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A NEW FORMULATION AND SOLUTION OF THE SEQUENCING PROBLEM: ALGORITHM

In paper [2] the mathematical model and properties of the general sequencing problem have been presented. This paper contains an algorithm.

It follows from [2] that the solution of this problem is equivalent to finding a minimaximal path in the disjunctive graph $\bar{D} = \langle A, U; V\rangle$. An algorithm for finding this path is similar to that presented in [1]. It works using the branch-and-bound method. In this paper the notation and the numeration of formulas are continuation of those from paper [2].

The minimaximal path of the disjunctive graph D is obtained by generating a sequence of circuit graphs $D_r \in R'_D$ and finding the critical path for each D_r in the sequence.

Let S^+ be a complete initial selection such that $S^+ \in R$ and the graph $D^+ = \langle A, U \cup S^+\rangle$ has no circuits. A disjunctive arc $\langle y, x\rangle \in V$ is called *normal* if $\langle y, x\rangle \in S^+$. The complement of a normal arc is called a *reverse arc*.

Starting with the graph $D^+ = \langle A, U \cup S^+\rangle$, we generate a sequence of graphs $D_r = \langle A, U \cup S_r\rangle$. Each graph D_s is obtained from a certain graph D_r of the sequence by complementing one normal arc. This process is presented in the form of a solution tree H. Each node in H corresponds to a pair of graphs D_r, D_s such that D_s is obtained from D_r by complementing one disjunctive arc from $\langle y, x\rangle \in S_r$. Then the arc $\langle D_r, D_s\rangle$ in H represents the complement $\langle u, v\rangle \in S_s$ of $\langle y, x\rangle$. We say that D_r is the *predecessor* of D_s (and D_s is the *successor* of D_r) if there is a path in H between D_r and D_s. The initial graph $D^+ = \langle A, U \cup S^+\rangle$ is the *root* in the solution tree H. The generation of a new branch in H is connected with the choice of a certain normal arc for complementing to obtain S_r. This choice is called the *operation of choice.*

For each graph D_r from the sequence we perform the *operation of testing* to check the possibility of the generation of the graph $D_s \in R'_D$ with a smaller critical path than that already found. If such a D_s does not exist, we abandon the considered graph D_r and backtrack the tree H

to the predecessor D_p from which the graph D_r was generated. If a new graph D_s is generated from the graph D_r by complementing the normal arc $\langle y, x\rangle \in S_r$, we temporarily fix a reverse arc $\langle u, v\rangle \in S_s$ in D_s. This arc cannot be complemented in any successor D_s of D_r in H. However, if we backtrack to D_r from a successor D_s, we fix the normal arc $\langle y, x\rangle$ the complementing of which has generated D_s. So, for each graph D_r we set a subset $F_r \subset S_r$ of disjunctive arcs. The reverse arcs in F_r are those which represent the path between the root and D_r in H, normal arcs in F_r being those the complements of which were abandoned during the backtracking process.

1. Operation of testing. The basic task of the operation of testing is the computation of the lower bound of the critical path for every possible successor $D_t \in R'_D$ generated from the graph D_r. We want to obtain the greatest possible value of this bound to be able to abandon in the algorithm a greater number of successors in H.

Let

$$D(F_r) = \langle A, U \cup F_r\rangle$$

be the graph generated from the set F_r of fixed disjunctive arcs and let

$$\bar{D}(F_r) = \langle A, U \cup F_r; V_r\rangle, \quad \text{where } V_r = V - [F_r \cup F'_r],$$

be the disjunctive graph of the graph $D(F_r)$, F'_r being the set of such disjunctive arcs the complements of which belong to F_r.

It is readily seen that the process of generating successors of D_r in H is equivalent to finding the minimaximal path of the disjunctive graph $\bar{D}(F_r)$. The initial selection is the set $E_r = S_r - F_r$ which is called the *set of free arcs* of the graph D_r.

Let us consider the sets of disjunctive graphs

$$R_{\bar{D}} = \{\bar{D}^k = \langle A, U; V^k\rangle\} \quad (k \in Q)$$

and

$$R_{\bar{D}(F_r)} = \{\bar{D}^k(F_r) = \langle A, U \cup F_r; V_r^k\rangle\} \quad (k \in Q),$$

where each graph $\bar{D}^k$ is d-partial and d-connected, and $V_r^k = V^k \cap V_r$. It follows from the definition of the sets V^k and V_r^k that the set V_r^k is the full subset of the set V. So, each graph $\bar{D}^k(F_r)$ is the d-partial graph of the graph $\bar{D}(F_r)$. Let $L_0(F_r)$ and $L^k(F_r)$ be the lengths of the minimaximal paths of the graphs $\bar{D}(F_r)$ and $\bar{D}^k(F_r)$, respectively. Then we have (see Theorem 3)

$$L_0(F_r) \geq \max_{k\in Q} L_0^k(F_r). \tag{51}$$

It follows from the definition of V_r^k that the d-partial graph $\bar{D}^k(F_r)$, $k \in Q$, is d-connected and we have

$$A^k(F_r) \subset X^k \quad \text{and} \quad B^k(F_r) \subset Y^k.$$

In the general case, no $D^k(F_r)$ is either z-symmetrical or 0-symmetrical, i.e. $D^k(F_r)$ satisfies neither relation (32) nor relation (50). In order to obtain a z-symmetrical or 0-symmetrical graph we proceed in the following way.

It follows from Property 2 that all paths starting from the node $x \in X$ begin with the arc $\langle x, \Gamma x\rangle$ the length of which is $c(x, \Gamma x)$.

Let $P(x, y)$ be the set of all paths from the node x to the node y in D. In order for the graph $\bar{D}^k(F_r)$ to be z-symmetrical we assume:

(a) The lengths of all arcs of the paths $P(\Gamma x_j, z)$ are equal to zero for each $x_j \in A^k(F_r)$.

(b) The lengths of all arcs of the paths $P(\Gamma x_j, x_i)$ are equal to zero for each pair $x_j, x_i \in A^k(F_r)$ for which there exists a path in $D(F_r)$.

In order for the graph $\bar{D}(F_r)$ to be 0-symmetrical we assume:

(a) The lengths of all arcs of the paths $P(0, \Gamma^{-1}x_j)$ are equal to zero for each $x_j \in B^k(F_r)$.

(b) The lengths of all arcs of the paths $P(x_j, \Gamma^{-1}x_i)$ are equal to zero for each pair $x_j, x_i \in B^k(F_r)$ for which there exists a path in $D(F_r)$.

The graph $D^k(F_r)$ with reduced weights for z-symmetry and 0-symmetry is denoted by $D^{kz}(F_r)$ and $D^{k0}(F_r)$, respectively. It is obvious that these graphs have already satisfied all assumptions of Theorem 4 and this will allow us to determine an optimal representation on the basis of Theorem 4. We can also determine the minimaximal paths $L_0^{kz}(F_r)$ and $L_0^{k0}(F_r)$. It can be readily seen that $L_0^k(F_r) \geqslant L_0^{kz}(F_r)$ and $L_0^k(F_r) \geqslant L_0^{k0}(F_r)$ for each $k \in Q$. By the above and by (51) we have

$$L_0(F_r) \geqslant \max_{k\in Q} L_0^{kz}(F_r) = L_d^z(F_r) \tag{51'}$$

and

$$L_0(F_r) \geqslant \max_{k\in Q} L_0^{kz}(F_r) = L_d^0(F_r). \tag{51''}$$

The values $L_d^z(F_r)$ and $L_d^0(F_r)$ are the lower bounds of the minimaximal paths of the graph $\bar{D}(F_r)$, i.e. the lower bounds of all successors of D_r.

Let us denote by $L_d(F_r)$ the length of the critical path of the graph $D(F_r)$. Since $D(F_r)$ is the partial graph of each of the successors of the graph D_r, $L_d(F_r)$ is also the lower bound of the length of the critical paths of the successors. Therefore,

$$L(F_r) = \max[L_d^z(F_r), L_d^0(F_r), L_d(F_r)]$$

is the lower bound of all successors of the graph D_r. Observe that in [1] the lower bound is estimated on the basis of the value $L_d(F_r)$, but our lower bound is stronger.

Let L^* be the length of the shortest critical path found so far. Then, if $L(F_r) \geqslant L^*$, we can reject the graph D_r and all its successors. The value L^* is the upper bound of the length of a minimaximal path in $\bar{D}$.

2. Operation of choice. The purpose of the operation of choice is to point out the normal arc for complementing and the generation of the successor D_s in H. The arcs $E_r = S_r - F_r$ are free. We complement only arcs of this set which belongs to the current critical path, i.e.

$$K_r = E_r \cap C_r. \tag{52}$$

We want to choose that arc the complementing of which generates a successor with the shortest possible critical path. This is especially important for the operation of testing. The choice criterion for an arc of K_r is the expression $\Delta_r[(y, x), (u, v)]$ defined by (28). On the basis of Theorem 2 the arc with the least value of $\Delta_r[(y, x), (u, v)]$ should be chosen for complementing.

3. The algorithm (from [1]). We start with $D_1 = \langle A, U \cup S^+\rangle$, $F_1 = 0$, and $L^* = \infty$. The graph D_1 represents the root of the solution tree H.

Let $D_r = \langle A, U \cup S_r\rangle$ be the current graph and let F_r be the current set of fixed disjunctive arcs in the r-th iteration of the algorithm.

Step 1. Test step. Compute the lower bound $L(F_r)$ of the graph $D(F_r) = \langle A, U \cup F_r\rangle$. If $L(F_r) \geqslant L^*$, then go to Step 4. Otherwise, go to Step 2.

Step 2. Evaluation step. For each $y \in A$, compute $L_r(0, y)$ by (23). If $L_r(0, z) < L^*$, then set $L^* = L_r(0, z)$.

Identify a critical path C_r and the set K_r defined by (52). If $K_r = \emptyset$, then go to Step 4. Otherwise, compute $\Delta_r[(y, x), (u, v)]$ for each $\langle y, x\rangle \in K_r$ and go to Step 3.

Step 3. Forward step. Choose $\langle y, x\rangle \in K_r$ such that

$$\Delta_r[(y, x), (u, v)] = \min_{\langle a,b\rangle \in K_r} \Delta_r[(a, b), (c, d)].$$

Then generate a new graph $D_s = \langle A, U_s\rangle$ by complementing the arc $\langle y, x\rangle$ and fixing the arc $\langle u, v\rangle$, i.e.

$$U_s = [U_r - \{\langle y, x\rangle\}] \cup \{\langle u, v\rangle\} \quad \text{and} \quad F_s = F_r \cup \{\langle u, v\rangle\}.$$

Simultaneously, add to the solution tree H a new node D_s and a new arc $\langle D_r, D_s\rangle$ associated with the arc $\langle u, v\rangle$ of the disjunctive graph $\bar{D}$. Then go to Step 1.

Step 4. Backtracking step. Backtrack to the predecessor D_p of D_r in H. If D_r has no predecessor, then the algorithm terminates, the representation S^* associated with the current L^* is optimal, and the longest path in D^* is minimaximal in $\bar{D}$. Otherwise, drop data of D_r and update the data for D_p, i.e.

$$K_p = K_p - \{\langle y, x\rangle\} \quad \text{and} \quad F_p = F_p \cup \{\langle y, x\rangle\}.$$

Then go to Step 3.

Now, we present the proof of Theorem 4 from [2].

THEOREM 4. *Let $\bar{D}^a = \langle A, U; V^a \rangle$ be a disjunctive d-partial, d-connected and z-symmetrical graph. Then the set S_0^{az} defined as*

$$(50) \quad S_0^{az} = \{\langle y, x\rangle \in B^a \times A^a \mid (y \neq \Gamma x) \wedge \wedge [\mathrm{War}(y, x) = 0 \vee \mathrm{War}(x, y) = 0] \wedge [\hat{L}(0, \Gamma^{-1}y) \leqslant \hat{L}(0, x)]\}$$

is the optimal solution (optimal selection) of the disjunctive graph $\bar{D}^a$ with minimaximal path L_0^{az}, where $\hat{L}(0, x)$, $x \in A$, is the maximal path from the node 0 *to the node x of the graph D.*

Proof. The assumptions of the theorem are well defined, since from the fact that the graph D has no circuits it follows that the longest path $C(0, x)$ exists for each $x \in A$.

It follows from (50) that

(i) S_0^{az} is a complete selection in $\bar{D}^a$,

(ii) the graph $D_0^a = \langle A, U \cup S_0^{az}\rangle$ has no circuits.

Assume that S_0^{az} is not the optimal representation. Now we should prove that there exists another selection, say S_t^{az}, for which the graph $D_t^a = \langle A, U \cup S_t^{az}\rangle$ has a critical path shorter than L_0^a. In order to obtain the selection S_t^a we generate a sequence of graphs $D_r^a \in R'_{D^a}$ and find the critical path for each D_r^a.

The generation and checking process of graphs proceed just as in the solution algorithm with the following modifications:

S_0^{az} is the initial selection (S^+),

we do not perform the test step,

in the evaluation step we determine only the set K_r of free arcs belonging to the critical path,

in the forward step we choose for complementing any arc of the set K_r.

In the algorithm we complement only the disjunctive arcs belonging to the critical path, therefore the constructed solution tree H represents all graphs D_r^a of the family R'_{D^a} which do not contain a shorter path than in the initial graph $D_0^a = \langle A, U \cup S_0^{az}\rangle$.

Let $D_r^a = \langle A, U \cup S_r^a\rangle$ be any given graph (node in H) generated as above and let F_r^a be the set of fixed arcs. Let K_r^a be a non-empty set of normal arcs. Now we prove that the complementing any arc of the set K_r^a does not cause the generating the successor D_s^a with a critical path longer than in the graph D_r^a. Let us notice that if the arc $\langle y, x\rangle \in K_r^a$ is preceded on the path C_r^a by arcs of the set $S_r^a \cap C_r^a$, then, by Conclusion 1, the generated successor has a critical path not longer than in the graph D_r^a. It should still be proved that if the complemented arc $\langle y, x\rangle$ is preceded by no arc of the set $S_r^a \cap C_r^a$, then the generated successor D_s^a has no critical path shorter than in the graph D_r^a. We conclude from Lemma 3 that for the arc $\langle y, x\rangle \in C_r^a \cap S_r^a$ there exists a node $v \in A^a \cup \{z\}$ which belongs to the critical path and which satisfies

$$C_r^a = C(0, v) \cup C(v, z). \tag{53}$$

However, by the assumptions of Theorem 4 we have (see Fig. 10)

$$C(0, v) = C(0, \Gamma^{-1}y) \cup \{\langle \Gamma^{-1}y, y\rangle\} \cup \{\langle y, x\rangle\} \cup C(x, v).$$

Now let us consider the path $d(0, v)$ of the graph obtained from complementing the arc $\langle y, x\rangle \in K_r^a$:

$$d(0, v) = d(0, x) \cup \{\langle x, \Gamma x\rangle\} \cup \{\langle \Gamma x, \Gamma^{-1}y\rangle\} \cup d(\Gamma^{-1}y, y). \tag{54}$$

The path $d(0, v)$ — if it exists — does not contain a disjunctive arc $\langle y, x\rangle$, however, it contains the complement of $\langle \Gamma x, \Gamma^{-1}y\rangle$. Therefore, $d(0, v)$ is a path in D_s^a, the successor of D_r^a.

Now, we prove that (a) the path $d(0, v)$ exists, and (b) the length of this path is not shorter than the length $L(0, v)$ of the path $C(0, v)$.

(a) In order to prove the existence of the path it should be proved that the first and the last components on the right-hand side of relation (54) exist. It follows from Property 1 that for any node of the set X, particularly for the node x, there exists a path in the graph D. Since D is a partial graph of any graph $D_r^a \in R'_{D^a}$, there exists a path $d(0, x)$ in each of those graphs. Moreover, it follows from the assumptions of the theorem that the length of the maximal path in D is equal to $\hat{L}(0, x)$. Further, by Lemma 3 there exists a path $d(\Gamma^{-1}y, v)$ the length of which is equal to $c(\Gamma^{-1}y, y)$.

(b) On the basis of the above consideration, the length of the path $d(0, v)$ may be expressed by

$$l^d(0, v) = \hat{L}(0, x) + c(x, \Gamma x) + 0 + c(\Gamma^{-1}y, y).$$

However, the length of the path $C(0, v)$ is

$$L(0, v) = L(0, \Gamma^{-1}y) + c(\Gamma^{-1}y, y) + 0 + c(x, \Gamma x).$$

Since the disjunctive arc $\langle y, x\rangle$ is not preceded by any arc of the set $S_r^a \in C_r^a$, the path $C(0, \Gamma^{-1}y)$ is a path in D. By the assumptions of the theorem, the length of the path is $\hat{L}(0, \Gamma^{-1}y)$, i.e.

$$L(0, \Gamma^{-1}y) = \hat{L}(0, \Gamma^{-1}y). \tag{55}$$

Further, since $\langle y, x\rangle \in K_r^a \cap S_0^{az}$, by (50) we have

$$\hat{L}(0, x) \geqslant \hat{L}(0, \Gamma^{-1}y), \tag{56}$$

therefore, by (55) and (56) we obtain

$$l^d(0, v) \geqslant L(0, v). \tag{57}$$

Further, since

$$d(0, z) = d(0, v) \cup C(v, z) \tag{58}$$

is a path from the node 0 to the node z in the graph D_s^a, we have

$$L_s^a(0, z) \geqslant l^d(0, z).$$

By (53), (56) and (58) we have

$$L_s^a(0, z) \geqslant l^d(0, v) + L(v, z) \geqslant L(0, v) + L(v, z) = L_r^a(0, z).$$

To sum up, for any graph D_r^a, the complementing any arc of the set K_r^a does not cause the generating a successor with a critical path shorter than in D_r^a.

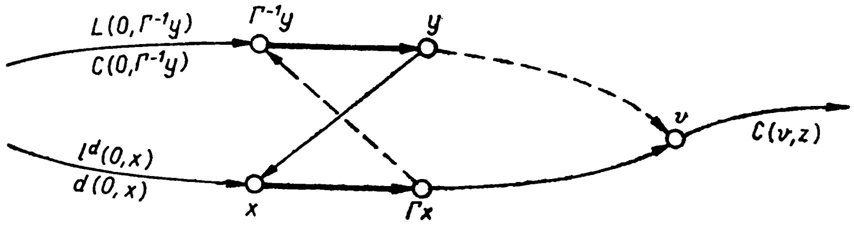

Fig. 10

4. Example. This part of the paper contains an example and the comparison of the solution with the solution obtained and presented in [1].

The data are the following:

Operation No	1	2	3	4	5	6
Processing time of operation	7	11	10	5	9	8

The disjunctive graph for this example is shown in Fig. 11.

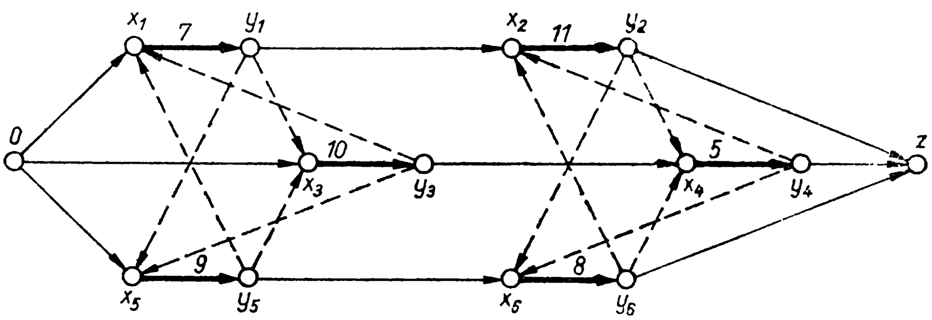

Fig. 11

We have

$$N = \{1, 2, 3, 4, 5, 6\}, \quad Q = \{1, 2\}, \quad X^1 = \{x_1, x_3, x_5\},$$

$$Y^1 = \{y_1, y_2, y_5\}, \quad X^2 = \{x_2, x_4, x_6\}, \quad Y^2 = \{y_2, y_4, y_6\}.$$

It is easily seen that the graph $D^1 = \langle A, U; V^1\rangle$ with the subset X^1 is d-connected and 0-symmetrical. The graph $D^2 = \langle A, U; V^2\rangle$ with the full subset X^2 is d-connected and z-symmetrical. The reduction of weight for both of these graphs is not necessary. Only the graph D^2 with its z-symmetrical properties is used in the algorithm. It can be easily seen that $L_d^z(F_r) \geqslant L_d(F_r)$, i.e.

$$L(F_r) = L_0^z(F_r).$$

The graphs $D_r \in R'_D$ obtained in the sequence of iterations are shown in Figs. 12-17. Broken lines represent the critical paths and the fixed arcs are indicated by additional arrows. The numbers in the rectangles for the nodes $x_j \in X^k$ represent $L'_r(0, x_j)$ and $L_r(0, x_j)$, and those for the nodes $y_j \in Y^k$ represent $L'_r(y_j, z)$ and $L_r(y_j, z)$. In the triangles, the lengths of paths $\hat{L}(0, x_j)$ are given for the nodes $x_j \in X^2$ in the graph $D^2(F_r)$. These path lengths are needed to determine the selection $S_0^{az}(F_r)$ according to Theorem 4. The solution tree H is shown in Fig. 18.

We start from the graph $D_1 = \langle A, U \cup S^+\rangle$, where S^+ is the selection which gives the same initial sequence of the operations as in [1] (Fig. 12).

Iteration 1. $F_1 = \emptyset$ and $L^* = \infty$ (Fig. 12).

(a) Test step:

$$L(F_r) = L_d^z(F_r) = 31 < \infty.$$

(b) Evaluation step: $L(0, z) = 34$, set $L^* = 34$, $K_1 = \{\langle y_1, x_3\rangle, \langle y_3, x_5\rangle\}$,

$$\Delta_1[(y_1, x_3), (y_3, x_1)] = \max[-7, -4, 10+7-7-3] = 7,$$

$$\Delta_1[(y_3, x_5), (y_5, x_3)] = \max[-10, -4, 9+10-10-4] = 5.$$

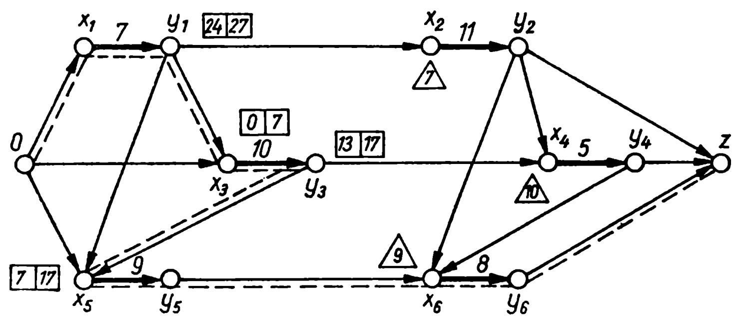

Fig. 12

(c) Forward step: Choose $\langle y_3, x_5\rangle$ and generate D_2 by fixing $\langle y_5, x_3\rangle$.

Iteration 2. $F_2 = \{\langle y_5, x_3\rangle\}$ (Fig. 13).

(a) Test step:

$$L(F_r) = L_d^z(F_r) = 31 < 34.$$

(b) Evaluation step: $L_z(0, z) = 39$, $K_2 = \{\langle y_1, x_5\rangle, \langle y_4, x_5\rangle\}$,

$$\Delta_2[(y_1, x_5), (y_5, x_1)] = \max[-7, -8, 9+7-7-8] = 1,$$

$$\Delta_2[(y_4, x_6), (y_6, x_4)] = \max[-13, -8, 8+5-13-8] = -8.$$

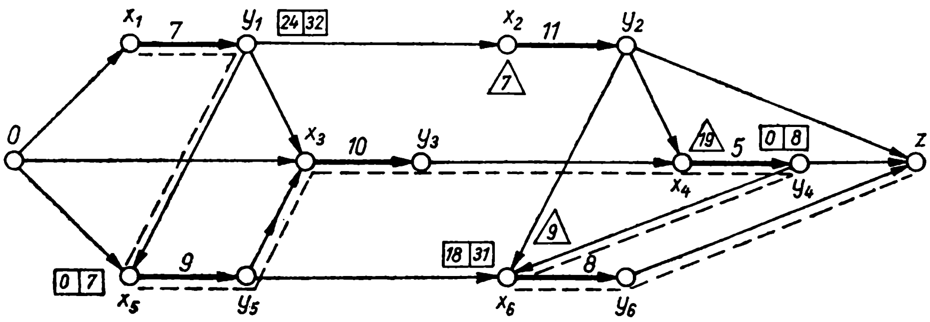

Fig. 13

(c) Forward step: Choose $\langle y_4, x_6\rangle$ and generate D_3 by fixing $\langle y_6, x_4\rangle$.

Iteration 3. $F_3 = \{\langle y_5, x_3\rangle, \langle y_6, x_4\rangle\}$ (Fig. 14).

(a) Test step:

$$L(F_r) = L_d^z(F_r) = 31 < 34.$$

(b) Evaluation step: $L_3(0, z) = 31$, set $L^* = 31$, $K_3 = \{\langle y_2, x_6\rangle\}$,

$$\Delta_3[(y_2, x_6), (y_6, x_2)] = \max[-2, -8, 8+11-2-8] = 9.$$

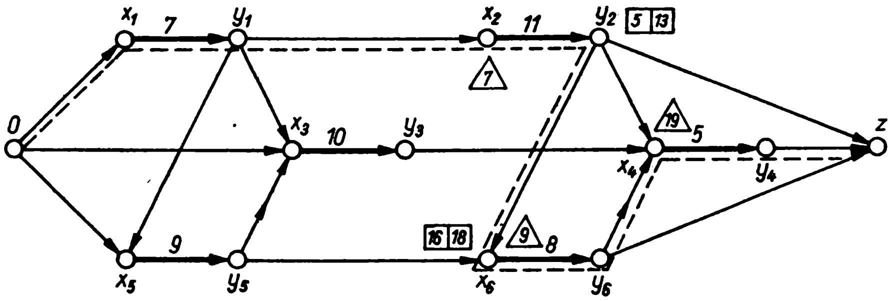

Fig. 14

(c) Forward step: Choose $\langle y_2, x_6\rangle$ and generate D_4 by fixing $\langle y_6, x_2\rangle$.

Iteration 4. $F_4 = \{\langle y_5, x_3\rangle, \langle y_6, x_4\rangle, \langle y_6, x_2\rangle\}$ (Fig. 15).

(a) Test step:

$$L(F_r) = L_d^z(F_r) = 33 > 31.$$

(b) Backtracking step: Backtrack. Introduce $\langle y_2, x_6\rangle$ into F_3. Then $K_3 = \emptyset$, hence again: Backtrack. Introduce $\langle y_4, x_6\rangle$ into F_2. Then $K_2 = \{\langle y_1, x_5\rangle\}$.

(c) Forward step: Generate D_5 by fixing $\langle y_5, x_1\rangle$.

Iteration 5. $F_5 = \{\langle y_5, x_3\rangle, \langle y_4, x_6\rangle, \langle y_5, x_1\rangle\}$ (Fig. 16).

(a) Test step:

$$L(F_r) = L_d^z(F_r) = 40 > 31.$$

(b) Backtracking step: Backtrack. Introduce $\langle y_3, x_5\rangle$ into F_1. Then $K_1 = \{\langle y_1, x_3\rangle\}$.

(c) Forward step: Generate D_6 by fixing $\langle y_3, x_1\rangle$.

Iteration 6. $F_6 = \{\langle y_3, x_5\rangle, \langle y_3, x_1\rangle\}$ (Fig. 17).

(a) Test step:

$$L(F_r) = L_d^z(F_r) = 36 > 31.$$

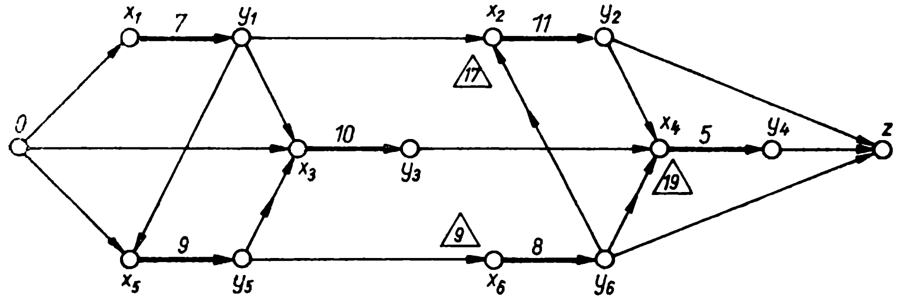

Fig. 15

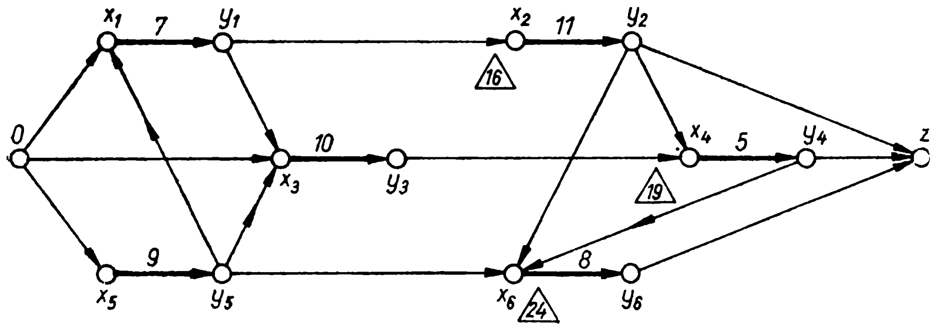

Fig. 16

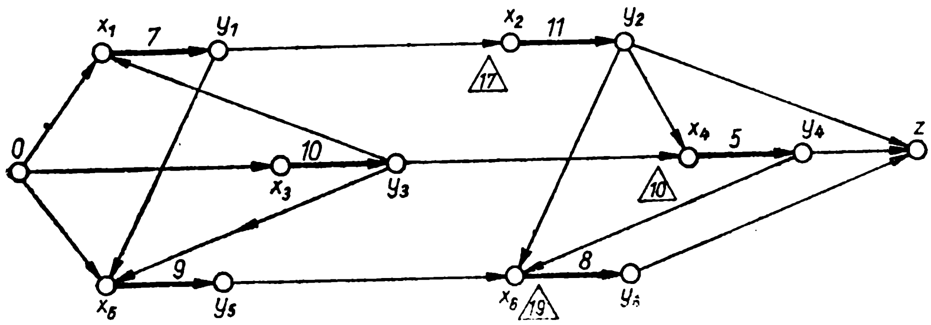

Fig. 17

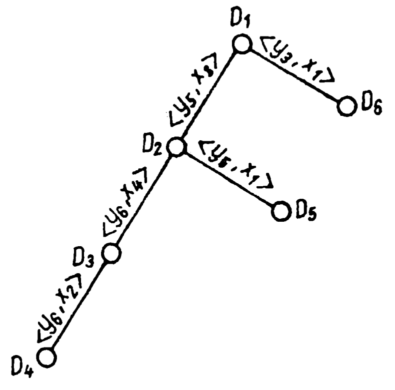

Fig. 18

(b) Backtracking step: Backtrack. Introduce $\langle y_1, x_3\rangle$ into F_1. Then $K_1 = \emptyset$, hence: Backtrack. End.

5. Remarks and conclusions.

1. The solution in our example was obtained after six iterations, while in [1] it was obtained after twice more iterations. It is worth mentioning that if the initial selection S^+ is an optimal representation (i.e. the selection S_3 obtained after three iterations), to obtain the solution one has to perform only two iterations. There is no such possibility in the algorithm given in [1]. The problem still needs twelve iterations.

2. It seems that there may appear computational difficulties in the reduction of the graphs and in the determination of the selections S_0^{a0} and S_0^{az}. In order to obtain a z-symmetrical or 0-symmetrical graph the reductions (a) and (b) should be executed only once at the beginning of the computation, i.e. for $F_r = \emptyset$, $A^k(F_r) = X^k$ and $B^k(F_r) = Y^k$. This follows from the fact that for any $F_r \neq \emptyset$ we have $A^k(F_r) \subset X^k$ and $B^k(F_r) \subset Y^k$.

There exist many sequencing problems, where for $k_1 \in Q$ the graph $D^{k_1}(F_r)$ is z-symmetrical, whereas for $k_2 \in Q$ the graph $D^{k_2}(F_r)$ is 0-symmetrical. Then relations (51′) and (51″) can be reduced to $k = k_1$ and $k = k_2$. Furthermore, we can aim to obtain a graph which is either z-symmetrical or 0-symmetrical, which, in turn, allows to increase considerably the efficiency of the algorithm (see the Example).

3. We can always find such $k_1, k_2 \in Q$ and $A^{a_1} \subset X^{k_1}$, $B^{a_2} \subset Y^{k_2}$ that for each sequencing problem the graph $D^{a_1} = \langle A, U; V^{a_1}\rangle$ is 0-symmetrical and the graph $D^{a_2} = \langle A, U; V^{a_2}\rangle$ is z-symmetrical. The set A^{a_1} contains the nodes representing the start of the operations which have no predecessors (the set $N_x \cap N^{k_1}$). The set B^{a_2} contains the nodes representing the finish of the operations which have no successors (the set $N_y \cap N^{k_2}$). The graphs $D^{a_1}(F_r)$ and $D^{a_2}(F_r)$ generated in such a way need no weight reductions in any algorithm iteration.

References

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[2] J. Grabowski, *A new formulation and solution of the sequencing problem: mathematical model*, Zastosow. Matem. 15 (1976), p. 325-343.

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NOWE SFORMUŁOWANIE I ROZWIĄZANIE ZAGADNIENIA KOLEJNOŚCIOWEGO: ALGORYTM

STRESZCZENIE

W pracy [2] przedstawiono nowe sformułowanie zagadnienia kolejnościowego, prowadzące do nowej konstrukcji grafu dysjunktywnego. W niniejszej pracy przedstawia się algorytm rozwiązania tego zagadnienia kolejnościowego, wykorzystując własności opisane w pracy [2].
