to a simple cycle of B (see [3]).

LEMMA 3. *If Q is a proper quasi-component of G, $w = c(Q)$, B is a block of G and $V(B)\cap V(Q) \neq \emptyset$, then either $V(B)\cap V(Q) = \{w\}$ or $V(B)\cap V(Q) = V(B)$.*

Proof. Since Q is a proper quasi-component, $|V(G)| > 1$ and $|V(B)| > 1$. If $V(B)\cap V(Q) = \{u\}$, then there exists $v\in \Gamma(u)\cap V(B)$. Thus $u = w$.

Let $|V(B)\cap V(Q)| > 1$. Then there exist $u_1, u_2 \in V(B)\cap V(Q)$ such that $u_1 \neq u_2$. If $|V(B)| = 2$, then the proof is completed. Let $|V(B)| \geqslant 3$. Assume that $v\in V(B)\setminus V(Q)$. Then $u_1 \neq w$ or $u_2 \neq w$. Let $u_1 \neq w$; the proof in the other case is analogous. Since $u_1, v\in V(B)$, there exists a simple cycle C in B such that $u_1, v\in V(C)$. The cycle C must be of the form

$$u_1 = v_1, v_2, \ldots, v_k, v, v_{k+1}, \ldots, v_n, u_1.$$

Since C is simple, each of the paths $u_1, v_2, \ldots, v_k, v$ and $u_1, v_n, \ldots, v_{k+1}, v$ is simple. By Lemma 1 and by the assumption that $u_1 \neq w$, the control point w must occur among the vertices $v_1, \ldots, v_k$ and among the vertices $v_{k+1}, \ldots, v_n$. Hence C is not simple, a contradiction. Thus the condition $v\in V(B)\setminus V(Q)$ leads to a contradiction and, consequently, $v\in V(B)\cap V(Q)$.

COROLLARY 1. *Every quasi-component of a graph G is either a one-element subgraph of G or a connected union of blocks of G.*

LEMMA 4. *Let Q and Q' be proper quasi-components of a connected graph G, where*

$$V(Q)\setminus V(Q') \neq \emptyset \neq V(Q')\setminus V(Q) \quad \textit{and} \quad V(Q)\cap V(Q') \neq \emptyset.$$

Then $c(Q)$, $c(Q')\in V(Q)\cap V(Q')$.

Proof. Let $u_1\in V(Q)\setminus V(Q')$, $u_2\in V(Q')\setminus V(Q)$, and $z\in V(Q)\cap V(Q')$. Since Q is connected, there exists a simple path L from z to u_1 whose vertices belong to $V(Q)$. By Lemma 1, the control point $c(Q')$ belongs to $V(L)$. But $V(L)\subset V(Q)$, so

$$c(Q')\in V(Q)\cap V(Q').$$

Analogously, $c(Q)\in V(Q')\cap V(Q)$.

THEOREM 1. *If Q and Q' are quasi-components of a connected graph G, then $Q\cup Q'$ is a quasi-component of G iff*

$$V(Q)\cap V(Q') \neq \emptyset.$$

Moreover, if $Q\cup Q'$ is a proper quasi-component of G, then

1° $c(Q\cup Q') = c(Q')$ *when* $Q\subset Q'$,

2° $c(Q\cup Q') = c(Q) = c(Q')$ *when* $V(Q)\setminus V(Q') \neq \emptyset \neq V(Q')\setminus V(Q)$.

Proof. $\Rightarrow$. If $V(Q) \cap V(Q') = \emptyset$, then $Q \cup Q'$ is not a quasi-component since it is not connected (see Lemma 2).

$\Leftarrow$. If one of Q and Q' coincides with G or $Q \cup Q' = G$, then the statement is obvious.

If $Q \subset Q'$ and $Q' \neq G$, then $Q \cup Q' = Q'$ and $c(Q \cup Q') = c(Q')$. Assume that

$$Q \cup Q' \neq G \quad \text{and} \quad V(Q) \setminus V(Q') \neq \emptyset \neq V(Q') \setminus V(Q).$$

By Lemma 4, $c(Q), c(Q') \in V(Q) \cap V(Q')$.

Let $s \in V(G) \setminus V(Q \cup Q')$. Take a simple path L in G from z to s. Let L be of the form $z = v_0, v_1, \ldots, v_n = s$. By Lemma 1, $c(Q), c(Q') \in V(L)$. Let $c(Q) = v_i$, $c(Q') = v_j$. If $i < j$, then $c(Q') = v_j \notin V(Q)$, by Lemma 1. This contradicts the fact that

$$c(Q') \in V(Q) \cap V(Q').$$

Analogously, if $j < i$, then $c(Q) \notin V(Q')$, a contradiction. Thus $c(Q) = c(Q')$.

Lemma 5. *If Q and Q' are quasi-components of G and*

$$V(Q) \cap V(Q') \neq \emptyset,$$

then $Q \cap Q'$ is a connected subgraph of G.

Proof. If $|V(Q) \cap V(Q')| = 1$ or one of Q and Q' coincides with G, then the statement is obvious by Lemma 2.

Let Q and Q' be proper and $z_1, z_2 \in V(Q) \cap V(Q')$, $z_1 \neq z_2$. Since Q is connected, there exists a simple path L in Q from z_1 to z_2. Let L be of the form

$$z_1 = v_0, v_1, \ldots, v_n = z_2.$$

We show that $\{v_0, v_1, \ldots, v_n\} \subset V(Q')$. Otherwise, there exists $i \in \{1, 2, \ldots, n-1\}$ such that $v_i \notin V(Q')$. Since L is simple, each of the paths $v_0, \ldots, v_i$ and $z_2, v_{n-1}, \ldots, v_{i+1}, v_i$ is simple. By Lemma 1, the control point $c(Q')$ has to occur among $v_0, v_1, \ldots, v_{i-1}$ and among the vertices $z_2, \ldots, v_{i+1}$, which contradicts the fact that L is a simple path.

Lemma 6. *If Q and Q' are proper quasi-components of G, and*

$$(1) \qquad V(Q) \setminus V(Q') \neq \emptyset \neq V(Q') \setminus V(Q), \qquad V(Q) \cap V(Q') \neq \emptyset,$$

then $Q \cap Q'$ is a quasi-component of G iff $c(Q) = c(Q')$.

Proof. Assume that (1) holds. Then, by Lemma 4, we have $c(Q), c(Q') \in V(Q) \cap V(Q')$ and

$$\Gamma(c(Q)) \cap (V(G) \setminus V(Q)) \neq \emptyset \neq \Gamma(c(Q')) \cap (V(G) \setminus V(Q')).$$

Hence

$$\Gamma(c(Q)) \cap (V(G) \setminus (V(Q) \cap V(Q'))) \neq \emptyset \neq \Gamma(c(Q')) \cap (V(G) \setminus (V(Q) \cap V(Q'))).$$

Thus $Q \cap Q'$ is a quasi-component iff $c(Q) = c(Q')$.

THEOREM 2. *If Q and Q' are quasi-components of a connected graph G, then $Q \cap Q'$ is a quasi-component of G iff one of the following three cases holds:*

1° $V(Q) \subset V(Q')$;

2° $V(Q') \subset V(Q)$;

3° *both Q and Q' are proper,* (1) *holds and* $c(Q) = c(Q')$.

Proof. We shall consider all possibilities for quasi-components Q and Q'.

If one of the assumptions of 1° and 2° holds, then $Q \cap Q'$ is a quasi-component of G.

If $V(Q) \cap V(Q') = \emptyset$, then obviously $Q \cap Q'$ is not a quasi-component.

If (1) holds, then, by Lemma 6, $Q \cap Q'$ is a quasi-component of G iff $c(Q) = c(Q')$.

Case 3° is illustrated in Fig. 2a. The condition $c(Q) = c(Q')$ is essential, which is explained in Fig. 2b.

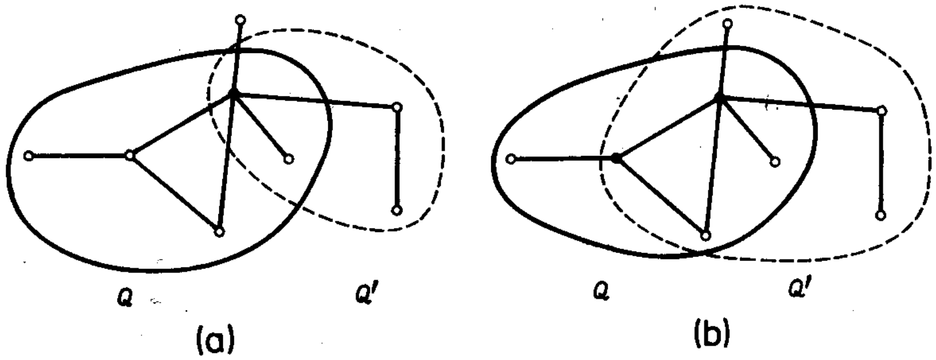

Fig. 2

Remark 1. The set $Q(G)$ of all quasi-components of G is obviously a poset with respect to the relation $\preccurlyeq$ defined by $Q_1 \preccurlyeq Q_2$ if $V(Q_1) \subseteq V(Q_2)$. However, $Q(G)$ is not a lattice. In fact, the example in Fig. 1 shows that the infimum

$$\inf(\langle\{v_1, v_2, v_3, v_4\}\rangle, \langle\{v_3, v_4, v_5, v_6, v_7\}\rangle)$$

does not exist. Similarly, $\sup(\langle\{v_3\}\rangle, \langle\{v_4\}\rangle)$ does not exist. We shall obtain a lattice for q-partitions considered in the next section. However, the structure of $Q(G)$ is also interesting. For example, we have

THEOREM 3. *Let L be a chain of non-empty subsets of $V(G)$ ordered by inclusion and such that, for every $A \in L$, $\langle\{A\}\rangle$ is a quasi-component of G.*

(a) *If $\bigcap L \neq \emptyset$, then the graph $\langle\bigcap L\rangle$ is a quasi-component of G. Moreover, if $\bigcap L \neq V(G)$, then for some $A_0 \in L$ we have:*

if $B \subseteq A_0$ and $B \subseteq L$, then $c(\langle B\rangle) = c(\langle\bigcap L\rangle)$.

(b) *$\langle\bigcup L\rangle$ is a quasi-component of G. Moreover, if $\bigcup L \neq V(G)$, then for some $A_0 \in L$ we have:*

if $B \supseteq A_0$ and $B \in L$, then $c(\langle B\rangle) = c(\langle\bigcup L\rangle)$.

Proof. Assume that $\langle\bigcap L\rangle$ is not a quasi-component. Then there exist $w_1, w_2, x_1, x_2 \in V(G)$ such that $w_1, w_2 \in \bigcap L$, $w_1 \neq w_2$, $x_1, x_2 \in V(G)\setminus\bigcap L$, where $\{w_1, x_1\}, \{w_2, x_2\} \in E(G)$. Hence there exist $A_1, A_2 \in L$ such that $x_1 \notin A_1$, $x_2 \notin A_2$. Put $A' = A_1 \cap A_2$. Then $A' \notin L$ and $w_1, w_2 \in A'$, $x_1, x_2 \notin A'$, $\{w_1, x_1\}$, $\{w_2, x_2\} \in E(G)$. Hence $\langle A'\rangle$ is not a quasi-component, a contradiction. Thus $\bigcap L$ is a quasi-component. If $\bigcap L \neq G$, then there exists $y \in V(G)\setminus\bigcap L$ such that

$$\{c(\langle\bigcap L\rangle), y\} \in E(G).$$

There exists $A_0 \in L$ such that $y \notin A_0$. But $c(\langle\bigcap L\rangle) \in A_0$ and A_0 is a quasi-component, so $c(\langle\bigcap L\rangle) = c(\langle A_0\rangle)$. If $B \subseteq A_0$ and $B \in L$, then again $y \notin B$ and $c(\langle\bigcap L\rangle) \in B$. Thus $c(\langle\bigcap L\rangle) = c(\langle B\rangle)$.

The proof of (b) is similar.

2. The lattice of q-partitions of a graph. Let V be a non-empty set. Recall that a family $P \subseteq (2^V\setminus\{\emptyset\})$ is a *partition* of V if

$$\bigcup_{A\in P} A = V$$

and all members in P are pairwise disjoint. For $a \in V$ we denote by $[a]_P$ the unique set in P to which a belongs. The set $[a]_P$ is called the *class of a*. We denote by $\mathscr{P}(V)$ the set of all partitions of V. For $P_1, P_2 \in \mathscr{P}(V)$ we have:

$$P_1 \leqslant P_2 \quad \text{if } [a]_{P_1} \subseteq [a]_{P_2} \text{ for every } a \in V.$$

It is known that the set $\mathscr{P}(V)$ with the relation $\leqslant$ is a complete lattice (see [1]).

For a family $\{P_r\}_{r\in R}$ of partitions of the set V we put

$$\bigwedge_{r\in R} P_r = \inf\{P_r\}_{r\in R} \quad \text{and} \quad \bigvee_{r\in R} P_r = \sup\{P_r\}_{r\in R}.$$

Obviously,

$$\bigwedge_{r\in R} P_r = \{\bigcap_{r\in R} [a]_{P_r} \colon a \in V\}.$$

The partition $\bigvee_{r\in R} P_r$ can be described as follows:

For $a, b \in V$ we have:

$$b \in [a]_{\bigvee_{r\in R} P_r}$$

if there exists a non-negative integer n and there exist $r_0, r_1, \ldots, r_n \in R$ and $a_0, a_1, \ldots, a_n \in V$ such that $a_0 = a$, $a_n = b$ and

$$[a_0]_{P_{r_0}} \cap [a_1]_{P_{r_1}} \neq \emptyset, \quad \ldots, \quad [a_{n-1}]_{P_{r_{n-1}}} \cap [a_n]_{P_{r_n}} \neq \emptyset$$

(see [1]).

The partition $\bigvee_{r\in R} P_r$ can be explained in terms of graphs. Let

$$S = \bigcup_{r\in R} P_r.$$

Denote by $\Omega(S)$ the intersection graph of the family S (two sets $A, B\in S$ are adjacent if $A\cap B \neq \emptyset$). For $D\in S$ we denote by $K(D)$ the component of the vertex D in the graph $\Omega(S)$. Then

$$\bigvee_{r\in R} P_r = \Big\{ \bigcup_{F\in V(K(D))} F\colon\ D\in S\Big\}.$$

This means that a class of the partitions $\bigvee_{r\in R} P_r$ is the union of all sets belonging to the same component of the graph $\Omega(S)$. Since every component of a graph is connected, the last definition is equivalent to the classical one.

Let G be a graph. A partition $P\in \mathscr{P}(V(G))$ will be called a *q-partition* of G if every class of P induces a quasi-component of G. For example,

$$\big\{\{v_1, v_2, v_3\}, \{v_4, v_5, v_6, v_7\}\big\}$$

is a q-partition of the graph in Fig. 1, whereas

$$\big\{\{v_1, v_2\}, \{v_3, v_4, v_5\}, \{v_6, v_7\}\big\}$$

and

$$\big\{\{v_1, v_7\}, \{v_2, v_3, \ldots, v_6\}\big\}$$

are partitions but not q-partitions. In practice, finding a q-partition of some area is nothing else but splitting this area into disjoint pieces with one control point in each.

Let us denote by $\mathscr{P}_q(G)$ the set of all q-partitions of G. Obviously, the set $\mathscr{P}_q(G)$ with the relation $\leqslant$ defined above is a poset.

LEMMA 7. *Let $\{P_r\}_{r\in R}$ be a family of q-partitions of a graph G. Then*

$$\bigvee_{r\in R} P_r \in \mathscr{P}_q(G).$$

Proof. Let

$$D\in \bigvee_{r\in R} P_r.$$

If $D = V(G)$, then $\langle D\rangle$ is a quasi-component of G. Let $D \neq V(G)$ and assume that $\langle D\rangle$ is not a quasi-component. Then there exist $u, v\in D$ such that $u\neq v$ and

$$\Gamma(u)\cap\big(V(G)\setminus D\big) \neq \emptyset \neq \Gamma(v)\cap\big(V(G)\setminus D\big).$$

By the definition of the partition $\bigvee_{r\in R} P_r$, there exist $r_0, r_1, \ldots, r_n\in R$ and $a_0, a_1, \ldots, a_n\in V(G)$ such that $a_0 = u$, $a_n = v$ and

$$[a_0]_{P_{r_0}} \cap [a_1]_{P_{r_1}} \neq \emptyset, \quad \ldots, \quad [a_{n-1}]_{P_{r_{n-1}}} \cap [a_n]_{P_{r_n}} \neq \emptyset.$$

Put

$$D' = [a_0]_{P_{r_0}} \cup [a_1]_{P_{r_1}} \cup \ldots \cup [a_n]_{P_{r_n}}.$$

Since every graph $\langle [a_k]_{P_{r_k}} \rangle$ is a quasi-component of G for $k = 0, 1, \ldots, n$, by Theorem 1 each of the graphs

$$\langle [a_0]_{P_{r_0}} \cup [a_1]_{P_{r_1}} \cup \ldots \cup [a_t]_{P_{r_t}} \rangle$$

for $t = 2, 3, \ldots, n$ is a quasi-component. Hence $\langle D' \rangle$ is a quasi-component, which gives a contradiction since $u, v \in D'$ and

$$\Gamma(u) \cap (V(G) \setminus D') \neq \emptyset \quad \text{and} \quad \Gamma(v) \cap (V(G) \setminus D') = \emptyset.$$

THEOREM 4. 1. *The set $\mathscr{P}_q(G)$ is a complete join subsemilattice of the semilattice $(\mathscr{P}(G); \vee)$.*

2. *The poset $(\mathscr{P}_q(G); \leqslant)$ is a complete lattice.*

Proof. The first statement follows from Lemma 7. The second statement follows from Lemma 7, from the existence of the least q-partition $\{\{v\}: v \in V(G)\}$ in $\mathscr{P}_q(G)$ and from the fact that every complete join semilattice $(L; \leqslant)$ with zero is a complete lattice.

For the family $\{P_r\}_{r \in R}$ of q-partitions of G, let us denote by $\bigwedge_{q\, r \in R} P_r$ the infimum of $\{P_r\}_{r \in R}$ in the set $\mathscr{P}_q(G)$. Obviously,

$$\bigwedge_{r \in R}{}_q P_r \leqslant \bigwedge_{r \in R} P_r.$$

The following example shows that we may have

$$\bigwedge_{r \in R}{}_q P_r \neq \bigwedge_{r \in R} P_r.$$

EXAMPLE 2. In the graph in Fig. 3 we have

$$P_1 = \{\{1\}, \{2, 3, \ldots, 12\}\}, \quad P_2 = \{\{1, 2, \ldots, 9\}, \{10, 11, 12\}\}.$$

$$\bigwedge_{r \in \{1,2\}} P_r = \{\{1\}, \{2, 3, \ldots, 9\}, \{10, 11, 12\}\},$$

$$\bigwedge_{r \in \{1,2\}}{}_q P_r = \{\{1\}, \{2, 3, 4, 5\}, \{6\}, \{7, 8, 9\}, \{10, 11, 12\}\}.$$

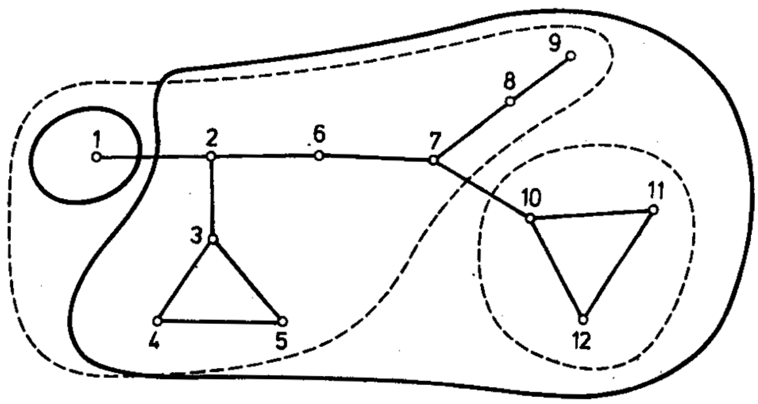

Fig. 3

The interesting question is what is an explicit form of the infimum of two q-partitions. We deal with this problem in the next section.

3. Infimum of two q-partitions of a graph. First recall that if G is a connected graph, then the graph $\mathrm{ba}(G)$ is defined as follows (see [2]): $V(\mathrm{ba}(G))$ consists of all blocks and all articulation points of G. The set $E(\mathrm{ba}(G))$ consists of all pairs $\{B, a\}$, where B is a block of G, and a is an articulation point of G such that $a \in V(B)$.

Let P_1 and P_2 be two q-partitions of a graph G. In the poset $(\mathscr{P}_q(G); \leqslant)$ we want to find $\inf\{P_1, P_2\}$ denoted by $P_1 \wedge_q P_2$. Obviously, we have

$$P_1 \wedge_q P_2 \leqslant P_1 \wedge P_2 = \{[a]_{P_1} \cap [a]_{P_2}\colon a \in V(G)\}.$$

If $[a]_{P_1} \cap [a]_{P_2}$ induces a quasi-component, then it can be taken as $[a]_{P_1 \wedge_q P_2}$. Hence we get into trouble if for some $a \in V(G)$ the graph $\langle [a]_{P_1} \cap [a]_{P_2} \rangle$ is not a quasi-component of G. Then it must be decomposed into as large as possible disjoint quasi-components of G. We present a procedure for such a situation.

Write

$$\begin{aligned} &Q = \langle [a]_{P_1} \rangle, \quad Q' = \langle [a]_{P_2} \rangle, \\ &H = Q \cap Q', \quad w = c(Q), \quad w' = c(Q'). \end{aligned} \tag{2}$$

Since $Q \cap Q'$ is not a quasi-component, by Lemma 6 and Theorem 2 we have

$$V(Q)\setminus V(Q') \neq \emptyset \neq V(Q')\setminus V(Q), \quad w, w' \in H \quad \text{and} \quad w \neq w'. \tag{3}$$

Hence

$$\Gamma(w) \cap \big(V(Q')\setminus V(Q)\big) \neq \emptyset \neq \Gamma(w') \cap \big(V(Q)\setminus V(Q')\big). \tag{4}$$

By (3) we obtain $|V(H)| > 1$, $|V(Q)| > 1$ and $|V(Q')| > 1$. Consequently, both vertices w and w' are articulation points of the graph G.

Consider the graph $\mathrm{ba}(G)$. By Lemma 3 we can distinguish the subgraph D of the graph $\mathrm{ba}(G)$ induced by all blocks and articulation points belonging to H. The graph $\mathrm{ba}(G)$ is a tree (see [2]), and H is a connected subgraph of G by Lemma 5. Hence D is a subtree of $\mathrm{ba}(G)$. Consequently, in D there is exactly one path connecting the articulation points w and w'. Denote this path by

$$a_0, B_1, a_1, B_2, \ldots, B_k, a_k, \quad \text{where } a_0 = w,\ a_k = w'. \tag{5}$$

Then we must have $k \geqslant 1$.

Let H^* denote the subgraph of G obtained from H by removing all edges from the blocks $B_1, B_2, \ldots, B_k$. Then we disconnect the graph H.

We claim that $P_1 \wedge_q P_2$ is equal to the partition P^* whose classes are of two forms:

If for some $a \in V(G)$ the graph $\langle [a]_{P_1} \cap [a]_{P_2} \rangle$ is a quasi-component of G, then $[a]_{P_1} \cap [a]_{P_2} = [a]_{P^*}$.

If $H = \langle [a]_{P_1} \cap [a]_{P_2} \rangle$ is not a quasi-component of G, then every component K^* of H^* is a quasi-component of G. We take as $[a]_{P^*}$ the component of H^* to which a belongs.

To show this we need two lemmas.

LEMMA 8. *Let H be a graph defined in* (2) *and assume that H is not a quasi-component of G. If Q_0 is a quasi-component of G and Q_0 is a subgraph of H, then Q_0 is a subgraph of some component of the graph H^*.*

Proof. It is enough to show that Q_0 contains no edge belonging to $E(H) \backslash E(H^*)$. Assume it is not the case and let

$$e \in (E(H) \backslash E(H^*)) \cap E(Q_0).$$

Then e belongs to some block B_i from (5), $i \in \{1, 2, \ldots, k\}$. By Lemma 3, the block B_i is a subgraph of Q_0.

Let

$$i_1 = \min \{i \in \{1, 2, \ldots, k\}\colon B_i \subset Q_0\},$$

$$i_2 = \max \{i \in \{1, 2, \ldots, k\}\colon B_i \subset Q_0\}.$$

Then $\Gamma(a_{i_1-1}) \cap (V(G) \backslash V(Q_0)) \neq \emptyset$. In fact, if $i_1 = 1$, then $a_{i_1-1} = w = c(Q)$ and, by (4),

$$\Gamma(w) \cap (V(Q') \backslash V(Q_0)) \neq \emptyset.$$

If $i_1 > 1$, then $\Gamma(a_{i_1-1}) \cap V(B_{i_1-1}) \neq \emptyset$. Analogously,

$$\Gamma(a_{i_2}) \cap (V(G) \backslash V(Q_0)) \neq \emptyset.$$

Since $a_{i_1-1} \neq a_{i_2}$ and $a_{i_1-1}, a_{i_2} \in V(Q_0)$, we get a contradiction with the assumption that Q_0 is a quasi-component of G.

LEMMA 9. *Every component K of the graph H^* is a quasi-component of G.*

Proof. If $|V(K)| = 1$, then the statement is obvious. Let $|V(K)| > 1$ and assume that K is not a quasi-component of G. Then there exist $u_1, u_2 \in V(K)$ such that $u_1 \neq u_2$ and

$$\Gamma(u_k) \cap (V(G) \backslash V(K)) \neq \emptyset \quad \text{for } k = 1, 2.$$

Put $M = B_1 \cup B_2 \cup \ldots \cup B_k$. We show that $u_k \in V(M)$ for $k = 1, 2$. In fact, let $\{u_1, v\} \in E(G)$ and $v \notin V(K)$. If $v \notin V(H)$, then $u_1 = c(Q)$ or $u_1 = c(Q')$, so $u_1 \in V(M)$ by Lemma 3. If $v \in V(H)$, then $\{u_1, v\} \in E(M)$, so $u_1 \in V(M)$. Analogously we prove that $u_2 \in V(M)$.

Since K is connected, there exists a simple path $u_1, v_1, \ldots, v_n = u_2$ in K. Let u_3 be the first vertex in this path belonging to $V(M)$ such that $u_3 \neq u_1$. Then $u_3 = v_s$ for some $s \in \{1, 2, \ldots, n\}$. Obviously, $\{u_1, v_1\} \notin E(M)$. On the other hand, since M is connected, there exists a simple path $u_1 = z_0$, $z_1, \ldots, z_p = u_3$ in M. So $\{u_1, z_1\} \in E(M)$. Thus

$$u_1, v_1, \ldots, v_s, z_{p-1}, \ldots, z_1, u_1$$

is a simple cycle in H. We have $\{u_1, z_1\} \in B_i$ for some $i \in \{1, 2, \ldots, k\}$. This however gives a contradiction since then $\{u_1, v_1\} \in B_i$ and, consequently, $\{u_1, v_1\} \in E(M)$.

Now we are able to do the proof.

THEOREM 5. *For $P_1, P_2 \in \mathscr{P}_q(G)$ we have $P_1 \wedge_q P_2 = P^*$.*

Proof. Obviously, P^* is a partition of G. By Lemma 9, P^* is a q-partition of G. We have $P^* \leqslant P_1$ and $P^* \leqslant P_2$ since $P^* \leqslant P_1 \wedge P_2$. Let $P_0 \in \mathscr{P}_q(G)$, $P_0 \leqslant P_1$, $P_0 \leqslant P_2$ and $a \in V(G)$. Then

$$[a]_{P_0} \subseteq [a]_{P_1} \cap [a]_{P_2}.$$

If $[a]_{P_1} \cap [a]_{P_2}$ is a quasi-component of G, then $[a]_{P_0} \subseteq [a]_{P^*}$. If $[a]_{P_1} \cap [a]_{P_2}$ is not a quasi-component of G, then by Lemma 8 there exists a component K of the graph

$$H^* = \langle [a]_{P_1} \cap [a]_{P_2} \rangle^*$$

such that $\langle [a]_{P_0} \rangle \subseteq K$. Since $a \in V(K)$, by the definition of P^* we have $\langle [a]_{P^*} \rangle = K$ and, consequently, $[a]_{P_0} \subseteq [a]_{P^*}$. Thus $P_0 \leqslant P^*$.

References

[1] G. Grätzer, *General Lattice Theory*, Akademie-Verlag, Berlin 1978.
[2] F. Harary, *Graph Theory*, Reading, Mass., 1969.
[3] O. Ore, *Theory of graphs*, Amer. Math. Soc. Colloq. Publ., 1962.

INSTITUTE OF MATHEMATICS
PEDAGOGICAL UNIVERSITY
45-951 OPOLE

ŚLĘŻNA 9 m. 6
53-301 WROCŁAW