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Z. W. BIRNBAUM and B. P. LIENTZ (Seattle)

EXACT DISTRIBUTIONS FOR SOME RÉNYI-TYPE STATISTICS (1)

1. Introduction. Let $X_1, \ldots, X_n$ be a random sample of size n with common continuous distribution function F, and let F_n denote the empirical distribution function of this sample. Let $U_1, \ldots, U_n$ be the values of $F(X_1), \ldots, F(X_n)$ ordered increasingly, i.e. so that U_1 is the smallest of $F(X_i)$ and U_n the largest of $F(X_i)$, $1 \leqslant i \leqslant n$. Since $U = F(X)$ has the uniform distribution on $[0, 1]$, $U_1, \ldots, U_n$ are an ordered sample from a uniformly distributed random variable on $[0, 1]$.

The purpose of this paper is to develop the exact distributions of various statistics defined in terms of F_n, F, and truncation or censoring values.

In his basic paper [7], A. Rényi obtained the limiting distributions of the statistics

$$\sup_{a \leqslant F(x) \leqslant b} \frac{F_n(x) - F(x)}{F(x)} \quad \text{and} \quad \sup_{F(x) \geqslant a} \frac{F_n(x) - F(x)}{F(x)}.$$

M. Csörgö [4] later found the limiting distributions of

$$\sup_{a \leqslant F_n(x) \leqslant b} \frac{F_n(x) - F(x)}{F_n(x)}, \quad \sup_{F_n(x) \geqslant a} \frac{F_n(x) - F(x)}{F_n(x)}, \quad \sup_{a \leqslant F(x) \leqslant b} \frac{F_n(x) - F(x)}{1 - F(x)},$$

$$\sup_{F(x) \leqslant b} \frac{F_n(x) - F(x)}{1 - F(x)}, \quad \sup_{a \leqslant F_n(x) \leqslant b} \frac{F_n(x) - F(x)}{1 - F_n(x)}, \quad \text{and} \quad \sup_{F_n(x) \leqslant b} \frac{F_n(x) - F(x)}{1 - F_n(x)}.$$

In [3] Csörgö obtained the exact distributions of

$$\sup_{F_n(x) \geqslant a} \frac{F_n(x) - F(x)}{F_n(x)} \quad \text{and} \quad \sup_{a \leqslant F(x)} \frac{F_n(x) - F(x)}{F(x)}.$$

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Both of the distributions given require corrections (see next section, Corollary 2.4.1 and Corollary 2.6.1).

In this paper we consider the following Rényi-type statistics:

$$\sup_{a \leq F(x) \leq b} \frac{F_n(x)-F(x)}{F(x)}, \tag{1.1.1}$$

$$\sup_{a \leq F(x) \leq b} \big(F_n(x)-F(x)\big), \tag{1.1.2}$$

$$\sup_{a \leq F(x) \leq b} \frac{F_n(x)-F(x)}{1-F(x)}, \tag{1.1.3}$$

$$\sup_{F(x) \leq b} \frac{F_n(x)-F(x)}{F(x)}, \tag{1.2.1}$$

$$\sup_{F(x) \leq b} \big(F_n(x)-F(x)\big), \tag{1.2.2}$$

$$\sup_{F(x) \leq b} \frac{F_n(x)-F(x)}{1-F(x)}, \tag{1.2.3}$$

$$\sup_{F(x) \geq a} \frac{F_n(x)-F(x)}{F(x)}, \tag{1.3.1}$$

$$\sup_{F(x) \geq a} \big(F_n(x)-F(x)\big), \tag{1.3.2}$$

$$\sup_{F(x) \geq a} \frac{F_n(x)-F(x)}{1-F(x)}, \tag{1.3.3}$$

$$\sup_{F_n(x) \leq b} \frac{F_n(x)-F(x)}{F_n(x)}, \tag{1.4.1}$$

$$\sup_{F_n(x) \leq b} \big(F_n(x)-F(x)\big), \tag{1.4.2}$$

$$\sup_{F_n(x) \leq b} \frac{F_n(x)-F(x)}{1-F_n(x)}, \tag{1.4.3}$$

$$\sup_{F_n(x) \geq a} \frac{F_n(x)-F(x)}{F_n(x)}, \tag{1.5.1}$$

$$\sup_{F_n(x) \geq a} \big(F_n(x)-F(x)\big), \tag{1.5.2}$$

$$\sup_{F_n(x) \geq a} \frac{F_n(x)-F(x)}{1-F_n(x)}. \tag{1.5.3}$$

In the remainder of the paper we obtain the exact probability distributions for these statistics. The distributions for (1.3.1) and (1.5.1), presented in [3], are given here in a corrected version. The statistic (1.5.3), listed above for the sake of completeness, is of no practical interest since one can easily verify that the probability of its assuming a value $\leqslant c$ is 0 for every $0 \leqslant c < \infty$.

2. Exact Distributions. The following lemma was obtained by R. Pyke [6]. For the sake of completeness its proof is outlined here.

LEMMA 2.1. *For any real d, s, and positive integer k, let*

$$f(s; d, k) = P[\max_{1 \leqslant j \leqslant k} (dj - U_j) \leqslant s].$$

Then

$$(2.1) \quad f(s; d, k) =$$

$$\begin{cases} 0 & \text{if} \quad kd - s \geqslant 1, \\ & \phantom{\text{if}} \quad 0 \leqslant d \leqslant 1, \\ (1+s-kd) \sum\limits_{j=0}^{\rfloor s/d \lfloor} \binom{k}{j} (jd-s)^j (1+s-jd)^{k-j-1} & \text{if} \quad 0 \leqslant kd - s < 1, \\ & \phantom{\text{if}} \quad 0 \leqslant d \leqslant 1, \\ 1 & \text{if} \quad kd - s < 0, \\ & \phantom{\text{if}} \quad 0 \leqslant d \leqslant 1, \end{cases}$$

where $\rfloor p \lfloor$ denotes the greatest integer smaller than p if $p > 0$ and $\rfloor p \lfloor = 0$ if $p \leqslant 0$.

For example $\rfloor 3.6 \lfloor = 3$, $\rfloor 3 \lfloor = 2$, and $\rfloor -2 \lfloor = 0$.

Proof. The first and the last line of (2.1) follow immediately from

$$P[\max_{1 \leqslant j \leqslant k} (dj - U_j) \leqslant s] = P[U_j \geqslant dj - s, \text{ for } 1 \leqslant j \leqslant k]$$

and

$$P[U_k \geqslant 1] = 0, \quad P[U_1 > 0] = 1.$$

For $0 \leqslant kd - s < 1$, $0 \leqslant d \leqslant 1$, one has

$$f(s; d, k) = k! \int\limits_{U_k = kd-s}^{1} \int\limits_{U_{k-1}=(k-1)d-s}^{U_k} \cdots \int\limits_{U_L = Ld-s}^{U_{L+1}} \int\limits_{U_{L-1}=0}^{U_L} \cdots \int\limits_{U_1=0}^{U_2} dU_1 \ldots \ldots dU_{L-1} dU_L \ldots dU_{k-1} dU_k$$

where $L = \rfloor s/d \lfloor + 1$. Applying the method used in [2] one obtains the expression in the middle line of (2.1).

THEOREM 2.2. *For $0 \leqslant a \leqslant b \leqslant 1$, d real, and $0 < \Delta < b-a$, one has*

$$
(2.2)\quad P[U_j > j\Delta + d, \text{ for } a \leqslant U_j \leqslant b,\ 1 \leqslant j \leqslant n] = \sum_{k=0}^{n} \binom{n}{k} a^k (1-b)^{n-k} +
$$

$$
+ \sum_{\substack{k=0 \\ (k+1 \leqslant m)}}^{S-1} \sum_{m=R+1}^{S} \frac{n!}{k!(m-k)!(n-m)!} a^k (b-a)^{m-k} (1-b)^{n-m} \frac{(b-d-m\Delta)}{(b-a)^{m-k}} \times
$$

$$
\times \sum_{j=0}^{]\frac{a-d-k\Delta}{\Delta}[} \binom{m-k}{j} \big((j+k)\Delta + d - a\big)^j \big(b-d-(j+k)\Delta\big)^{m-k-j-1} +
$$

$$
+ \sum_{\substack{k=0 \\ (k+1 \leqslant m)}}^{R-1} \sum_{m=1}^{R} \frac{n!}{k!(m-k)!(n-m)!} a^k (b-a)^{m-k} (1-b)^{n-m}
$$

where

$$
(2.2.0.1)\qquad R = \left]\frac{a-d}{\Delta}\right[, \qquad S = \min\left\{\left]\frac{b-d}{\Delta}\right[, n\right\}.
$$

If $R = 0$, define the last sum to be 0.

Proof. By decomposition one has

$$
P[U_j > j\Delta + d, \text{ for } a \leqslant U_j \leqslant b,\ 1 \leqslant j \leqslant n] = \sum_{k=0}^{n} P[U_k < a,\ U_{k+1} > b] +
$$

$$
\sum_{\substack{k=0 \\ (k+1 \leqslant m)}}^{n-1} \sum_{m=1}^{n} P[U_j > j\Delta + d,\ k+1 \leqslant j \leqslant m,\ U_k < a \leqslant U_{k+1} \text{ and } U_m \leqslant b < U_{m+1}].
$$

For R and S defined by (2.2.0.1) one has $R\Delta + d < a \leqslant (R+1)\Delta + d$ and $S\Delta + d < b \leqslant (S+1)\Delta + d$. Also,

$$
\begin{aligned}
&P[U_j > j\Delta + d, \text{ for } k+1 \leqslant j \leqslant m,\ U_k < a \leqslant U_{k+1} \text{ and } U_m \leqslant b < U_{m+1}] \\
&= P[U_j > j\Delta + d, \text{ for } k+1 \leqslant j \leqslant m \mid U_k < a \leqslant U_{k+1} \text{ and } U_m \leqslant b < U_{m+1}] \times \\
&\quad \times P[U_k < a \leqslant U_{k+1} \text{ and } U_m \leqslant b < U_{m+1}].
\end{aligned}
$$

The general summation can now be split into parts using

$$
P[U_j > j\Delta + d, \text{ for } k+1 \leqslant j \leqslant m \mid \ldots\}
$$

which is a consequence of the definition of R and S. Noting that

$$P[U_k < a \leqslant U_{k+1},\ U_m \leqslant b < U_{m+1}] = \frac{n!}{k!(m-k)!(n-m)!}\, a^k (b-a)^{m-k}(1-b)^{n-m}$$

one obtains, for $m \leqslant R$, the last sum in (2.2).

For $R < m \leqslant S$, one obtains

$$P[U_j > j\Delta + d,\ \text{for } k+1 \leqslant j \leqslant m \mid \ldots] = P\left[\max_{1\leqslant j-k\leqslant m-k} \left\{\frac{(j-k)\Delta}{b-a} - \frac{U_j - a}{b-a}\right\} < \frac{a-d-k\Delta}{b-a} \mid \ldots\right].$$

Set $U'_{j-k} = (U_j - a)/(b-a)$ and replace $j-k$ by j. This amounts to a normalization of those U_j where $k+1 \leqslant j \leqslant m$ so that the resulting U'_j are the order statistics of a sample of size $m-k$ from a uniform distribution in $(0,1)$. By Lemma 2.1 for $0 < \Delta/(b-a) < 1$, one obtains

$$(2.2.0.2)\quad P\left[\max_{1\leqslant j\leqslant m-k} \left(\frac{j\Delta}{b-a} - U'_j\right) < \frac{a-d-k\Delta}{b-a} \mid \ldots\right]$$

$$= \frac{b-d-m\Delta}{(b-a)^{m-k}} \sum_{j=0}^{\rfloor\frac{a-d-k\Delta}{\Delta}\lfloor} \binom{m-k}{j} [(j+k)\Delta - a + d]^j [b-d-(j+k)\Delta]^{m-k-j-1}$$

for $(a-d)/\Delta \leqslant m \leqslant (b-d)/\Delta$, $(R < m \leqslant S)$, and the probability (2.2.0.2) is 0 for $m \geqslant (b-d)/\Delta$, $(m > S)$ and is 1 for $m < (a-d)/\Delta$, $(m \leqslant R)$. Using these expressions one obtains the middle sum of (2.2). The first sum is the probability that none of the U_j fall into $[a, b]$.

COROLLARY 2.2.1. *For $0 < a < b < 1$ and $0 < 1/n(1+c) < b-a$, one has for the probability distribution function of the statistic* (1.1.1) *the expression*

$$(2.2.1)\quad P\left[\sup_{a\leqslant F(x)\leqslant b} \frac{F_n(x) - F(x)}{F(x)} < c\right] = \sum_{k=0}^{n} \binom{n}{k} a^k (1-b)^{n-k} +$$

$$+ \sum_{\substack{k=0\\(k+1\leqslant m)}}^{R-1} \sum_{m=1}^{R} \frac{n!}{k!(m-k)!(n-m)!} a^k (b-a)^{m-k}(1-b)^{n-m} +$$

$$+ \sum_{\substack{k=0\\(k+1\leqslant m)}}^{S-1} \sum_{m=R+1}^{S} \frac{n!}{k!(m-k)!(n-m)!} a^k (1-b)^{n-m} \left(b - \frac{m}{n(1+c)}\right) \times$$

$$\times \sum_{j=0}^{\rfloor na(1+c)-k\lfloor} \binom{m-k}{j} \left(\frac{j+k}{n(1+c)} - a\right)^j \left(b - \frac{j+k}{n(1+c)}\right)^{m-k-j-1}$$

where $R = \rfloor an(1+c)\lfloor$ and $S = \min\{\rfloor bn(1+c)\lfloor, n\}$.

Proof. First, note that

$$P\left[\sup_{a\leqslant F(x)\leqslant b}\frac{F_n(x)-F(x)}{F(x)}<c\right]=P\left[F(x)>\frac{F_n(x)}{1+c}\ \text{and}\ a\leqslant F(x)\leqslant b\right].$$

Since the supremum occurs at the sample points, one has

$$P\left[\sup_{a\leqslant F(x)\leqslant b}\frac{F_n(x)-F(x)}{F(x)}<c\right]=P\left[U_j>\frac{j}{n(1+c)}\ \text{and}\ a\leqslant U_j\leqslant b\right].$$

Setting $d=0$ and $\Delta=1/n(1+c)$ in (2.2) one obtains (2.2.1).

COROLLARY 2.2.2. *For* $0<a<b<1$ *and* $0<1/n(b-a)<1$, *one has for the probability distribution function of the statistic* (1.1.2) *the expression*

$$P\Big[\sup_{a\leqslant F(x)\leqslant b}\big(F_n(x)-F(x)\big)<c\Big]=\sum_{k=0}^{n}\binom{n}{k}a^k(1-b)^{n-k}+ \tag{2.2.2}$$

$$+\sum_{k=0}^{R-1}\sum_{\substack{m=1\\(k+1\leqslant m)}}^{R}\frac{n!}{k!(m-k)!(n-m)!}a^k(b-a)^{m-k}(1-b)^{n-m}+$$

$$+\sum_{k=0}^{S-1}\sum_{\substack{m=R+1\\(k+1\leqslant m)}}^{S}\frac{n!}{k!(m-k)!(n-m)!}a^k(1-b)^{n-m}\left(b+c-\frac{m}{n}\right)\times$$

$$\times\sum_{j=0}^{\lfloor n(a+c)-k\rfloor}\binom{m-k}{j}\left(\frac{j+k}{n}-c-a\right)^j\left(b+c-\frac{j+k}{n}\right)^{m-k-j-1}$$

where $R=\lfloor n(a+c)\rfloor$ *and* $S=\min\{\lfloor n(b+c)\rfloor,n\}$.

Proof. As in the proof of 2.2.1, one has

$$P\Big[\sup_{a\leqslant F(x)\leqslant b}\{F_n(x)-F(x)\}<c\Big]=P\big[F(x)>F_n(x)-c\ \text{and}\ a\leqslant F(x)\leqslant b\big]$$
$$=P\left[U_j>\frac{j}{n}-c\ \text{and}\ a\leqslant U_j\leqslant b\right].$$

Substituting $\Delta=1/n$ and $d=-c$ in (2.2), one obtains (2.2.2).

COROLLARY 2.2.3. *For* $0<a<b<1$ *and* $0<1/n(1-c)<b-a$, *one has for the probability distribution function of* (1.1.3)

$$P\left[\sup_{a\leqslant F(x)\leqslant b}\frac{F_n(x)-F(x)}{1-F(x)}<c\right]=\sum_{k=0}^{n}\binom{n}{k}a^k(1-b)^{n-k}+ \tag{2.2.3}$$

$$+\sum_{k=0}^{R-1}\sum_{\substack{m=1\\(k+1\leqslant m)}}^{R}\frac{n!}{k!(m-k)!(n-m)!}a^k(b-a)^{m-k}(1-b)^{n-m}+$$

$$+\sum_{\substack{k=0\\(k+1\leqslant m)}}^{S-1}\sum_{m=R+1}^{S}\frac{n!}{k!(m-k)!(n-m)!}a^k(1-b)^{n-m}\left(b+\frac{c}{1-c}-\frac{m}{n(1-c)}\right)\times$$

$$\times\sum_{j=0}^{\lfloor an(1-c)+nc-k\rfloor}\binom{m-k}{j}\left(\frac{j+k}{n(1-c)}-a-\frac{c}{1-c}\right)^j\left(b+\frac{c}{1-c}-\frac{j+k}{n(1-c)}\right)^{m-k-j-1}$$

where $R=\lfloor an(1-c)+nc\rfloor$ *and* $S=\min\{\lfloor bn(1-c)+nc\rfloor, n\}$.

Proof. As in 2.2.1 one has

$$P\left[\sup_{a\leqslant F(x)\leqslant b}\frac{F_n(x)-F(x)}{1-F(x)}<c\right]=P\left[F(x)>\frac{F_n(x)}{1-c}-\frac{c}{1-c}\right.$$

$$\left.\text{and } a\leqslant F(x)\leqslant b\right]=P\left[U_j>\frac{j}{n(1-c)}-\frac{c}{1-c}\text{ and } a\leqslant U_j\leqslant b\right].$$

Substituting $\Delta=1/n(1-c)$ and $d=-c/(1-c)$ in (2.2) gives (2.2.3).

The following Theorems 2.3 and 2.4 are immediate consequences of Theorem 2.2 and each in turn has a number of useful corollaries.

THEOREM 2.3. *For* $0<b<1$, d *real, and* $0<\Delta<b$

$$P[U_j>j\Delta+d,\ U_j\leqslant b,\ 1\leqslant j\leqslant n]=(1-b)^n+ \tag{2.3}$$

$$+\sum_{m=R+1}^{S}\binom{n}{m}(1-b)^{n-m}(b-d-m\Delta)\times$$

$$\times\sum_{j=0}^{\lfloor -d/\Delta\rfloor}\binom{m}{j}(j\Delta+d)^j(b-d-j\Delta)^{m-j-1}+\sum_{m=1}^{R}\binom{n}{m}b^m(1-b)^{n-m}$$

where $S=\min\left\{\left\lfloor\frac{b-d}{\Delta}\right\rfloor, n\right\}$ *and* $R=\left\lfloor\frac{-d}{\Delta}\right\rfloor$.

Proof. Set $a=0$, and $k=0$ in theorem 2.2 to obtain (2.3). Note that the last sum in (2.2) disappears.

COROLLARY 2.3.1. *For* $0<b<1$ *and* $0<1/n(1+c)<b$, *one has for the statistic* (1.2.1) *the probability distribution function*

$$P\left[\sup_{F(x)\leqslant b}\frac{F_n(x)-F(x)}{F(x)}<c\right]=(1-b)^n+ \tag{2.3.1}$$

$$+\sum_{m=1}^{S}\binom{n}{i}(1-b)^{n-m}\left(b-\frac{m}{n(1+c)}\right)\cdot\left\{\sum_{j=0}^{0}\binom{m}{j}\left(\frac{j}{n(1+c)}\right)^j\left(b-\frac{j}{n(1+c)}\right)^{m-j-1}\right\}$$

$$=(1-b)^n+\sum_{m=1}^{S}\binom{n}{m}(1-b)^{n-m}b^{m-1}\left(b-\frac{m}{n(1+c)}\right)$$

where $S=\min\{n, \lfloor bn(1+c)\rfloor\}$.

Proof. As in Corollary 2.2.1, one has

$$P\left[\sup_{F(x)\leqslant b}\left\{\frac{F_n(x)-F(x)}{F(x)}\right\}<c\right]=P\left[F(x)>\frac{F_n(x)}{(1+c)}\text{ and }F(x)\leqslant b\right]$$

$$=P\left[U_j>\frac{j}{n(1+c)},\ U_j\leqslant b\right].$$

Set $\Delta=1/n(1+c)$ and $d=0$ in (2.3) to obtain (2.3.1).

COROLLARY 2.3.2. *For $0<b<1$ and $0<1/n<b$, one has for the statistic* (1.2.2) *the probability distribution function*

$$P\left[\sup_{F(x)\leqslant b}\left(F_n(x)-F(x)\right)<c\right]=(1-b)^n+\sum_{i=1}^{S}\binom{n}{i}(1-b)^{n-m}\times \tag{2.3.2}$$

$$\times\left(b+c-\frac{m}{n}\right)\sum_{j=0}^{]nc[}\binom{m}{j}\left(\frac{j}{n}-c\right)^j\left(b+c-\frac{j}{n}\right)^{m-j-1}$$

where $S=\min\{n,\ ]n(b+c)[\}$.

Proof.

$$P[\sup_{F(x)\leqslant b}\{F_n(x)-F(x)\}<c]=P[F(x)>F_n(x)-c\text{ and }F(x)\leqslant b]$$

$$=P\left[U_j>\frac{j}{n}-c\text{ and }U_j\leqslant b\right].$$

Putting $\Delta=1/n$ and $d=-c$ in (2.3) one obtains (2.3.2).

COROLLARY 2.3.3. *For $0<b<1$ and $0<1/n(1-c)<b$, one has for the statistic* (1.2.3) *the probability distribution function*

$$P\left[\sup_{F(x)\leqslant b}\left\{\frac{F_n(x)-F(x)}{1-F(x)}\right\}<c\right]=(1-b)^n+\sum_{m=1}^{S}\binom{n}{m}(1-b)^{n-m}\times \tag{2.3.3}$$

$$\times\left(b+\frac{c}{1-c}-\frac{m}{n(1-c)}\right)\sum_{j=0}^{]nc[}\binom{m}{j}\left(\frac{j}{n(1-c)}-\frac{c}{1-c}\right)^j\times$$

$$\times\left(b+\frac{c}{1-c}-\frac{j}{n(1-c)}\right)^{m-j-1}$$

where $S=\min\{]bn(1-c)+nc[,\ n\}$.

Proof.

$$P\left[\sup_{F(x)\leqslant b}\frac{F_n(x)-F(x)}{1-F(x)}<c\right]=P\left[F(x)>\frac{F_n(x)}{1-c}-\frac{c}{1-c}\text{ and }F(x)\leqslant b\right]$$

$$=P\left[U_j>\frac{j}{n(1-c)}-\frac{c}{1-c}+U_j\leqslant b\right].$$

Set $\Delta=1/n(1-c)$ and $d=-c/(1-c)$ in Theorem 2.3 to produce (2.3.3).

THEOREM 2.4. *For $0 < a < 1$ and $0 < \Delta/(1-a) < 1$*

$$(2.4)\qquad P[U_j > j\Delta + d,\ U_j \geqslant a,\ 1 \leqslant j \leqslant n] = a^n + \sum_{k=0}^{S-1} \binom{n}{k} a^k (1 - n\Delta - d) \times$$

$$\times \sum_{j=0}^{\lfloor \frac{a-d-k\Delta}{\Delta} \rfloor} \binom{n-k}{j} \big((j+k)\Delta - a + d\big)^j \big(1 - d - (j+k)\Delta\big)^{n-k-j-1}$$

where $S = \min\left\{\left\lfloor \dfrac{1-d}{\Delta} \right\rfloor, n\right\}$.

Proof. Set $b = 1$, $m = n$ and note that $\sum\limits_{k=0}^{n} P[a > U_k\ ,\ U_{k+1} > 1] = a^n$ in (2.2). Then the last sum in (2.2) disappears and the result is (2.4).

COROLLARY 2.4.1. *For $0 < a < 1$ and $0 < c/(1+c) < b$, the probability distribution function of the statistic* (1.3.1) *is*

$$(2.4.1)\qquad P\left[\sup_{F(x)\geqslant a} \frac{F_n(x) - F(x)}{F(x)} < c\right] = a^n + \sum_{k=0}^{n-1} \binom{n}{k} a^k \frac{c}{1+c} \times$$

$$\times \sum_{j=0}^{\lfloor an(1+c)-k \rfloor} \binom{n-k}{j} \left(\frac{j+k}{n(1+c)} - a\right)^j \left(1 - \frac{j+k}{n(1+c)}\right)^{n-k-j-1}.$$

Proof.

$$P\left[\sup_{F(x)\geqslant a} \frac{F_n(x) - F(x)}{F(x)} < c\right] = P\left[F(x) > \frac{F_n(x)}{1+c},\ F(x) \geqslant a\right]$$

$$= P\left[U_j > \frac{j}{n(1+c)},\ U_j \geqslant a\right].$$

Set $\Delta = 1/n(1+c)$ and $d = 0$ in (2.4) to obtain (2.4.1) (2).

COROLLARY 2.4.2. *For $0 < a < 1$, the probability distribution function of the statistic* (1.3.2) *is*

$$(2.4.2)\qquad P[\sup_{F(x)\geqslant a} \{F_n(x) - F(x)\} < c] = a^n +$$

$$+ \sum_{k=0}^{n-1} \binom{n}{k} a^k (c) \sum_{j=0}^{\lfloor n(a+c)-k \rfloor} \binom{n-k}{j} \left(\frac{j+k}{n} - a - c\right)^j \left(1 + c - \frac{j+k}{n}\right)^{n-k-j-1}$$

where $R = \lfloor n(a+c) \rfloor$.

(2) As mentioned in the introduction, M. Csörgö in [3] obtained for the exact distribution of the random variable in 2.4.1. the expression

$$\sum_{j=K+1}^{n} \binom{n}{j} \left(1 - \frac{j}{n(1+c)} - \frac{c}{1+c}\right)^{n-j} \left(\frac{j}{n(1+c)} + \frac{c}{1+c}\right)^{j-1} \frac{c}{1+c}.$$

It is easily verified that this is not the same as (2.4.1). For example for $n = 3$, $c = 1/3$, $a = 1/2$, (2.4.1) and direct calculation gives a probability of 45/64 while Csörgö's expression yields 25/64.

Proof.

$$P[\sup_{F(x)\geqslant a}\{F_n(x)-F(x)\}<c] = P[F(x)>F_n(x)-c \text{ and } F(x)\geqslant a]$$
$$= P\left[U_j>\frac{j}{n}-c,\, U_j\geqslant a\right].$$

Substitution of $\Delta = 1/n$ and $d = -c$ in (2.4) yields (2.4.2).

COROLLARY 2.4.3. *For $0<a<1$, the probability distribution function of the statistic* (1.3.3) *is*

$$P\left[\sup_{F(x)\geqslant a}\frac{F_n(x)-F(x)}{1-F(x)}<c\right] = a^n. \tag{2.4.3}$$

Proof.

$$P\left[\sup_{F(x)\geqslant a}\frac{F_n(x)-F(x)}{1-F(x)}<c\right] = P\left[U_j>\frac{j}{n(1-c)}-\frac{c}{1-c},\, U_j\geqslant a\right]+a^n.$$

Since

$$P\left[U_j>\frac{j}{n(1-c)}-\frac{c}{1-c},\, U_j\geqslant a\right]\leqslant P\left[U_n>\frac{n}{n(1-c)}-\frac{c}{1-c}=1\right]=0,$$

one has (2.4.3).

THEOREM 2.5. *For v integer, $1\leqslant v\leqslant n$, $0<\Delta/(v\Delta+d)<1$, $0<v\Delta+d<1$, and $\Delta>0$,*

$$P[U_j>j\Delta+d,\, 1\leqslant j\leqslant v] \tag{2.5}$$
$$= (1-v\Delta-d)^n+\sum_{i=1}^{M-1}\binom{n}{i}(v\Delta+d)^i(1-v\Delta-d)^{n-i}+$$
$$+\sum_{i=M}^{v-1}\binom{n}{i}(1-v\Delta-d)^{n-i}\big((v-i)\Delta\big)\sum_{j=0}^{[-d/\Delta]}\binom{i}{j}(j\Delta+d)^j\big((v-j)\Delta\big)^{i-j-1}$$

where M is such that $M\Delta+d\geqslant 0\geqslant(M-1)\Delta+d$, and if $M=0$ or 1, set the first sum equal to zero.

Proof: The pattern of the proof follows that of Theorem 2.2. However, instead of "splitting" at b, the method will be to "split" at $L\Delta+d$ which by hypothesis lies in $(0,1)$.

One notes that if $v\Delta+d<0$ there is nothing to compute and if $v\Delta+d>1$, then set the new v equal to the maximum integer $q<v$ such that $q\Delta+d<1$. The same argument that appears below then applies to this new v.

By decomposition one has

$$(2.5.0.1)\qquad P[U_j > j\Delta+d, 1 \leqslant j \leqslant v] = \sum_{i=1}^{v-1} P[U_j > j\Delta+d, 1 \leqslant j \leqslant i \mid U_i < v\Delta+d < U_{i+1}]P[U_i \leqslant v\Delta+d < U_{i+1}]+P[U_1 > v\Delta+d].$$

Examining the general conditional probability term, one has

$$P[U_j > j\Delta+d, 1 \leqslant j \leqslant i \mid \ldots] = P\left[\max_{1\leqslant j\leqslant i}\left\{\frac{j\Delta}{v\Delta+d} - \frac{U_j}{v\Delta+d}\right\} < \frac{-d}{v\Delta+d}\,\Big|\ldots\right].$$

Applying Lemma (2.1) one obtains for this term the values

$$\begin{array}{ll} 1 & \text{if } \dfrac{i\Delta+d}{v\Delta+d} < 0, \\ 0 & \text{if } \dfrac{i\Delta+d}{v\Delta+d} > 1, \\ \dfrac{(v-i)\Delta}{(v\Delta+d)^i} \displaystyle\sum_{j=0}^{[-d/\Delta]} \binom{i}{j} (j\Delta+d)^j\big((v-j)\Delta\big)^{i-j-1} & \text{if } 0 < \dfrac{i\Delta+d}{v\Delta+d} \leqslant 1. \end{array}$$

Summing in (2.5.0.1) gives (2.5).

COROLLARY 2.5.1. *For $0 < b < 1$, one has for the probability distribution function of the statistic* (1.4.1) *the expression*

$$(2.5.1)\qquad P\left[\sup_{F_n(x)\leqslant b} \frac{F_n(x)-F(x)}{F_n(x)} < c\right] = \left[1-\frac{v}{n}(1-c)\right]^n + \sum_{i=1}^{v-1}\binom{n}{i}\left[1-\frac{v}{n}(1-c)\right]^{n-i}\frac{v-i}{n}(1-c)\left[\frac{v}{n}(1-c)\right]^i$$

where $v = [nb]$.

Proof. Follows from Theorem 2.5 by setting $\Delta = \frac{1}{n}(1-c)$, $d = 0$.

COROLLARY 2.5.2. *For $0 < b < 1$, one has for the probability distribution function of the statistic* (1.4.2) *the expression*

$$(2.5.2)\qquad P[\sup_{F_n(x)\leqslant b}\{F_n(x)-F(x)\} < c] = \left(1-\frac{v}{n}+c\right)^n + \sum_{i=M}^{v-1}\binom{n}{i}\left(1-\frac{v}{n}+c\right)^{n-i}\frac{v-i}{n}\sum_{j=0}^{[nc]}\binom{i}{j}\left(\frac{j}{n}-c\right)^j\left(\frac{v-j}{n}\right)^{i-j-1} + \sum_{i=1}^{M-1}\binom{n}{i}\left(1-\frac{v}{n}+c\right)^{n-i}\left(\frac{v}{n}-c\right)^i$$

where M is such that $M/n-c > 0 > (M-1)/n-c$ and $v = [nb]$.

Proof.

$$P[\sup_{F_n(x)\leqslant b}\{F_n(x)-F(x)\} < c] = P\left[U_j > \frac{j}{n} - c, j \leqslant v = [nb]\right],$$

and set $\Delta = 1/n$, $d = -c$ in (2.5) to obtain (2.5.2).

COROLLARY 2.5.3. *For* $0 < b < 1$, *one has for the probability distribution function of the statistic* (1.4.3) *the expression*

$$(2.5.3)\qquad P\left(\sup_{F_n(x)\leqslant b}\frac{F_n(x)-F(x)}{1-F_n(x)} < c\right) = \left(1-\frac{v}{n}(1+c)+c\right)^n +$$
$$+\sum_{i=1}^{M-1}\binom{n}{i}\left(\frac{v}{n}(1+c)\right)^i\left(1-\frac{v}{n}(1+c)+c\right)^{n-i} +$$
$$+\sum_{i=M}^{v-1}\binom{n}{i}\left(1-\frac{v}{n}(1+c)+c\right)^{n-i}\left(\frac{v-i}{n}(1+c)\right)\sum_{j=0}^{\left[\frac{nc}{1+c}\right]}\binom{i}{j}\times$$
$$\times\left(\frac{j}{n}(1+c)-c\right)^j\left(\frac{v-j}{n}(1+c)\right)^{i-j-1}$$

where $v = [nb]$ *and* M *is such that* $\frac{M}{n}(1+c) > c \geqslant \frac{(M-1)}{n}(1+c)$.

Proof.

$$P\left(\sup_{F_n(x)\leqslant b}\frac{F_n(x)-F(x)}{1-F_n(x)} < c\right) = P\left(U_j > \frac{j}{n}(1+c)-c, j \leqslant v = [nb]\right),$$

and set $\Delta = \frac{1}{n}(1+c)$, $d = -c$ in (2.5) to obtain (2.5.3).

THEOREM 2.6. *For* k *integer*, $1 \leqslant k+1 \leqslant n$, $\Delta > 0$,

$$(2.6)\qquad P\left(U_j < (j-1)\Delta + d, 1 \leqslant j \leqslant k+1\right)$$
$$= 1 - \sum_{j=0}^{k}\binom{n}{j}(1-d-j\Delta)^{n-j}(d+j\Delta)^{j-1}d.$$

Proof. This theorem is an immediate consequence of a theorem in [2] since

$$P\left(U_j < (j-1)\Delta + d,\ j \leqslant k+1\right)$$
$$= n!\int_{U_1=0}^{d}\int_{U_2=U_1}^{\Delta+d}\cdots\int_{U_{k+1}=U_k}^{k\Delta+d}\int_{U_{k+2}=U_{k+1}}\cdots\int_{U_n=U_{n-1}}^{1}\times$$
$$\times dU_n\ldots dU_{k+2}dU_{k+1}\ldots dU_2dU_1.$$

COROLLARY 2.6.1. *For* $0 < a < 1$, *the probability distribution function of the statistic* (1.5.1) *is*

$$(2.6.1)\quad P\left(\sup_{F_n(x)\geqslant a}\frac{F_n(x)-F(x)}{F_n(x)}<c\right)=1-\sum_{j=0}^{k}\binom{n}{j}\times$$
$$\times\left(1-c-\frac{j}{n}(1-c)\right)^{n-j}\left(\frac{j}{n}(1-c)+c\right)^{j-1}c$$

where $k = \min\{0, [n(1-a)]-1\}$.

Proof. By replacing $F(x)$ by $1-F(-x)$, $F_n(x)$ by $1-F_n(-x)$, and $-x$ by x in the statistic we obtain

$$P\left(\sup_{F_n(x)\geqslant a}\frac{F_n(x)-F(x)}{F_n(x)}<c\right)=P\left(\sup_{F_n(x)\leqslant 1-a}\frac{F(x)-F_n(x)}{1-F_n(x)}<c\right)$$
$$=P\left(U_j<\frac{j-1}{n}(1-c)+c,\ j\leqslant[n(1-a)]=k+1\right).$$

(2.6.1) is obtained by substituting $\Delta=\dfrac{1-c}{n}d=c$ in (2.6). (3)

COROLLARY 2.6.2. *For* $0 < a < 1$, *the probability distribution function of the statistic* (1.5.2) *is*

(2.6.2)
$$P\left(\sup_{F_n(x)\geqslant a}F_n(x)-F(x)<c\right)=1-\sum_{j=0}^{k}\binom{n}{j}\left(1-c-\frac{j}{n}\right)^{n-j}\left(\frac{j}{n}+c\right)^{j-1}c$$

where $k = \min\{0, [n(1-a)]-1\}$.

Proof. By replacing $F(x)$ by $1-F(-x)$, $F_n(x)$ by $1-F_n(-x)$, and $-x$ by x in the statistic we obtain

$$P\left(\sup_{F_n(x)\geqslant a}F_n(x)-F(x)<c\right)=P\left(\sup_{F_n(x)\leqslant 1-a}F(x)-F_n(x)<c\right)$$
$$=P\left(U_j<\frac{j-1}{n}+c,\ j\leqslant[n(1-a)]\right).$$

(2.6.2) is obtained by substituting $\Delta=\dfrac{1}{n}d=c$ in (2.6).

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(3) As mentioned in the introduction, M. Csörgö in [3] obtained for the exact distribution of the random variable in 2.6.1. the expression

$$1-\sum_{j=0}^{k+1}\binom{n}{j}\left(1-\frac{j}{n}-c\left(1-\frac{j}{n}\right)\right)^{n-j}\left(\frac{j}{n}+c\left(1-\frac{j}{n}\right)\right)^{j-1}c.$$

It is easily verified that this is not the same as (2.6.1). For example for $n = 4$, $c = 1/2$, $a = 1/2$, (2.6.1) and direct computation give a probability of 213/256 while Csörgö's expression yields 177/256.

3. Concluding Remarks. Some of the Rényi-type statistics discussed above are of particular importance for estimating the probability distribution functions, or for testing hypotheses dealing with probability distribution functions, of random variables for which only truncated or censored samples are available. This is typically the case for observations of life-lengths when n specimens are tested but observation is terminated after k of them have failed (censoring), or when the experiment is stopped after a preassigned time regardless of the number of failures occurring within that time (truncation). An extensive program is now being carried out which includes the derivation and calculation of limiting distributions and the numerical tabulation of the exact probability distributions (for finite sample size n) for these statistics, as well as a study of the power of the corresponding tests. It is hoped that the results of these studies will be ready for publication in the near future.

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UNIVERSITY OF WASHINGTON
SEATTLE, WASHINGTON

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