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SOME ANALYTIC ASPECTS OF PROBABILISTIC POTENTIAL THEORY

1. It is well known that not only can the Wiener-Perron extension of classical (Newtonian) potential theory be conveniently formulated in terms of the Brownian motion process but that also extensions to a much wider class of sets become possible.

The idea which underlies this development goes back to Kakutani and consists in defining the capacitory potential at y of a set K as the probability that the Brownian motion starting from y will ultimately hit K.

If instead of probability of hitting one considers the probability of penetration i.e. the probability of spending positive time in K (having started from y) one is lead to a different theory.

This theory is applicable to a narrower class of sets (i.e. one has to assume that the Lebesgue measure of K is positive otherwise probability of penetration is trivially zero) and it may thus appear to be of lesser interest. But what it looses in generality it gains in expliciteness and one is rewarded by a number of new analytic formulas and facts.

2. Let K be a compact set in Euclidean three-space R^3 of positive (three-dimensional) Lebesgue measure.

Let y be a point in R^3 and let $T_K(y)$ be the total time the Brownian motion starting from y spends in K. One can easily show that the random variable $T_K(y)$ is finite with probability 1 (in fact, the expectation of $T_K(y)$ is finite).

It is also not difficult to calculate the expectation

$$E\{e^{-uT_K(y)}\}, \quad u \geqslant 0, \tag{2.1}$$

and the answer is expressible in terms of eigenvalues and eigenfunctions of the integral equation

$$\frac{1}{2\pi}\int_K \frac{\varphi(x)}{|x-y|}\,dx = \lambda\varphi(y), \quad y \in K, \tag{2.2}$$

where dx is a volume element and $|x-y|$ the Euclidean distance between the points x and y.

In fact, one has (see e.g. [3])

$$(2.3) \qquad 1-E\{e^{-uT_K(y)}\} = \sum_{j=1}^{\infty} \frac{u}{1+u\lambda_j} \int_K \varphi_j(x)\,dx \frac{1}{2\pi} \int_K \frac{\varphi_j(x)}{|x-y|}\,dx,$$

where $\lambda_1, \lambda_2, \ldots$ are the eigenvalues and $\varphi_1, \varphi_2, \ldots$ the coresponding normalized eigenfunctions of the kernel

$$(2.4) \qquad \frac{1}{2\pi}\frac{1}{|x-y|}$$

taken over the set K. The kernel (2.4) is well known to be positive definite a fact which simplifies the discussion considerably.

If one lets u approach infinity increasingly ($u\nearrow\infty$) one obtains

$$(2.5) \qquad U(y) = \text{Prob.}\{T_K(y) > 0\} =$$

$$\lim_{u\nearrow\infty} \sum_{j=1}^{\infty} \frac{u}{1+u\lambda_j} \int_K \varphi_j(x)\;dx \frac{1}{2\pi} \int_K \frac{\varphi_j(x)}{|x-y|}\,dx$$

for every y in R^3.

Outside K ($y \notin K$) $U(y)$ is harmonic and $U(y)$ approaches 0 as the distance between y and a fixed point in space approaches infinity.

If K_0 is the interior of the compact set K then $U(y) = 1$ for $y \in K_0$ (this is an immediate consequence of continuity of Brownian paths). One therefore sees that $U(y)$ is in some sense the solution of the exterior Dirichlet problem with the boundary function being identically equal to 1.

If y is on the boundary of K one has (as one has also in K_0)

$$(2.6) \qquad U(y) = \text{Prob.}\{T_K(y) > 0\} = \lim_{u\nearrow\infty} \sum_{j=1}^{\infty} \frac{\lambda_j}{\frac{1}{u}+\lambda_j} \left(\int_K \varphi_j(x)\,dx\right) \varphi_j(y).$$

3. The question now arises whether

$$U(y) = 1 \quad \text{if} \quad y \in K \setminus K_0?$$

The answer depends on certain regularity properties of the boundary point y. To state the pertinent result it is best to introduce the concept of an s-regular point (see [2]).

First define $T_t(y)$ as the time spent in K up to time t. Then define the first penetration time $\tau(y)$ as the least upper bound of t's for which $T_t(y) = 0$ i.e.

$$(3.1) \qquad \tau(y) = \text{l.u.b.}\{t \colon T_t(y) = 0\}.$$

One can then show using Blumenthal's zero-one law (see [1]) that

$$\text{Prob.}\{\tau(y) > 0\}$$

is either 0 or 1. For example if $y \notin K$, $\text{Prob.}\{\tau(y) = 0\} = 0$ but if $y \in K_0$ then $\text{Prob.}\{\tau(y) = 0\} = 1$. On the boundary both alternatives are possible and one calls a boundary point y s-regular if $\text{Prob.}\{\tau(y) = 0\} = 1$. Otherwise it is called s-irregular.

There is simple geometric sufficient condition for s-regularity [2]: if the symmetric Lebesgue upper density of K at y is positive, then y is s-regular. Since clearly $U(y) = 1$ if y is s-regular one infers immediately that the subset of K where $U(y) < 1$ is of Lebesgue measure zero and even more is true, the set of all $y \in K$ which are s-irregular is of Lebesgue measure zero.

For a wide class of sets K (e.g. if K is star-shaped) the concept of s-regularity coincides with that of regularity in the sense of the Wiener-Perron potential theory. But now because of (2.6) we are led to a purely analytic criterion of regularity.

Thus if K is, for example, star-shaped a boundary point y is regular if and only if

$$1 = \lim_{\delta \searrow 0} \sum_{j=1}^{\infty} \frac{\lambda_j}{\lambda_j + \delta} \left(\int_K \varphi_j(x)\,dx \right) \varphi_j(y). \tag{3.2}$$

Since

$$\sum_{j=1}^{\infty} \left(\int_K \varphi_j(x)\,dx \right) \varphi_j(y) \tag{3.3}$$

is clearly the Fourier series of the function $f(x) = 1$, $x \in K$, with respect to the orthonormal set $\{\varphi_j\}$, and $\lambda_j/(\delta + \lambda_j)$ are the convergence factors which define a summability method one can say that y is regular if and only if the Fourier series (3.3) of $f(x) = 1$, $x \in K$, is summable to what it should be (i.e. to 1) by a certain summability method.

4. Nothing striking is found when one extends the theory sketched above to spaces of dimensionality higher than three but in the case of the plane (R^2) one finds new and interesting analytic phenomena.

Restricting ourself again to compact sets K of non-zero (two-dimensional) Lebesgue measure we no longer can work with the total time $T_K(y)$ spent in K since now with probability 1

$$T_K(y) = \infty. \tag{4.1}$$

However $T_t(y)$[1] is perfectly well defined and so is the concept of s-regularity.

Instead of (2.1) we now consider

$$E\{e^{-uT_t(y)}\} \tag{4.2}$$

and it is convenient to work with the Laplace transform

$$G(y; s, u) = s\int_0^\infty e^{-st}E\{e^{-uT_t(y)}\}\,dt. \tag{4.3}$$

The function $G(y; s, u)$ satisfies the integral equation

$$G(y; s, u) = 1-\frac{u}{\pi}\int_K K_0(\sqrt{2s}\,|x-y|)G(x; s, u)\,dx, \tag{4.4}$$

where K_0 is the Bessel function of second kind of imaginary argument.

One can show that

$$\lim_{s\searrow 0} G(y; s, u)\log\sqrt{\frac{2}{s}} = Q(y; u) \tag{4.5}$$

exists for every y and that Q satisfies the integral equation

$$Q(y; u) = R(u)-\frac{u}{\pi}\int_K \log\frac{1}{|x-y|}\,Q(x; u)\,dx, \tag{4.6}$$

where $R(u)$ is defined as the limit

$$R(u) = \lim_{s\searrow 0}\left\{1-\frac{u}{\pi}\int_K G(x; s, u)\,dx\right\}\log\sqrt{\frac{2}{s}} \tag{4.7}$$

which can be shown to exist.

As a consequence of (4.5) and the existence of the limit in (4.7) one gets also that

$$1 = \frac{u}{\pi}\int_K Q(x; u)\,dx. \tag{4.8}$$

The derivation of these results though not overly difficult is not entirely trivial and requires certain amount of care.

5. It can be shown that

$$\lim_{u\nearrow\infty} R(u) = R < \infty \tag{5.1}$$

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[1] As above we drop the subscript K but the reader will remember that unless otherwise stated T refers to the time spent in K.

exists and so does

$$\lim_{u \nearrow \infty} Q(y; u) = U(y). \tag{5.2}$$

Also (because of (4.8)) one can show that the measures

$$\mu_u(dx) = \frac{u}{\pi} Q(x; u)\, dx \tag{5.3}$$

approach weakly as $u \nearrow \infty$ a measure $\mu(dx)$

$$\mu_u(dx) \Rightarrow \mu(dx) \tag{5.4}$$

and also that

$$U(y) = \lim_{u \nearrow \infty} Q(y; u) = R - \int_K \log \frac{1}{|x-y|} \mu(dx). \tag{5.5}$$

From (4.8) one has the normalization

$$\int_K \mu(dx) = 1. \tag{5.6}$$

It follows quite easily that for $y \in K_0$ $U(y) = 0$ and it is clear that for $y \notin K$ $U(y)$ is harmonic. Moreover $U(y)$ has a logarithmic pole at infinity and it is not too difficult to conjecture that for wide class of K's $U(y)$ is the capacitory logarithmic potential of K at y. Indead for sets K with sufficiently smooth boundaries $U(y)$ is the familiar logarithmic potential.

The precise statement is that

$$U(y) \to 0$$

whenever y approaches an s-regular point and that for a wide class of sets (e.g. for star-shaped K's) the concept of s-regularity is the same as that of ordinary regularity in potential theory.

6. The question which comes up now quite naturally is whether an analytic criterion for s-regularity analogous to (3.2) can be established.

The integral equation analogous to (2.2) is now

$$\frac{1}{\pi} \int_K \varphi(x) \log \frac{1}{|x-y|}\, dx = \lambda \varphi(y), \qquad y \in K, \tag{6.1}$$

but the kernel

$$\frac{1}{\pi} \log \frac{1}{|x-y|}, \qquad x \in K, \qquad y \in K, \tag{6.2}$$

need no longer be positive definite.

In fact, one can establish the following result:

(a) if $R > 0$ the kernel (6.2) is positive definite (so that all eigenvalues are positive),

(b) if $R = 0$ the kernel (6.2) is non-negative definite and moreover if zero is an eigenvalue then it must be simple,

(c) if $R < 0$ the kernel (6.2) has exactly one simple negative eigenvalue and all other eigenvalues are positive ([2]).

One is faced here (for wide class of K's) with something like "mystery of the vanishing eigenfunction" where the word vanishing is used in the mystery story sense of disappearing into thin air rather than in the mathematical sense of being equal to zero.

Suppose, in fact, that K is such that $R = 0$([3]) and consider a descending sequence of sets K_n

$$K_1 \supset K_2 \supset K_3 \supset \ldots$$

such that

$$\lim_{n\to\infty} K_n = K \qquad \text{and} \qquad R(K_n) < R(K) = R.$$

It is also supposed that our potentials corresponding to the K_n's and K are identical with the classical capacitory potentials of these sets.

Clearly $R(K_n) < 0$ and consequently by (c) above the kernel (6.2) considered over K_n has for each n one negative eigenvalue. Let the corresponding normalized eigenfunction be denoted by

$$\psi(x; K_n), \qquad x \in K_n.$$

The eigenfunctions corresponding to the positive eigenvalues of (6.2) over K_n approach the eigenfunctions of (6.2) over K but $\psi(x; K_n)$ cannot approach in any sensible way a non-vanishing L^2 function over K for this would be in contradiction with completeness of the eigenfunctions of (6.2) over K. What does then happen to $\psi(x; K_n)$?

The answer is simple, not unexpected and one which would be guessed by discussing the case of the circle.

It is that the measure $\mu_n(dx)$ defined by the formula

$$\mu_n(dx) = \frac{\psi(x; K_n)}{\int_{K_n} \psi(x; K_n)\,dx}\,dx \tag{6.3}$$

([2]) While this paper was in print we have learned that zero is not an eigenvalue.

([3]) Since it is no doubt clear to the reader that for "reasonable" sets R is the Robin constant (i.e. minus the logarithm of the logarithmic capacity) the simplest example of a set with $R = 0$ is provided by a circle of radius 1.

(it follows easily that the denominator does not vanish) is non-negative and it approaches weakly as $n \to \infty$ the measure μ defined in the formula (5.4). This measure as can be seen from (5.5) and from the fact that $U(y)$ is the logarithmic potential is concentrated on the boundary of K. Moreover one shows that except for a set of logarithmic capacity zero

$$\int_{K_n} \log\frac{1}{|y-z|}\mu_n(dz) \to \int_K \log\frac{1}{|y-z|}\mu(dz).$$

Speaking somewhat loosely we can thus say that as $n \to \infty$ $\psi(x; K_n)$ becomes an eigendistribution belonging to eigenvalue zero since for $R = 0$ equation (5.5) becomes

$$0 = \int_K \log\frac{1}{|x-y|}\mu(dx) \tag{6.4}$$

for μ almost all y.

7. Returning to the question posed at the begining of section 6 as to whether an analogue of formula (3.2) holds for plane sets the answer is in the afermative if $R \neq 0$.

In fact, one has the following more general result([4]):

If $R \neq 0$ *and* $f(x)$ *is continuous on* K *then*

$$f(x) = \lim_{\delta\searrow 0}\sum_{j=1}^{\infty}\frac{\lambda_j}{\lambda_j+\delta}\Big(\int_K \varphi_j(y)f(y)\,dy\Big)\varphi_j(x) \tag{7.1}$$

at every s-regular point. Conversely, if (7.1) *holds for all* f *continuous on* K*, then* x *is s-regular.*

The situation is radically different if $R = 0$ as can be seen by considering the case when K is a circle of radius 1.

For a circle of radius a (and center at the origin) one finds by direct calculation that

$$\frac{Q(x;u)}{R(u)} = \frac{I_0(\sqrt{2u}\,|x|)}{I_0(\sqrt{2ua})-\sqrt{2ua}\log a\, I_1(\sqrt{2ua})}, \tag{7.2}$$

where I_0 and I_1 are the familiar Bessel functions of first kind of imaginary argument.

([4]) A similar generalization holds also in R^3 (see [4]).

If $a = 1$ and x is on the circumference of the circle ($|x| = 1$) one has

$$\frac{Q(x;u)}{R(u)} = 1$$

which implies by letting $u \nearrow \infty$ that

$$0 = \lim_{\delta \searrow 0} \sum_{j=1}^{\infty} \frac{\lambda_j}{\lambda_j + \delta} \Big(\int_K \varphi_j(y)\, dy \Big) \varphi_j(x).$$

Thus the Fourier series of the function $f = 1$ is summable on the circumference of the circle to 0!

If $|x| < 1$ nothing out of the ordinary happens and one has

$$1 = \lim_{\delta \searrow 0} \sum_{j=1}^{\infty} \frac{\lambda_j}{\lambda_j + \delta} \Big(\int_K \varphi_j(y)\, dy \Big) \varphi_j(x)$$

as expected.

A heuristic explanation is simple and immediate. On the boundary of the circle we have the eigendistribution discussed in section 6 which must be included in the expansion.

In fact, for functions f which are orthogonal to the eigendistribution, which in our case means that

$$\int_0^{2\pi} f(\theta)\, d\theta = 0,$$

formula (7.1) can be shown to hold.

It is highly probable, though we do not have a proof yet, that (7.1) holds for general sets whose Robin constant is zero provided

$$\int_K f(x)\, \mu(dx) = 0. \tag{7.3}$$

However we can prove that (7.1) holds always in the interior of K.

Our pleasure in being able to dedicate the note to Professor Steinhaus on the occasion of his eightieth birthday is hightened by the realization that it seems to fulfill a dictum he voiced to one of us over thirty years ago in Lwów.

He said that no general theorem on sets or curves is interesting unless it is already interesting for a circle.

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