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THE SUPERPOSITION OF TWO INDEPENDENT MARKOV RENEWAL PROCESSES

1. INTRODUCTION

The superposition of two or more stochastic streams to form a single stream has been examined in previous research. The paper by Çinlar [6] is an excellent review of this area. Most of this previous work has been concerned with the superposition of the renewal process and attention has been focused on the counting processes. In this paper* we provide a structure for the interval process which results when two independent Markov renewal processes defined on countable state spaces are superposed. The corresponding process obtained by superposing independent renewal processes follows from these results. However, Cherry and Disney [3] provide a study of that problem. The usefulness of the characterization is related to the theory of queueing networks, as well as to reliability problems as is shown in Section 4.

2. BACKGROUND AND PREVIOUS RESULTS

2.0. Introduction. This section presents a discussion of pertinent aspects of the theory of semi-Markov or Markov renewal processes defined on a state space which is the cross product of a denumerable set and the non-negative real numbers. For the most part the discussion is a presentation of results from the literature which is useful in establishing the characteristics of the particular processes we examine. However, we include a brief discussion of some of the difficulties or pathologies that can be associated with the stochastic processes studied and an explanation of how these difficulties can be avoided. In addition, in this section

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we define formally Markov renewal processes, and then describe the assumptions which are made throughout. The implications of these assumptions are briefly outlined. None of the results in this section is new; for a more complete discussion the works of Breiman [1], Çinlar [5], Doob [8], and Serfozo [13] are useful.

2.1. The state space. The most familiar examples of Markov renewal processes are those defined on countable state spaces. Informally, such a process is a stochastic process in which the sequence of states visited is determined by a homogeneous Markov chain, while the time spent in a particular state between transitions is a random variable with distribution function dependent on both the current state and the state reached at the next transition. A thorough discussion of the theory of Markov renewal processes defined on countable state spaces is available in [4]. The most significant feature of the countable state space is that renewal theory can be used to prove many of the results connected with such processes, since the recurrence times of any particular state form a (possibly delayed) renewal process.

The Markov renewal processes studied in this paper are defined on a state space which is the cross product of a countable set with the non-negative real numbers. The countable set, denoted in this section by J, may itself be the cross product of several countable sets, i.e., $J = J_1 \times J_2 \times J_3$. In any case, for purposes of discussion, by a suitable ordering scheme we may represent the elements of this space by ordered number pairs (j, u), where j is an integer and u is a non-negative real number. In order to make statements about the probability of events in this state space we must construct a suitable σ-algebra, that is, we cannot divorce the state space from considerations of measurability. This σ-algebra can be constructed in a natural way. Associate with J the distance function $|j_1 - j_2|$ and with R^+ the distance function $|u_1 - u_2|$. On $J \times R^+$ the distance function given by

$$d\big((j_1, u_1), (j_2, u_2)\big) = |j_1 - j_2| + |u_1 - u_2|$$

induces a topology on the space $J \times R^+$. The topology is the "product" of the discrete topology on J and the open sets of R^+. The smallest σ-algebra generated by the open sets of this topology is easily seen to be the product σ-algebra $2^J \otimes \boldsymbol{R}^+$, where 2^J is simply the power σ-algebra on J and $\boldsymbol{R}^+$ is the Borel algebra generated by the open sets of R^+. This σ-algebra has an attractive property. First, if A is any element of the σ-algebra, we may decompose A into a countable disjoint union (see [10])

$$A = \sum_{j \in J_A} (j \times A_j),$$

where J_A is the projection of A on J, A_j is the "j section" of A, and $\sum$ indicates disjoint union. It is clear that A_j is a measurable set in $\boldsymbol{R}^+$. Second, since this σ-algebra is generated by the open sets of $J \times R^+$, it suffices in dealing with probabilities defined on the space to restrict attention to sets of the form $A = j \times (u_1, u_2)$, that is, A is the cross product of j with an open interval of non-negative real numbers. This property is utilized throughout the paper. For convenience we denote, where possible, the state space of any process being considered by E and the associated σ-algebra by Ξ, that is, stochastic processes are defined on the measurable space (E, Ξ).

Finally, it should be noted that state spaces of the form outlined above are metric and, moreover, are Hausdorff, locally compact, complete, separable, and have a countable base. The significance of these properties [11] is that probabilities defined on such spaces are regular, i.e.,

$$\Pr[A] = \sup_C \left[\Pr[C];\ C \subseteq A,\ C \text{ closed},\ C, A \in \Xi\right]$$

and

$$\Pr[A] = \inf_U \left[\Pr[U];\ A \subseteq U,\ U \text{ open},\ U, A \in \Xi\right].$$

Thus it is possible to restrict attention to sequences of events, and in these sequences to sets of the form $A = j \times (u_1, u_2)$ or $A = j \times [u_1, u_2]$.

2.2. Markov renewal processes on (E, Ξ). In this section we give a formal definition of a Markov renewal process defined on (E, Ξ). We assume that $E = J \times R^+$ and $\Xi = 2^J \otimes \boldsymbol{R}^+$, where J is some countable set, i.e., E is the space and Ξ the associated σ-algebra. We also state without proof several properties of such a process. The definition and properties are analogous to those of a Markov renewal process defined on a countable state space; the primary differences being due to the absence of renewal properties present when the state space is countable. The definition follows those of Çinlar [5] and Serfozo [13].

Let $(\Omega, \mathscr{A}, P)$ be a complete probability space and suppose the following functions are defined:

(1) A measurable function S_n: $\Omega \to E$ defined for $0 \leqslant n < \infty$.

(2) A measurable function T_n: $\Omega \to R^+$ defined for $0 \leqslant n < \infty$ such that

$$0 = T_0(w) \leqslant T_1(w) \leqslant \ldots$$

Definition 2.1. The stochastic process $\{S_n, T_n;\ n \geqslant 0\}$ is called a *Markov renewal process* induced by the kernel Q if

$$\Pr[S_{n+1} \in A, T_{n+1} \leqslant t \mid S_0, \ldots, S_n;\ T_0, \ldots, T_n]$$
$$= \Pr[S_{n+1} \in A, T_{n+1} \leqslant t \mid S_n, T_n] = Q(A \times [0, t - T_n] \mid S_n),$$

where $A \in \Xi$, $t \in [0, \infty)$, and $n \geqslant 0$. We assume Q does not depend on n. The process is homogeneous.

The kernel Q: $(\Xi \otimes \boldsymbol{R}^+) \times E \to [0, 1]$ has the following properties:

(1) for any $y \in E$, $Q(\cdot \mid y)$ is a probability on $\Xi \otimes \boldsymbol{R}^+$,

(2) for any $G \in \Xi \otimes \boldsymbol{R}^+$, $Q(G \mid \cdot)$ is Ξ-measurable.

Note that in Definition 2.1 the state of the process is represented by S_n while the corresponding transition times are given by T_n. This notation is followed throughout, provided that a more precise notation is not more convenient. Referring to the informal definition in Section 2.1, we see that the process proceeds along its trajectories, entering the state S_n at time T_n. The trajectories are thus right continuous.

The following propositions can be shown to apply to Markov renewal processes defined on (E, Ξ).

PROPOSITION 2.1. *The stochastic process* $\{S_n, T_n;\ n \geqslant 0\}$ *is a Markov chain defined on* $E \times R^+$ *with transition probabilities given by*

$$\Pr[S_{n+1} \in A, T_{n+1} \leqslant t \mid S_0 = y] = Q^{n+1}(A \times [0, t] \mid y),$$

where $Q^n(\cdot \mid \cdot)$ *is defined recursively by*

$$Q^0(A \times [0, t] \mid y) = \begin{cases} 1 & \text{for } y \in A, \\ 0 & \text{for } y \notin A, \end{cases}$$

and

$$Q^{n+1}(A \times [0, t] \mid y) = \int_{E \times [0,t]} Q^n(A \times [0, t-u] \mid z) Q\big(d(z \times [0, u]) \mid y\big).$$

PROPOSITION 2.2. *Let* $X_0 = T_0 = 0$ *and* $X_n = T_n - T_{n-1}$, $n \geqslant 1$. *Then, given* $n_1, \ldots, n_k$ *and* $t_1, \ldots, t_k$, *we have*

$$\Pr[X_{n_1} \leqslant t_1, \ldots, X_{n_k} \leqslant t_k \mid S_n;\ n \geqslant 0] = \prod_{i=1}^{k} \Pr[X_{n_i} \leqslant t_i \mid S_{n_i - 1}, S_{n_i}];$$

that is, the sojourn times $\{X_{n_i};\ i = 1, \ldots, k\}$ *are conditionally independent given* $\{S_{n_i - 1}, S_{n_i};\ i = 1, \ldots, k\}$.

Note that we may speak of the process $\{S_n, X_n;\ n \geqslant 0\}$ or $\{S_n, T_n;\ n \geqslant 0\}$ equivalently in light of the above definition. Depending upon methods to be used either notation is applied.

PROPOSITION 2.3. *The stochastic process* $\{S_n;\ n \geqslant 0\}$ *is a time homogeneous Markov chain defined on* (E, Ξ) *with transition probabilities given by*

$$\Pr[S_{n+1} \in A \mid S_n = y] = \lim_{t \to \infty} Q(A \times [0, t] \mid y) = Q\big(A \times [0, \infty) \mid y\big).$$

The stochastic process $\{S_n;\ n \geqslant 0\}$ is referred to as the *underlying* Markov chain of the Markov renewal process. Its importance is outlined

in the following section. With a slight change in notation the contents of the preceding two propositions may be clarified. Without loss of generality let

$$Q(A \times [0, t] \mid y) = \int_A K(y, dz) F_{yz}(t).$$

With this notation the transition probability of the underlying Markov chain $\{S_n;\ n \geqslant 0\}$ is given by

$$\Pr[S_{n+1} \in A \mid S_n = y] = \int_A K(y, dz) = K(y, A).$$

The kernel K is a mapping from $E \times \Xi$ to $[0, 1]$ and, in addition,
(1) $K(y, \cdot)$ is a probability on (E, Ξ) for all $x \in E$,
(2) $K(\cdot, A)$ is Ξ-measurable for any $A \in \Xi$.

The function $F_{yz}(t)$ defined on the non-negative real numbers for all $(y, z) \in E \times E$ becomes then the distribution function for the sojourn in state y, conditional on the next transition being to state z. In the developments that follow we make frequently use of this notation.

Without additional assumptions Markov renewal processes as defined above may be subject to certain "pathologies". We outline those which may be directly ascribed to the underlying Markov chain in Section 2.3. First we list three properties which are assumed to hold for the processes studied.

Definition 2.2. Let ζ be the random variable defined by

$$\zeta = \sup_n \{T_n\}$$

for the Markov renewal process $\{S_n, T_n;\ n \geqslant 0\}$. The process $\{S_n, T_n;\ n \geqslant 0\}$ is called *conservative* if

$$\Pr[\zeta < \infty] = 0.$$

Assuming that sojourn times are finite with probability one, a conservative process is one in which an infinite number of transitions take place, constrained by the fact that in any finite time interval only a finite number of transitions occur, that is, the process cannot "explode" in a finite time period. Note that this excludes neither instantaneous states nor the absorption of the process into a single state. A renewal process is an example of the latter situation.

Definition 2.3. For the Markov renewal process $\{S_n, T_n;\ n \geqslant 0\}$ define the random process $N(t)$ by $N(0) = 0$ and $N(t) = n$ for $T_n \leqslant t < T_{n+1}$. The process $\{S_n, T_n;\ n \geqslant 0\}$ is called *regular* if

$$\mathrm{E}[N(t)] < \infty \quad \text{for all } t < \infty.$$

Note that although regularity does not exclude the possibility of instantaneous states, it does ensure that if such states exist on any trajectory, only a finite number may be visited between visits to non-instantaneous states. Moreover, regularity implies that the expected number of transitions in any bounded time interval is bounded.

One further property is defined here for Markov renewal processes with countable state spaces.

Definition 2.4. Let $N_k(t)$ be the random process which counts the number of entrances to state k during the interval $[0, t)$ including, if applicable, the initial state at $T_0 = 0$. The Markov renewal process $\{S_n, T_n;\ n \geqslant 0\}$ with countable state space is said to be *normal* if, regardless of initial state,

$$\mathrm{E}[N_k(t)] < \infty \quad \text{for all } t \in [0, \infty).$$

A normal process may not be conservative, but the trajectory which "explodes" must do so without revisiting any state an infinite number of times. Almost all physical processes may be described as conservative, and non-normal processes on countable state spaces are rare in a physical sense. Regularity, however, need not be fulfilled, in particular, if instantaneous states are possible. In this paper we exclude conditions which could cause non-regularity and, in particular, instantaneous states, and assume that the Markov renewal processes to be studied are conservative.

As a consequence of the assumed absence of instantaneous states we may assert that to each Markov renewal process there corresponds a unique semi-Markov process $\{S(t);\ t \geqslant 0\}$ defined by $S(0) = S_0$ and $S(t) = S_n$ for $T_n \leqslant t < T_{n+1}$.

Çinlar [5] has shown that the Markov renewal process is conservative if and only if the minimal solution of the equation

$$\text{(2.1)}\quad \Pr[S(t) \in A \mid S(0) = y] = \chi_A(y)\, Q(E \times (t, \infty) \mid y) + \int_{E \times R^+} Q(d(z \times u) \mid y) \Pr[S(t-u) \in A \mid S(0) = z]$$

has the property

$$\Pr[S(t) \in E \mid S(0) = y] = 1,$$

where $\chi_A(\cdot)$ is the indicator function of $A \in \Xi$, and $Q(\cdot \mid \cdot)$ is the Markov renewal kernel.

2.3. The underlying Markov chain. Proposition 2.3 establishes that with the Markov renewal process $\{S_n, T_n;\ n \geqslant 0\}$ we have associated a time homogeneous Markov chain $\{S_n;\ n \geqslant 0\}$. Throughout this section

we represent the transition probability of this underlying chain by

$$\Pr[S_n \in A \mid S_{n-1} = y] = \int_A K(y, dz) = K(y, A)$$

for all $y \in E$ and all $A \in \Xi$.

The significance of the underlying Markov chain of a Markov renewal process defined on a countable state space is related to two types of properties. The first of these comprises the recurrence properties of states. A state in a normal Markov renewal process occurs infinitely often or finitely often in the Markov renewal process if it is persistent or transient, respectively, in the underlying chain [4]. The second type of these properties is connected with the existence of limits for the probability associated with the semi-Markov process, namely

$$\lim_{t\to\infty} \Pr[S(t) = j \mid S(0) = i],$$

and involves the existence of a unique stationary distribution for the underlying Markov chain. In this section we discuss briefly the extensions of these concepts, recurrence and stationarity, to Markov chains defined on non-countable state spaces. The primary references are [1] and [8].

We begin by assuming that the Markov chain $\{S_n;\ n \geqslant 0\}$ is defined on the measurable space (E, Ξ) and first give two definitions analogous to those for irreducible closed sets of a Markov chain defined on a countable state space.

Definition 2.5. The set $A \in \Xi$ is called *closed* if

$$\Pr[S_n \in A \mid S_{n-1} = y] = 1 \quad \text{for all } y \in A.$$

Definition 2.6. The Markov chain $\{S_n;\ n \geqslant 0\}$ is called *indecomposable* if there are no two disjoint closed sets A_1 and A_2 belonging to Ξ.

It is obvious that every Markov chain has at least one closed set, namely the state space E. For an indecomposable chain on a general state space, however, problems arise. In some sense we require the existence of a minimal closed set, analogous to the set of persistent states in the countable state space of a Markov chain with a single closed set of persistent states, and a set of transient states, each leading to the closed set. In a general case, it is possible only to define such a closed set up to an equivalence relationship depending on sets of zero measure where the zero measure in question is with respect to some auxiliary measure. The state space with which we deal in this paper and the associated transition probabilities do not in general meet the conditions under which such an auxiliary measure exists. However, and this overcomes a second problem, all open sets of the state spaces occur infinitely often with probability zero or one, regardless of initial state. This removes from

consideration the problem of a trajectory remaining outside a "minimal" closed set for more than a finite number of transitions. Thus we may label each open set in the state space as persistent or transient depending upon whether it is visited infinitely often or not. Instead of referring to points as being persistent or transient we refer to sets as having one or the other of these properties.

The primary objective of studies of Markov chains which make use of an auxiliary measure has been to determine conditions under which there exists a distribution $\Pi(A)$ defined on (E, Ξ) such that

$$\lim_{n\to\infty} \Pr[S_n \in A \mid S_0 = y] = \Pi(A).$$

Doob [8] devotes a chapter to this problem using Doeblin's condition as an assumption, namely:

There are a finite-valued measure Φ defined on Ξ with $\Phi(E) > 0$, an integer $\nu \geqslant 1$, and an $\varepsilon > 0$ such that

$$\Pr[S_\nu \in A \mid S_0 = y] \leqslant 1 - \varepsilon \quad \text{if } \Phi(A) \leqslant \varepsilon.$$

The auxiliary measure $\Phi(\cdot)$ thus imposes a uniformity on the transition probability. For our purposes, however, we require only that if a distribution $\Pi(\cdot)$ defined on (E, Ξ) is stationary for the Markov chain $\{S_n;\ n \geqslant 0\}$, that is

$$\Pi(A) = \int_E \Pr[S_n \in A \mid S_{n-1} = y]\Pi(dy) \qquad \text{for all } A \in \Xi \text{ and } n \geqslant 1, \tag{2.2}$$

then that distribution is unique. We obtain this result from [1] by suitably modifying the state space.

THEOREM 2.1. *Let the Markov chain $\{S_n;\ n \geqslant 0\}$ defined on (E, Ξ) be indecomposable. Then if a stationary distribution exists, it is unique.*

When dealing with a stationary distribution in the paper we take advantage of the definition (2.2) to assume that $\Pi(\cdot)$ is an initial distribution:

$$\Pi(A) = \Pr[S_0 \in A].$$

2.4. Limiting results. We have already defined the semi-Markov process $\{S(t);\ t \geqslant 0\}$ associated with the Markov renewal process $\{S_n, T_n;\ n \geqslant 0\}$. For any state space two results are associated with the semi-Markov process. The first is the solution of equation (2.1) and the second is the existence of the limit

$$\lim_{t\to\infty} \Pr[S(t) \in A \mid S(0) = y]$$

of the solution to (2.1).

Çinlar [5] examines both results for semi-Markov processes defined on arbitrary state spaces. As mentioned previously, the following result was proved:

The solution to (2.1) is unique if and only if the Markov renewal process $\{S_n, T_n;\ n \geqslant 0\}$ is conservative.

In the case of the limiting result, by modifying Çinlar's work slightly we obtain the following

THEOREM 2.2. *Suppose the semi-Markov process* $\{S(t),\ t \geqslant 0\}$ *is conservative and that, for any open set* A *of the topological state space* (E, Ξ),

$$\Pr[S_n \in A \text{ infinitely often}] = 1$$

regardless of initial state. In addition, suppose that for any $A \in \Xi$ *and any bounded time interval the expected number of visits to* A *by the Markov renewal process* $\{S_n, T_n;\ n \geqslant 0\}$ *during the time interval is bounded. Then if the process* $\{S_n, T_n;\ n \geqslant 0\}$ *is aperiodic, if the limit*

$$\lim_{t\to\infty} \mathrm{E}[\text{number of transitions in } (t-t_2, t-t_1)]$$

exists for $0 \leqslant t_1 \leqslant t_2 < \infty$, *and if* $\Pi(\cdot)$ *is a unique stationary measure for the underlying Markov chain* $\{S_n;\ n \geqslant 0\}$, *then*

$$\lim_{t\to\infty} \Pr[S(t) \in A \mid S(0) = y] = \int_A m(u)\Pi(du)\Big(\int_E m(u)\Pi(du)\Big)^{-1},$$

where $y \in E$, $A \in \Xi$, *and*

$$m(y) = \int_{t=0}^{\infty}\int_E K(y, dz)\big(1 - F_{yz}(t)\big)\,dt.$$

Note that $m(y)$ is the mean time spent between transitions in state y. This result applies to the limit of the *probability* that $S(t) \in A$ and implies that this limit is independent of initial conditions.

3. THE SUPERPOSITION OF TWO INDEPENDENT MARKOV RENEWAL PROCESSES

3.0. Introduction. In this section we study the superposition of two independent Markov renewal processes. Our approach involves the use of Markov renewal processes defined on state spaces which include a random variable corresponding to backward recurrence time. Several applications are suggested.

3.1. Assumptions and notation. In this section we examine the superposition of two independent Markov renewal processes

$$\{{}^1Z_n, X_n;\ n \geqslant 0\} \quad \text{and} \quad \{{}^2Z_m, Y_m;\ m \geqslant 0\},$$

subject to the constraints outlined. The processes are defined on countable state spaces J_1 and J_2, respectively, and assumed to be normal, regular, and conservative. We assume further that the state spaces J_1 and J_2 consist of a single closed set of persistent states with a finite number (possibly zero) of transient states, and that the two processes $\{{}^1Z_n, X_n; n \geqslant 0\}$ and $\{{}^2Z_m, Y_m; m \geqslant 0\}$ are aperiodic. Transition probabilities and associated quantities are denoted as follows:

(1) $\Pr[{}^1Z_{n+1} = j,\ X_{n+1} \leqslant x \mid {}^1Z_n = i] = A_{ij}(x)$,

$\Pr[{}^2Z_{m+1} = l,\ Y_{m+1} \leqslant y \mid {}^2Z_m = k] = B_{kl}(y)$.

(2) $\Pr[{}^1Z_{n+1} = j,\ X_{n+1} \in [x, x+dx) \mid {}^1Z_n = i] = a_{ij}(x)\,dx$,

$\Pr[{}^2Z_{m+1} = l,\ Y_{m+1} \in [y, y+dy) \mid {}^2Z_m = k] = b_{kl}(y)\,dy$.

(3) $\Pr[X_{n+1} \leqslant x \mid {}^1Z_n = i] = \sum_j A_{ij}(x) = F_i(x)$,

$\Pr[Y_{m+1} \leqslant y \mid {}^2Z_m = k] = \sum_l B_{kl}(y) = G_k(y)$.

(4) $\Pr[X_{n+1} \leqslant x \mid {}^1Z_{n+1} = j,\ {}^1Z_n = i] = A_{ij}(x)/A_{ij}(\infty) = F_{ij}(x)$, where $A_{ij}(\infty) \neq 0$,

$\Pr[Y_{m+1} \leqslant y \mid {}^2Z_{m+1} = l,\ {}^2Z_m = k] = B_{kl}(y)/B_{kl}(\infty) = G_{kl}(y)$, where $B_{kl}(\infty) \neq 0$.

(5) Let ${}^1N_j(t)$ be the number of visits to j in time t in process 1, and let ${}^2N_l(t)$ be the number of visits to l in time t in process 2. Then

$$\mathrm{E}[{}^1N_j(t) \mid {}^1Z_0 = i] = R_{ij}(t) \quad \text{with derivative } r_{ij}(t),$$

$$\mathrm{E}[{}^2N_l(t) \mid {}^2Z_0 = k] = S_{kl}(t) \quad \text{with derivative } s_{kl}(t).$$

(6) $\mathrm{E}[X_{n+1} \mid {}^1Z_n = i] = \int_0^\infty \big(1 - F_i(x)\big)\,dx = m_i$,

$\mathrm{E}[Y_{m+1} \mid {}^2Z_m = k] = \int_0^\infty \big(1 - G_k(y)\big)\,dy = n_k$.

We assume that all sojourn time probability distributions are non-singular and that the probability density functions corresponding to $F_{ij}(x)$ and $G_{kl}(y)$ are continuous on the intervals on which the distribution functions are not constant when $F_{ij}(x)$ and $G_{kl}(y)$ are defined, that is, where the transitions ${}^1Z_n = i$ to ${}^1Z_{n+1} = j$ and ${}^2Z_m = k$ to ${}^2Z_{m+1} = l$ have non-zero probability. Further, we suppose that m_i and n_k are bounded for all i and k belonging to J_1 and J_2, respectively. Finally, we assume that instantaneous transitions are impossible, that is, $A_{ij}(0)$ and $B_{kl}(0)$ are equal to zero for all transitions.

Associated with the irreducible persistent classes of states, say J_1' and J_2', respectively, in the underlying Markov chains of the two processes $\{{}^1Z_n, X_n;\ n \geqslant 0\}$ and $\{{}^2Z_m, X_m;\ m \geqslant 0\}$ are unique stationary probability vectors [9] which will be denoted by $\{\alpha_i\}_{i \in J_1'}$ and $\{\beta_k\}_{k \in J_2'}$, i.e.,

$$\alpha_j = \sum_{i \in J_1'} \alpha_i A_{ij}(\infty) \quad \text{and} \quad \beta_l = \sum_{k \in J_2'} \beta_k B_{kl}(\infty).$$

The process to be modelled is illustrated in Fig. 3.1. The times to the n-th and m-th events in the two processes are denoted by

$$\sum_{i=1}^{n} X_i = {}^1T_n \quad \text{and} \quad \sum_{i=1}^{m} Y_i = {}^2T_m,$$

where ${}^1T_0 = {}^2T_0 = 0$.

Two realizations $\{{}^1Z_n, {}^1T_n;\ n \geqslant 0\}$ and $\{{}^2Z_m, {}^2T_m;\ m \geqslant 0\}$ are *superposed* by re-ordering the union of the two sequences $\{{}^1T_n;\ n \geqslant 0\}$ and $\{{}^2T_m;\ m \geqslant 0\}$ to form a single monotone increasing sequence $\{U_k;\ k \geqslant 0\}$,

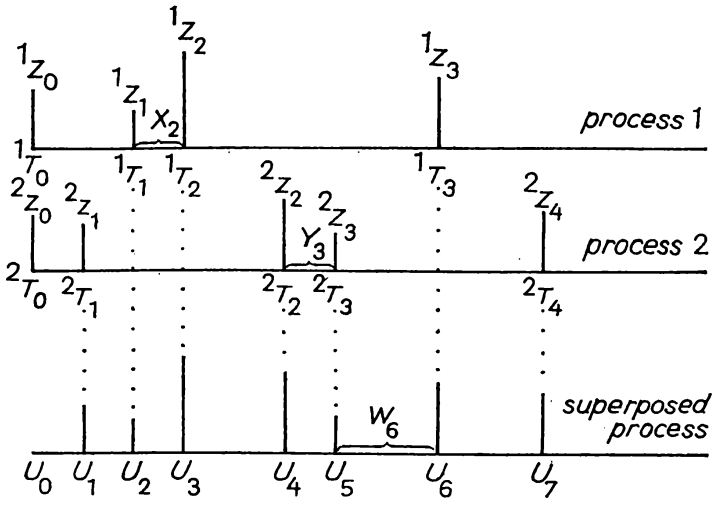

Fig. 3.1. The superposition of two Markov renewal processes

where the random variable U_k is the time of the k-th event in the superposed process. We then associate with U_k the following random variables:

${}^1Z_k$: the state of the first process at time U_k^+,

${}^2Z_k$: the state of the second process at time U_k^+,

I_k: the index of the process which produced the k-th event in the superposed process,

V_k: the time elapsed since the last event in the process which did not produce the k-th event of the superposed process,

W_k: the difference between U_{k+1} and U_k.

As a consequence of the non-singularity of the distributions $F_{ij}(x)$ and $G_{kl}(y)$ in the component processes, simultaneous transitions cannot take place in either process, i.e., with probability one a non-zero time is spent between transitions. Furthermore, the simultaneous occurrence of events in both processes has probability zero as a consequence of independence and non-singularity of distributions. Again therefore, we ignore the initial situation illustrated in Fig. 3.1 and take as initial conditions any event not corresponding to "start up" with ${}^1T_0 = {}^2T_0$, and take as a model of the superposition of two Markov independent renewal processes the stochastic process

$$\{{}^1Z_n, {}^2Z_n, I_n, V_n, U_n;\ n \geqslant 0\}. \tag{3.1}$$

3.2. Structure and properties. In this section we characterize the process (3.1) and examine certain properties which result from the assumptions made in the previous section. We begin with a characterization of the process.

THEOREM 3.1. *The stochastic process* (3.1) *is a Markov renewal process defined on the state space*

$$(J_1 \times J_2 \times \{1, 2\} \times R^+, 2^{J_1} \otimes 2^{J_2} \otimes 2^{\{1,2\}} \otimes \boldsymbol{R}^+).$$

Proof. The state space of the process clearly corresponds to that given in the theorem (Section 2.1). Referring to Fig. 3.2 and recalling

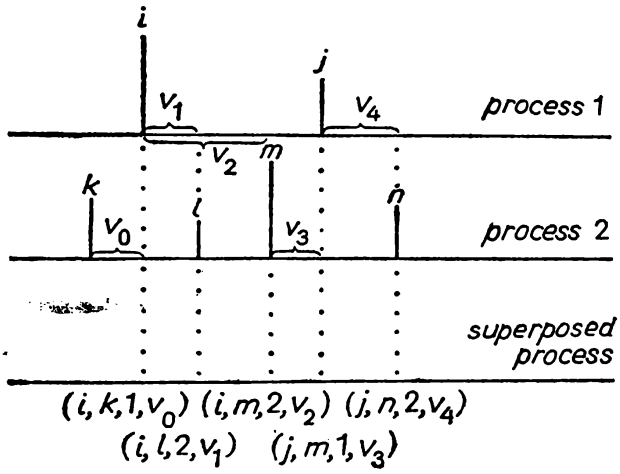

Fig. 3.2. The superposed process

from the previous section that the simultaneous occurrence of events in both processes $\{{}^1Z_n, {}^1T_n;\ n \geqslant 0\}$ and $\{{}^2Z_m, {}^2T_m;\ m \geqslant 0\}$ has probability zero, we find that it is sufficient to consider two cases:

(i) $\Pr\big[{}^1Z_n = j,\ {}^2Z_n = l,\ I_n = 1,\ V_n \in [v, v+dv),\ W_n \leqslant w \mid$
$({}^1Z_{n-1} = i,\ {}^2Z_{n-1} = k,\ I_{n-1} = 2,\ V_{n-1} = v_{n-1},\ W_{n-1} = w_{n-1}), \ldots$
$\ldots, ({}^1Z_0, {}^2Z_0, I_0, V_0, W_0)\big]$

$$= \begin{cases} \Pr\big[{}^1Z_{m+1} = j,\ X_{m+1} \in [v+v_{n-1}, v+v_{n-1}+dv) \mid {}^1Z_m = i,\ X_{m+1} > v_{n-1}\big] \times \\ \qquad \times \Pr[Y_{p+1} > v \mid {}^2Z_p = k]\,\delta_{kl} \quad \text{for } v \leqslant w, \\ 0 \quad \text{for } v > w, \end{cases}$$

where δ_{kl} is the Kronecker delta and m, p are arbitrary indices. This follows from the independence of the two Markov renewal processes.

(ii) $\Pr\big[{}^1Z_n = j, {}^2Z_n = l, I_n = 1, V_n \in [v, v+dv), W_n \leqslant w \mid$
$({}^1Z_{n-1} = i,\ {}^2Z_{n-1} = k, I_{n-1} = 1,\ V_{n-1} = v_{n-1}, W_{n-1} = w_{n-1}), \ldots$
$\ldots, ({}^1Z_0, {}^2Z_0, I_0, V_0, W_0)\big]$

$$= \begin{cases} \Pr\big[{}^1Z_m = j,\ X_m \in [v-v_{n-1}, v-v_{n-1}+dv) \mid {}^1Z_{m-1} = i\big] \times \\ \qquad \times \Pr[Y_{p+1} > v \mid {}^2Z_p = k,\ Y_p > v_{n-1}]\,\delta_{kl} \quad \text{for } v-v_{n-1} \leqslant w, \\ 0 \quad \text{for } v-v_{n-1} > w. \end{cases}$$

Again the result follows from the independence of the two component processes.

Similar results may be shown for those states in which $I_{n+1} = 2$ for $I_n = 1$ or 2. Thus we have, with q equal to 1 or 2,

$$\begin{aligned} &\Pr\big[{}^1Z_{n+1} = j, {}^2Z_{n-1} = l, I_{n+1} = q,\ V_{n+1} \in [v, v+dv), \\ &\qquad W_{n+1} \leqslant w \mid ({}^1Z_n, {}^2Z_n, I_n, V_n, W_n), \ldots, ({}^1Z_0, {}^2Z_0, I_0, V_0, W_0)\big] \\ &= \Pr\big[{}^1Z_{n+1} = j, {}^2Z_{n+1} = l, I_{n+1} = q,\ V_{n+1} \in [v, v+dv), \\ &\qquad\qquad W_{n+1} \leqslant w \mid ({}^1Z_n, {}^2Z_n, I_n, V_n)\big], \end{aligned}$$

and the stochastic process (3.1) is a Markov renewal process.

With reference to the notation in Section 3.1, Theorem 3.1, and making use of symmetry we obtain the following

COROLLARY 3.1. *The transition probabilities of the stochastic process*

$$\{{}^1Z_n, {}^2Z_n, I_n, V_n, W_n;\ n \geqslant 0\}$$

are given by

$$A(j, l, q, v, w \mid i, k, q_n, v_n) = \Pr\big[{}^1Z_{n+1} = j, {}^2Z_{n+1} = l, I_{n+1} = q,$$
$$V_{n+1} \in [v, v+dv),\ W_{n+1} \leqslant w \mid {}^1Z_n = i, {}^2Z_n = k, I_n = q_n, V_n = v_n\big],$$

where

$$(3.2)\quad A(j, l, 1, v, w \mid i, k, 1, v_n) = a_{ij}(v-v_n)\,dv\,\frac{\bar{G}_k(v)}{\bar{G}_k(v_n)}\,\delta_{kl}\,D\big(w-(v-v_n)\big),$$

$$(3.3)\qquad A(j,l,1,v,w \mid i,k,2,v_n) = \frac{a_{ij}(v+v_n)}{\bar{F}_i(v_n)}\, dv\, \bar{G}_k(v)\, \delta_{kl} D(w-v),$$

$$A(j,l,2,v,w \mid i,k,1,v_n) = \frac{b_{kl}(v+v_n)}{\bar{G}_k(v_n)}\, dv\, \bar{F}_i(v)\, \delta_{ij} D(w-v),$$

$$A(j,l,2,v,w \mid i,k,2,v_n) = b_{kl}(v-v_n)\, dv\, \frac{\bar{F}_i(v)}{\bar{F}_i(v_n)}\, \delta_{ij} D\big(w-(v-v_n)\big),$$

$D(s)$ being the Heaviside unit step function:

$$D(s) = \begin{cases} 1 & \text{for } s \geqslant 0, \\ 0 & \text{for } s < 0. \end{cases}$$

Theorem 3.1 and Corollary 3.1 completely specify the "motion" of the process (3.1). To describe the complete behavior of the trajectories we require only an initial distribution, that is, a probability measure over $({}^1Z_0, {}^2Z_0, I_0, V_0)$. This corresponds to selection of a state and a backward recurrence time for that state in one process, and to selection of a state alone in the other process.

We now turn to the properties of the Markov renewal process (3.1), beginning with an investigation of the underlying Markov chain $\{{}^1Z_0, {}^2Z_n, I_n, V_n;\ n \geqslant 0\}$. Many of the properties of the Markov renewal process result from the behavior of the underlying Markov chain. The transition probabilities of the chain are given in the following

PROPOSITION 3.1. *The transition probabilities of the underlying Markov chain $\{{}^1Z_n, {}^2Z_n, I_n, V_n;\ n \geqslant 0\}$ are given by the function $A(j,l,q_1,v \mid i,k,q_0,v_0)$, where*

$$A(j,l,q_{n+1},v \mid i,k,q_n,v_n) = \Pr\big[{}^1Z_{n+1} = j, {}^2Z_{n+1} = l, I_{n+1} = q_{n+1},$$
$$V_{n+1} \in [v, v+dv) \mid ({}^1Z_n = i, {}^2Z_n = k, I_n = q_n, V_n = v_n)\big]$$

and

$$A(j,l,1,v \mid i,k,1,v_n) = a_{ij}(v-v_n)\, dv\, \frac{\bar{G}_k(v)}{\bar{G}_k(v_n)}\, \delta_{kl},$$

$$A(j,l,1,v \mid i,k,2,v_n) = \frac{a_{ij}(v+v_n)}{\bar{F}_i(v_n)}\, dv\, \bar{G}_k(v)\, \delta_{kl},$$

$$A(j,l,2,v \mid i,k,2,v_n) = b_{kl}(v-v_n)\, dv\, \frac{\bar{F}_i(v)}{\bar{F}_i(v_n)}\, \delta_{ij},$$

$$A(j,l,2,v \mid i,k,1,v_n) = \frac{b_{kl}(v+v_n)}{\bar{G}_k(v_n)}\, dv\, \bar{F}_i(v)\, \delta_{ij}.$$

The transition probability is defined for $i, j \in J_1$, $k, l \in J_2$, q_n, q_{n+1} *in* $\{1, 2\}$, *and* $v, v_n \in R^+$.

Proof. By Proposition 2.3 the transition probabilities of the underlying Markov chain are the limits as w tends to infinity of the transition probabilities of the Markov renewal process. The proposition then follows from Corollary 3.1.

In determining the recurrence properties of the elements of the σ-algebra $2^{J_1} \otimes 2^{J_2} \otimes 2^{\{1,2\}} \otimes \boldsymbol{R}^+$ we can initially make use of the structures of the component processes. We first dispose of those sets which are entered finitely often with probability one.

THEOREM 3.2. *Any event of the form* $\{{}^1Z_n = j, {}^2Z_n = l, I_n, V_n\}$ *in which either one or both of the states* j *and* l *are transient in the component processes* $\{{}^1Z_n, {}^1T_n;\ n \geqslant 0\}$ *and* $\{{}^2Z_m, {}^2T_m;\ m \geqslant 0\}$ *is entered finitely often with probability one.*

Proof. Let us denote the persistent classes of $\{{}^1Z_n;\ n \geqslant 0\}$ and $\{{}^2Z_m;\ m \geqslant 0\}$ by J_1' and J_2', respectively. Then with probability one the number of transitions N_1 in J_1 prior to absorption into J_1' is finite regardless of initial state $j \in J_1$. Similarly, with probability one the number of transitions N_2 in J_2 prior to absorption into J_2' is finite regardless of initial state $l \in J_2$. Hence the random variable $N = \max\{N_1, N_2\}$ is finite with probability one, and N is clearly the number of transitions in the underlying chain prior to absorption in $J_1' \times J_2'$.

To investigate the recurrence properties of events of the form

$$\{{}^1Z_n = j, {}^2Z_n = l, I_n = q_n, V_n \in [v-\varepsilon, v+\varepsilon]\}$$

for states j and l persistent in their respective underlying process, we make use of the fact that the event $\{{}^1Z_n = j\}$ occurs infinitely often with probability one in the Markov chain $\{{}^1Z_n;\ n \geqslant 0\}$, and thus infinitely often with probability one in the underlying chain of the superposed process.

THEOREM 3.3. *If* j *and* l *are persistent in the underlying Markov chains* $\{{}^1Z_n;\ n \geqslant 0\}$ *and* $\{{}^2Z_m;\ m \geqslant 0\}$, *respectively, then the state* $\{{}^1Z_n = j, {}^2Z_n = l, I_n = q, V_n = v\}$ *is persistent in the sense that the event* $\{{}^1Z_n = j, {}^2Z_n = l, I_n = q, V_n \in [v-\varepsilon, v+\varepsilon]\}$ *occurs infinitely often with probability one, regardless of initial state. Here* $q \in \{1, 2\}$, $v \geqslant 0$ *is an admissible value, and* ε *is arbitrarily small.*

Proof. Suppose the initial state of the process is $({}^1Z_0 = j, {}^2Z_0 = k, I_0 = 1, V_0 = v_0)$, and denote by $f_{jj}'(t)$ the probability density function of the first passage time from j to j in the process $\{{}^1Z_n, {}^1T_n;\ n \geqslant 0\}$. Since j is persistent, the number of transitions between visits to j in the first component process is finite with probability one.

Then in the filtered process (see [4]) $\{{}^1H_n, {}^2H_n, I_n, V_n;\ n \geqslant 0\}$ with $H = \{{}^1Z = j\}$ we have

$$\Pr\left[{}^1H_1 = j, {}^2H_1 = l, I_1 = 1, V_1 \in [v-\varepsilon, v+\varepsilon] \mid {}^1H_0 = j, {}^2H_0 = k,\right.$$

$$\left. I_0 = 1, V_0 = v_0\right] = \int_{u=v-\varepsilon-v_0}^{v+\varepsilon-v_0} f'_{jj}(u)\,\frac{\bar{G}_k(v_0+u)}{\bar{G}_k(v_0)}\,du\,\delta_{kl} +$$

$$+ \int_{u=v-\varepsilon}^{v+\varepsilon}\int_{t=u}^{\infty} f'_{jj}(t)\,s'_{kl}(t-u)\,\bar{G}_l(u)\,du\,dt,$$

where

$$s'_{kl}(y) = \frac{b_{kl}(y)}{\bar{G}_k(v_0)} + \sum_{n=1}^{\infty}\sum_{r\in J_2}\int_{w=0}^{y}\frac{b_{kr}(v_0+w)}{\bar{G}_k(v_0)}\,b^n_{rl}(y-w)\,dw.$$

The function $s'_{kl}(y)$ is simply the derivative of the Markov renewal function for the modified or delayed process $\{{}^2Z_m, {}^2T_m;\ m \geqslant 0\}$ in which the first transition is biased by the presence of V_0, the backward recurrence time. Çinlar [4] shows that

$$\lim_{y\to\infty} s'_{kl}(y) = \Pi_l,$$

where Π_l is the inverse of the mean recurrence time of state l in the Markov renewal process $\{{}^2Z_m, {}^2T_m;\ m \geqslant 0\}$. Since l is persistent and the process is normal, $\Pi_l < \infty$. Since by assumption n_j is bounded, $\Pi_l > 0$. We have

$$\Pr\left[{}^1H_1 = j, {}^2H_1 = l, I_1 = 1, V_1 \in [v-\varepsilon, v+\varepsilon] \mid {}^1H_0 = j, {}^2H_0 = k,\right.$$

$$\left. I_0 = 1, V_0 = v_0\right] \geqslant \int_{u=v-\varepsilon}^{v+\varepsilon}\int_{t=u}^{\infty} f'_{jj}(t)\,s'_{kl}(t-u)\,\bar{G}_l(u)\,dt\,du,$$

which, after selecting Y^* such that $|s'_{kl}(y) - \Pi_l| < \delta$ for $y > Y^*$, is greater than or equal to

$$\bar{G}(v+\varepsilon)(\Pi_l - \delta)\cdot 2\varepsilon \int_{Y^*+v+\varepsilon}^{\infty} f'_{jj}(t)\,dt = 1-\varrho_l > 0.$$

That is, the transition probability in the filtered process has a non-zero lower bound independent of the initial state k and the initial backward recurrence time v_0.

Let P_N be the probability that the first entrance to $(j, l, 1, [v-\varepsilon, v+\varepsilon])$ takes place after N transitions in the filtered process. Then $P_N < \varrho_l^N$ regardless of the initial state $(j, k, 1, v_0)$ and $1-P_N \geqslant 1-\varrho_l^N$.

Taking limits as $N \to \infty$ we infer that with probability one the number of steps in the filtered process $\{{}^1H_n = j, {}^2H_n, I_n = 1, V_n;\ n \geqslant 0\}$ between occurrences of $\{{}^1H_n = j, {}^2H_n = l, I_n = 1, V_n \in [v-\varepsilon, v+\varepsilon]\}$ is finite.

We now must show that regardless of initial state the event $\{{}^1Z_n = j, I_n = 1\}$ is reached in a finite number of steps with probability one. First suppose $({}^1Z_0 = i, {}^2Z_0 = k, I_0 = 1, V_0 = v_0)$ is the initial state. Since j is persistent in $\{{}^1Z_n;\ n \geqslant 0\}$, the number of steps from i to j is finite with probability one. Let $f'_{ij}(t)$ be the probability density function for the first passage time from i to j in the Markov renewal process $\{{}^1Z_n, {}^1T_n;\ n \geqslant 0\}$. We consider only $i \neq j$. Then in the filtered process $H = \{{}^1Z_n \in \{i, j\}\}$ we have

$$\Pr[{}^1H_1 = j, {}^2H_1 = l, I_1 = 1, V_1 \leqslant v \mid {}^1H_0 = i, {}^2H_0 = k, I_0 = 1, V_0 = v_0]$$

$$= \int_{u=0}^{v} f'_{ij}(u) \frac{\bar{G}_k(v_0+u)}{\bar{G}_k(v_0)}\, du\, \delta_{kl} + \int_{u=0}^{v} \int_{t=u}^{\infty} f'_{ij}(t) s'_{kl}(t-u) \bar{G}_l(u)\, du\, dt$$

$$\geqslant v \bar{G}_l(v)(\Pi_l - \delta) \int_{Y^*+v}^{\infty} f'_{ij}(t)\, dt = 1 - \varrho_l > 0,$$

where Y^* and Π_l are defined above. Arguing as previously, we infer that the number of transitions prior to entrance to the event $\{{}^1Z_n = j, {}^2Z_n = l, I_n = 1, V_n \leqslant v\}$, given the initial state $({}^1Z_0 = i, {}^2Z_0 = k, I_0 = 1, V_0 = v_0)$, is finite with probability one.

Finally, given the initial state $({}^1Z_0 = i, {}^2Z_0 = k, I_0 = 2, V_0 = v_0)$ it is clear that the event $\{I_n = 1\}$ is reached in a finite number of transitions with probability one, since $\Pr[{}^1T_1 < \infty \mid {}^1Z_0 = i, {}^1T_1 > v_0]$ is one regardless of v_0, the initial backward recurrence time.

We have shown that regardless of initial state the event $\{{}^1Z_n = j, {}^2Z_n = l, I_n = 1, V_n \in [v-\varepsilon, v+\varepsilon]\}$ is reached in a finite number of transitions with probability one and that the event re-occurs after a finite number of steps with probability one. For the event $\{{}^1Z_n = j, {}^2Z_n = l, I_n = 2, v_n \in [v-\varepsilon, v+\varepsilon]\}$ we apply identical arguments to the filtered process with $H = \{{}^2Z_n = k\}$. Thus for j and l persistent in their respective component Markov renewal processes the events $\{{}^1Z_n = j, {}^2Z_n = l, I_n = 1, V_n \in [v-\varepsilon, v+\varepsilon]\}$ and $\{{}^1Z_n = j, {}^2Z_n = l, I_n = 2, V_n \in [v-\varepsilon, v+\varepsilon]\}$ occur infinitely often with probability one regardless of initial state.

Theorem 3.2 suggests that the states of the superposed process behave much like those of a discrete Markov chain, in that persistence and transience are in some sense "preserved" by superposition. The infinitely often occurrence of the events contained in $2^{J'_1} \otimes 2^{J'_2} \otimes 2^{\{1,2\}} \otimes \boldsymbol{R}^+$ also suggests the existence of a stationary probability measure for the Markov chain $\{{}^1Z_n, {}^2Z_n, I_n, V_n;\ n \geqslant 0\}$; it is given in the following

THEOREM 3.4. *The Markov chain* $\{{}^1Z_n, {}^2Z_n, I_n, V_n;\ n \geqslant 0\}$ *has a unique stationary distribution given by*

$$\Pr[{}^1Z_0 = i, {}^2Z_0 = k, I_0 = 1, V_0 \in [v_0, v_0 + dv_0)] = \frac{\alpha_i \beta_k}{\sum \alpha_i m_i + \sum \beta_j n_j} \bar{G}_k(v_0)\, dv_0 \tag{3.4}$$

and

$$\Pr[{}^1Z_0 = i, {}^2Z_0 = k, I_0 = 2, V_0 \in [v_0, v_0 + dv_0)] = \frac{\alpha_i \beta_k}{\sum \alpha_i m_i + \sum \beta_j n_j} \bar{F}_i(v_0)\, dv_0, \tag{3.5}$$

where α_i, β_k, m_i, *and* n_j *are defined in Section* 3.1.

Proof. By definition $\Pi(\cdot)$ is stationary for the process $\{S_n;\ n \geqslant 0\}$:

$$\Pi(A) = \int_E \Pr[S_1 \in A \mid S_0 = x]\, \Pi(dx).$$

Due to symmetry we need to consider only one case:

$$\begin{aligned}
&\Pr[{}^1Z_1 = j, {}^2Z_1 = l, I_1 = 1, V_1 \leqslant v] \\
&= \sum_i \sum_k \int_{v_0=0}^{v} \Pr[{}^1Z_1 = j, {}^2Z_1 = l, I_1 = 1, V_1 \\
&\leqslant v \mid {}^1Z_0 = i, {}^2Z_0 = k, I_0 = 1, V_0 \in [v_0, v_0 + dv_0)] \times \\
&\quad \times \Pr[{}^1Z_0 = i, {}^2Z_0 = k, I_0 = 1, V_0 \in [v_0, v_0 + dv_0)] + \\
&\quad + \sum_i \sum_k \int_{v_0=0}^{\infty} \Pr[{}^1Z_1 = j, {}^2Z_1 = l, I_1 = 1, \\
&\quad V_1 \leqslant v \mid {}^1Z_0 = i, {}^2Z_0 = k, I_0 = 2, V_0 \in [v_0, v_0 + dv_0)] \times \\
&\quad \times \Pr[{}^1Z_0 = i, {}^2Z_0 = k, I_0 = 2, V_0 \in [v_0, v_0 + dv_0)] \\
&= \sum_i \sum_k \int_{v_0=0}^{v} \int_{u=v_0}^{v} a_{ij}(u - v_0) \frac{\bar{G}_k(u)}{\bar{G}_k(v_0)}\, \delta_{kl}\, \frac{\alpha_i \beta_k}{\sum \alpha_i m_i + \sum \beta_j n_j} G_k(v_0)\, du\, dv_0 + \\
&\quad + \sum_i \sum_k \int_{v_0=0}^{\infty} \int_{u=0}^{v} \frac{a_{ij}(u + v_0)}{\bar{F}_i(v_0)}\, \bar{G}_k(u)\, \delta_{kl}\, \frac{\alpha_i \beta_k}{\sum \alpha_i m_i + \sum \beta_j n_j} \bar{F}_i(v_0)\, du\, dv_0 \\
&= \frac{\beta_l}{\sum \alpha_i m_i + \sum \beta_j n_j} \sum_i \int_{u=0}^{v} \bar{G}_l(u) \Big(\int_{v_0=0}^{u} a_{ij}(u - v_0)\, dv_0 \Big) \alpha_i\, du +
\end{aligned}$$

$$+\frac{\beta_l}{\sum a_i m_i+\sum \beta_j n_j}\sum_i \int_{u=0}^{v} \bar{G}_l(u)\Big(\int_{v_0=0}^{\infty} a_{ij}(u+v_0)\,dv_0\Big)a_i\,du$$

$$=\frac{\beta_l}{\sum a_i m_i+\sum \beta_j n_j}\sum_i \int_{u=0}^{v} \bar{G}_l(u)\big(A_{ij}(u)-A_{ij}(0)\big)a_i\,du+$$

$$+\frac{\beta_l}{\sum a_i m_i+\sum \beta_j n_j}\sum_i \int_{u=0}^{v} \bar{G}_l(u)\big(A_{ij}(\infty)-A_{ij}(u)\big)a_i\,du$$

$$=\frac{\beta_l}{\sum a_i m_i+\sum \beta_j n_j}\int_{u=0}^{v} \bar{G}_l(u)\Big[\sum_i a_i A_{ij}(\infty)\Big]du$$

$$=\frac{\beta_l a_j}{\sum a_i m_i+\sum \beta_j n_j}\int_{u=0}^{v} \bar{G}_l(u)\,du = \Pr[{}^1Z_0 = j, {}^2Z_0 = l, I_0 = 1, V_0 \leqslant v].$$

The equality holds since $A_{ij}(0) = 0$ by assumption. An identical argument suffices for $I_0 = 2$. The uniqueness of the distribution follows from the indecomposability of the state space and Theorem 2.1.

Comparison of the distribution exhibited in Theorem 3.4 with its counterpart for the renewal case given by Theorem 3.2 in [3] reveals that the quantity $\big(\sum a_i m_i+\sum \beta_j n_j\big)^{-1}$ plays the part of the renewal density function, and that the distribution is formed by multiplying this "density" function first by the stationary, in this case, equilibrium probabilities a_j and β_l of being in states j and l, respectively, and then by an analogue of the equilibrium backward recurrence time distribution. The only noticeable relationship between the two processes occurs in the selection of the backward recurrence time distribution which establishes the value of I_0.

We now turn to the semi-Markov process $\{{}^1Z(t), {}^2Z(t), I(t), V(t); t \geqslant 0\}$ associated with the Markov renewal process (3.1). In particular, we derive the limit of the probability as t grows large:

$$\Pr[{}^1Z(t) = j, {}^2Z(t) = l, I(t) = q, V(t) \in (v-\varepsilon, v+\varepsilon) \mid {}^1Z_0 = i, \\ {}^2Z_0 = k, I = q_0, V_0 = v_0].$$

THEOREM 3.5. *For the semi-Markov process* $\{{}^1Z(t), {}^2Z(t), I(t), V(t); t \geqslant 0\}$ *we have*

$$\lim_{t\to\infty}\Pr[{}^1Z(t) = j, {}^2Z(t) = l, I(t) = q, \\ V(t) \in (v-\varepsilon, v+\varepsilon) \mid {}^1Z_0 = i, {}^2Z_0 = k, I_0 = q_0, V_0 = v_0] \\ = \int_{(j,l,q,(v-\varepsilon,v+\varepsilon))} m(x)\Pi(dx)\Big(\int_E m(x)\Pi(dx)\Big)^{-1},$$

where $m\big((i, k, q, v_n)\big) = \mathrm{E}[W_{n+1} \mid {}^1Z_n = i, {}^2Z_n = k, I_n = q, V_n = v_n]$, *and* $\Pi(\cdot)$ *is a stationary distribution for the underlying Markov chain.*

Proof. By Theorem 3.4 the stationary distribution $\Pi(\cdot)$ exists and is unique. Moreover, $\Pi(\cdot)$ is non-zero only on sets which occur infinitely often with probability one, hence the limit in question is non-zero only on the set $\{j \in J_1', l \in J_2'\}$.

Let

$$\boldsymbol{T}f(x, t) = \int_{E \times R^+} f(y, t-u) K(x, dy) G_{xy}(du)$$

for $f(x, t)$ bounded and measurable and let

$$\mathscr{T}f(x, t) = \sum_{k=0}^{\infty} \boldsymbol{T}^k f(x, t).$$

Then for $f(x, t) = \chi_{A \times (t_1, t_2)}(x, t)$ we have

$$\mathscr{T}f(x, t) = \chi_{A \times (t_1, t_2)} + \sum_{k=1}^{\infty} \Pr[S_n \in A, t-t_2 \leqslant T_k \leqslant t-t_1 \mid S_0 = x]$$

or $\mathscr{T}\chi_{A \times (t_1, t_2)}(x, t)$ is the expected number of visits to the set A in the interval $[t-t_2, t-t_1]$. Due to the structure of the state space it is sufficient to consider the set

$$A = \{{}^1Z_n = j, {}^2Z_n = l, I_n = q, V_n \in (v-\varepsilon, v+\varepsilon)\}$$

and the initial state $x = (i, k, q_0, v_0)$. Then $\mathscr{T}f(x, t)$ becomes the expected number of visits to the set A in the interval $(t-t_2, t-t_1)$ if the initial state is (i, k, q_0, v_0). This function is clearly bounded by the expected number of visits to the state j in the first component process during the same interval, and by Section 3.1 this bound is finite since $\{{}^1Z_n, {}^1T_n; n \geqslant 1\}$ is normal. $\mathscr{T}f(x, t)$ is clearly measurable.

Now we have

$$\Pr[S_1 \in A, t-t_2 \leqslant V_1 \leqslant t-t_1 \mid S_0 = x]$$

$$= \int_{u=v-\varepsilon}^{v+\varepsilon} a_{ij}(u-v_0) \frac{\bar{G}_k(u)}{\bar{G}_k(v_0)} \delta_{kl} D\big((t-t_1)-(u-v_0)\big)\, du -$$

$$- \int_{u=v-\varepsilon}^{v+\varepsilon} a_{ij}(u-v_0) \frac{\bar{G}_k(u)}{\bar{G}_k(v_0)} \delta_{kl} D\big((t-t_2)-(u-v_0)\big)\, du$$

and, for $n \geqslant 2$,

$$\Pr[S_n \in A,\, t-t_2 \leqslant U_n \leqslant t-t_1 \mid S_0 = x]$$

$$= \int_{w=t-t_1}^{t-t_2} \sum_{s=1}^{n-1} \int_{u=v-\varepsilon}^{v+\varepsilon} a_{ij}^{s}(w)\, b_{kl}'^{n-s}(w-u)\, \bar{G}_l(u)\, du\, dw +$$

$$+ \int_{u=v-\varepsilon}^{v+\varepsilon} a_{ij}^{n}(u-v_0)\, \frac{\bar{G}_k(u)}{\bar{G}_k(v_0)}\, \delta_{kl}\, D\big((t-t_2)-(u-v_0)\big)\, du -$$

$$- \int_{u=v-\varepsilon}^{v+\varepsilon} a_{ij}^{n}(u-v_0)\, \frac{\bar{G}_k(u)}{\bar{G}_k(v_0)}\, \delta_{kl}\, D\big((t-t_1)-(u-v_0)\big)\, du.$$

Thus

$$\sum_{n=1}^{\infty} \Pr[S_n \in A,\, t-t_2 \leqslant U_n \leqslant t-t_1 \mid S_0 = x]$$

$$= \sum_{n=1}^{\infty} \int_{u=v-\varepsilon}^{v+\varepsilon} a_{ij}^{n}(u-v_0)\, \frac{\bar{G}_k(u)}{\bar{G}_k(v_0)}\, \delta_{kl} \big[D\big((t-t_1)-(u-v_0)\big) - D\big((t-t_2)-$$

$$- (u-v_0)\big)\big]\, du + \sum_{n=2}^{\infty} \sum_{s=1}^{n-1} \int_{w=t-t_2}^{t-t_1} \int_{u=v-\varepsilon}^{v+\varepsilon} a_{ij}^{s}(w)\, b_{kl}'^{n-s}(w-u)\, \bar{G}_l(u)\, du\, dw$$

$$= \int_{u=v-\varepsilon}^{v+\varepsilon} r_{ij}(u-v_0)\, \frac{\bar{G}_k(u)}{\bar{G}_k(v_0)}\, \delta_{kl} \big[D\big((t-t_1)-(u-v_0)\big) - D\big((t-t_2)-$$

$$- (u-v_0)\big)\big]\, du + \int_{w=t-t_2}^{t-t_1} \int_{u=v-\varepsilon}^{v+\varepsilon} r_{ij}(w)\, s_{kl}'(w-u)\, \bar{G}_l(u)\, du\, dw.$$

As y grows large, we have (see [4]) $r_{ij}(y) \to 1/\mu_j$ and $s_{kl}'(y) \to 1/\nu_l$, where μ_j and ν_l are the mean recurrence times of states j and l, respectively. Further

$$D\big((t-t_1)-(u-v_0)\big) - D\big((t-t_2)-(u-v_0)\big) = 1$$

only for $t-t_2 < u-v_0 < t-t_1$. But the variable u is restricted by $v-\varepsilon < u < v+\varepsilon$, implying

$$\lim_{t\to\infty} \sum_{n=1}^{\infty} \Pr[S_n \in A,\, t-t_2 \leqslant U_n \leqslant t-t_1 \mid S_0 = x]$$

$$= \lim_{t\to\infty} \int_{w=t-t_2}^{t-t_1} \int_{u=v-\varepsilon}^{v+\varepsilon} r_{ij}(w)\, s_{kl}'(w-u)\, \bar{G}_l(u)\, du\, dw.$$

For t sufficiently large, $r_{ij}(w)$ and $s_{kl}'(w-u)$ are arbitrarily close to $1/\mu_j$ and $1/\nu_l$, respectively. Thus the limit is independent of the initial

conditions. Applying Theorem 2.2 we obtain

$$\lim_{t\to\infty} \Pr[{}^1Z(t) = j, {}^2Z(t) = l, I(t) = 1, V(t) \in (v-\varepsilon, v+\varepsilon) \mid {}^1Z_0 = i,$$

$${}^2Z_0 = k, I_0 = 1, V_0 = v_0] = \int_{(j,l,1,(v-\varepsilon,v+\varepsilon))} m(x)\Pi(dx) \Big(\int_E m(x)\Pi(dx)\Big)^{-1}.$$

Identical arguments apply to the initial state $(i, k, 2, v_0)$ and the terminal state $\big(j, k, 2, (v-\varepsilon, v+\varepsilon)\big)$ taken together or paired with the states above and the theorem follows.

We now derive the distribution of the mean time spent in any state and then combine this distribution with Theorem 3.5 to obtain the limiting probability results.

PROPOSITION 3.2. *The distribution function for the sojourn time in state* $({}^1Z_n = i, {}^2Z_n = k, I_n = q, V_n = v_0)$ *is given by*

$$\Pr[W_{n+1} \leqslant w \mid {}^1Z_n = i, {}^2Z_n = k, I_n = 1, V_n = v_n]$$

$$= \frac{1}{\bar{G}_k(v_n)} \int_{z=0}^{w} \big(F_i(dz)\, \bar{G}_k(z+v_n) + G_k\big(d(z+v_n)\big)\, \bar{F}_i(z)\big)$$

and

$$\Pr[W_{n+1} \leqslant w \mid {}^1Z_n = i, {}^2Z_n = k, I_n = 2, V_n = v_n]$$

$$= \frac{1}{\bar{F}_i(v_n)} \int_{z=0}^{w} \big(G_k(dz)\, \bar{F}_i(z+v_n) + F_i\big(d(z+v_n)\big)\, \bar{G}_k(z)\big).$$

Proof. We have

$$\Pr[W_{n+1} \leqslant w \mid {}^1Z_n = i, {}^2Z_n = k, I_n = 1, V_n = v_n]$$

$$= \sum_j \sum_l \int_{v=v_n}^{w+v_n} a_{ij}(v-v_n) \frac{\bar{G}_k(v)}{\bar{G}_k(v_n)} \delta_{kl} D\big(w-(v-v_n)\big)\, dv +$$

$$+ \sum_j \sum_l \int_{v=0}^{w} \frac{b_k(v+v_n)}{\bar{G}_k(v_n)} \bar{F}_i(v)\, \delta_{ij} D(w-v)\, dv$$

$$= \sum_j \int_{v=v_n}^{w+v_n} a_{ij}(v-v_n) \frac{\bar{G}_k(v)}{\bar{G}_k(v_n)} dv + \sum_l \int_{v=0}^{w} \frac{b_{kl}(v+v_n)}{\bar{G}_k(v_n)} \bar{F}_i(v)\, dv$$

$$= \frac{1}{\bar{G}_k(v_n)} \Big[\sum_j \int_{z=0}^{w} a_{ij}(z)\, \bar{G}_k(z+v_n)\, dz + \sum_l \int_{z=0}^{w} b_{kl}(z+v_n)\, \bar{F}_i(z)\, dz \Big]$$

$$= \frac{1}{\bar{G}_k(v_n)} \int_{z=0}^{w} \big(F_i(dz)\, \bar{G}_k(z+v_n) + G_k\big(d(z+v_n)\big)\, \bar{F}_i(z)\big).$$

By symmetry we obtain the second equality.

Examination of the distributions given in Proposition 3.2 reveals that the random variable in question, namely W_{n+1}, is simply the minimum of two independent random variables — the sojourn times in states i and k, respectively, with the appropriate bias introduced by the presence of a backward recurrence time for one of the two. Thus we obtain

$$m(x) = \begin{cases} \int_{z=0}^{\infty} \bar{F}_i(z) \dfrac{\bar{G}_k(z+v_n)}{\bar{G}_k(v_n)}\, dz & \text{for } x = (i, k, 1, v_n), \\ \int_{z=0}^{\infty} \bar{G}_k(z) \dfrac{\bar{F}_i(z+v_n)}{\bar{F}_i(v_n)}\, dz & \text{for } x = (i, k, 2, v_n). \end{cases}$$

The limiting probability for the event $\{{}^1Z(t) = i, {}^2Z(t) = k, I(t) = q, V(t) \in (v-\varepsilon, v+\varepsilon)\}$ will now be explicitly derived.

PROPOSITION 3.3. *The limiting probability given by Theorem* 3.5 *takes the form*

$$\begin{aligned} (3.6)\quad & \lim_{t\to\infty} \Pr[{}^1Z(t) = i, {}^2Z(t) = k, I(t) = 1, \\ & V(t) \in (v-\varepsilon, v+\varepsilon) \mid {}^1Z_0 = j, {}^2Z_0 = l, I_0 = q, V_0 = v_0] \\ & = \int_{z=0}^{\infty} \int_{u=v-\varepsilon}^{v+\varepsilon} \bar{F}_i(z) \bar{G}_k(z+u) \frac{\alpha_i \beta_k}{(\sum \alpha_i m_i)(\sum \beta_j n_j)}\, du\, dz \end{aligned}$$

and

$$\begin{aligned} (3.7)\quad & \lim_{t\to\infty} \Pr[{}^1Z(t) = i, {}^2Z(t) = k, I(t) = 2, \\ & V(t) \in (v-\varepsilon, v+\varepsilon) \mid {}^1Z_0 = j, {}^2Z_0 = l, I_0 = q, V_0 = v_0] \\ & = \int_{z=0}^{\infty} \int_{u=v-\varepsilon}^{v+\varepsilon} \bar{F}_i(z+u) \bar{G}_k(z) \frac{\alpha_i \beta_k}{(\sum \alpha_i m_i)(\sum \beta_j n_j)}\, du\, dz. \end{aligned}$$

Proof. Let $I(t) = 1$. Then by Theorem 3.5 and Proposition 3.2 we have

$$\begin{aligned} & \lim_{t\to\infty} \Pr[{}^1Z(t) = i, {}^2Z(t) = k, I(t) = 1, \\ & V(t) \in (v-\varepsilon, v+\varepsilon) \mid {}^1Z_0 = j, {}^2Z_0 = l, I_0 = q, V_0 = v_0] \\ & = \int_{(i,k,1,(v-\varepsilon,v+\varepsilon))} m(x) \Pi(dx) \Big(\int_E m(x) \Pi(dx) \Big)^{-1}. \end{aligned}$$

Moreover, we obtain

$$\int\limits_{(i,k,1,(v-\varepsilon,v+\varepsilon))} m(x)\,\Pi(dx) = \int\limits_{u=v-\varepsilon}^{v+\varepsilon}\int\limits_{z=0}^{\infty} \overline{F}_i(z)\,\frac{\overline{G}_k(z+u)}{\overline{G}_k(u)}\,\frac{\alpha_i\beta_k}{\sum\alpha_i m_i+\sum\beta_j n_j}\times$$

$$\times\,\overline{G}_k(u)\,dz\,du = \int\limits_{z=0}^{\infty}\int\limits_{u=v-\varepsilon}^{v+\varepsilon} \overline{F}_i(z)\,\overline{G}_k(z+u)\,\frac{\alpha_i\beta_k}{\sum\alpha_i m_i+\sum\beta_j n_j}\,dz\,du$$

and

$$\int\limits_E m(x)\,\Pi(dx) = \sum_i\sum_k \int\limits_{u=0}^{\infty}\int\limits_{z=0}^{\infty} \overline{F}_i(z)\,\frac{\overline{G}_k(z+u)}{\overline{G}_k(u)}\,\frac{\alpha_i\beta_k\overline{G}_k(u)}{\sum\alpha_i m_i+\sum\beta_j n_j}\,du\,dz +$$

$$+\sum_i\sum_k \int\limits_{u=0}^{\infty}\int\limits_{z=0}^{\infty} \frac{\overline{F}_i(z+u)}{\overline{F}_i(u)}\,\overline{G}_k(z)\,\frac{\alpha_i\beta_k\overline{F}_i(u)}{\sum\alpha_i m_i+\sum\beta_j n_j}\,du\,dz$$

$$= \sum_i\sum_k \frac{\alpha_i\beta_k}{\sum\alpha_i m_i+\sum\beta_j n_j}\Big[\int\limits_{u=0}^{\infty}\int\limits_{z=0}^{\infty} \overline{F}_i(z)\,\overline{G}_k(z+u)\,du\,dz +$$

$$+\int\limits_{zu=0}^{\infty}\int\limits_{=0}^{\infty} \overline{F}_i(z+u)\,\overline{G}_k(z)\,du\,dz\Big].$$

Changing the variables of integration we obtain

$$\int\limits_E m(x)\,\Pi(dx) = \sum_i\sum_k \frac{\alpha_i\beta_k}{\sum\alpha_i m_i+\sum\beta_j n_j}\Big[\int\limits_{z=0}^{\infty}\int\limits_{w=z}^{\infty} \overline{F}_i(z)\,\overline{G}_k(w)\,dw\,dz +$$

$$+\int\limits_{z=0}^{\infty}\int\limits_{w=z}^{\infty} \overline{F}_i(w)\,\overline{G}_k(z)\,dw\,dz\Big]$$

$$= \sum_i\sum_k \frac{\alpha_i\beta_k}{\sum\alpha_i m_i+\sum\beta_j n_j}\Big[\int\limits_{z=0}^{\infty}\int\limits_{w=z}^{\infty} \overline{F}_i(z)\,\overline{G}_k(w)\,dw\,dz +$$

$$+\int\limits_{w=0}^{\infty}\int\limits_{z=w}^{\infty} \overline{F}_i(z)\,\overline{G}_k(w)\,dw\,dz\Big]$$

$$= \sum_i\sum_k \frac{\alpha_i\beta_k}{\sum\alpha_i m_i+\sum\beta_j n_j}\Big[\int\limits_{z=0}^{\infty}\int\limits_{w=z}^{\infty} \overline{F}_i(z)\,\overline{G}_k(w)\,dw\,dz +$$

$$+\int\limits_{z=0}^{\infty}\int\limits_{w=0}^{z} \overline{F}_i(z)\,\overline{G}_k(w)\,dw\,dz\Big] = \frac{\left(\sum\alpha_i m_i\right)\left(\sum\beta_k n_k\right)}{\sum\alpha_i m_i+\sum\beta_j n_j}.$$

Hence

$$\lim_{t\to\infty} \Pr[{}^1Z(t) = i, {}^2Z(t) = k, I(t) = 1, V(t) \in (v-\varepsilon, v+\varepsilon)]$$

$$= \int_{z=0}^{\infty} \int_{u=v-\varepsilon}^{v+\varepsilon} \bar{F}_i(z)\bar{G}_k(z+u) \frac{a_i\beta_k}{(\sum a_i m_i)(\sum \beta_j n_j)}\, dz\, du.$$

A similar result for $I(t) = 2$ follows by symmetry.

Allowing u to vary between zero and infinity and summing over the values of $I(t)$ we obtain the following

COROLLARY 3.2. *The limit of the marginal joint probability on* ${}^1Z(t)$ *and* ${}^2Z(t)$ *takes the form*

$$\lim_{t\to\infty} \Pr[{}^1Z(t) = i, {}^2Z(t) = k] = \frac{a_i m_i}{\sum a_i m_i} \frac{\beta_k n_k}{\sum \beta_k n_k}.$$

Obviously, this result, the product of the limiting probabilities for the two component processes, follows directly from the independence of the component processes.

4. APPLICATIONS AND ILLUSTRATIONS OF SUPERPOSITION

4.0. Introduction. In this section we illustrate the use of the theory developed in the preceding chapters with two examples. We first examine the superposition of the departure processes from two independent $M/G/1$ queues and give some specific results for the $M/M/1$ queue. We then turn to a reliability and maintenance model and illustrate the errors which can be introduced by approximating a Markov renewal process on a complex state space with a Markov renewal process on a countable state space.

4.1. Merging the outputs of two independent $M/G/1$ queues. The problem is illustrated in Fig. 4.1. The merged output stream is clearly a point process, that is, a sequence of departures separated by random

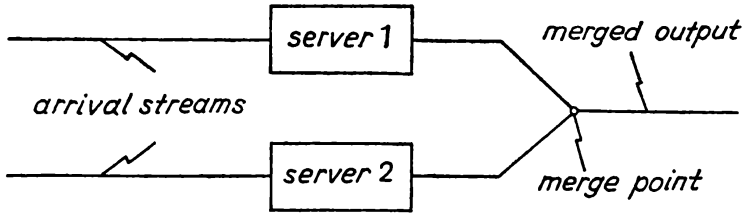

Fig. 4.1. Merging the outputs of two independent $M/G/1$ queues

time intervals. The departures may be homogeneous or may be characterized by the component process in which they receive service. We give a characterization of the merged output stream that establishes its structure and retains the ability to differentiate between customers depending upon the queue from which any particular customer departs.

We make the following assumptions:

(1) the arrival streams to servers 1 and 2 are Poisson with parameters λ_1 and λ_2, respectively;

(2) the service times for servers 1 and 2 are random variables with non-singular distribution functions $F(x)$ and $G(y)$, respectively;

(3) the expected service times μ_1 and μ_2 for servers 1 and 2 are finite and the products $\lambda_1\mu_1$ and $\lambda_2\mu_2$ are strictly less than one;

(4) the two queues are independent.

The departure processes or output streams from the two queues are denoted by $\{{}^1Z_n, {}^1T_n;\ n \geqslant 0\}$ and $\{{}^2Z_m, {}^2T_m;\ m \geqslant 0\}$, where ${}^1T_n$ and ${}^2T_m$ are the epochs of the n-th and m-th departures from servers 1 and 2, respectively, and ${}^1Z_n$ and ${}^2Z_m$ are the numbers of customers in systems 1 and 2 at times ${}^1T_n^+$ and ${}^2T_n^+$, respectively.

It is well known [7] that the stochastic processes $\{{}^1Z_n, {}^1T_n;\ n \geqslant 0\}$ and $\{{}^2Z_m, {}^2T_m;\ m \geqslant 0\}$ are Markov renewal processes with transition probabilities given by

$$A_{ij}(t) = \Pr[{}^1Z_n = j,\ {}^1T_n - {}^1T_{n-1} \leqslant t \mid {}^1Z_{n-1} = i], \quad n \geqslant 1,$$

$$B_{kl}(t) = \Pr[{}^2Z_m = l,\ {}^2T_m - {}^2T_{m-1} \leqslant t \mid {}^2Z_{m-1} = k], \quad m \geqslant 1.$$

The matrices of transition probabilities are given by

$$A_{0j}(t) = \int_{y=0}^{t} \left(1 - \exp[-\lambda_1(t-y)]\right) \exp[-\lambda_1 y] \frac{(\lambda_1 y)^j}{j!} F(dy) \quad \text{for } j \geqslant 0,$$

$$A_{ij}(t) = \begin{cases} \displaystyle\int_{y=0}^{t} \frac{(\lambda_1 y)^{j-i+1}}{(j-i+1)!} \exp[-\lambda_1 y] F(dy) & \text{for } i \geqslant 1,\ j \geqslant i-1, \\ 0 & \text{for } j < i-1 \end{cases}$$

and

$$B_{0l}(t) = \int_{y=0}^{t} \left(1 - \exp[-\lambda_2(t-y)]\right) \exp[-\lambda_2 y] \frac{(\lambda_2 y)^l}{l!} G(dy) \quad \text{for } l \geqslant 0,$$

$$B_{kl}(t) = \begin{cases} \displaystyle\int_{y=0}^{t} \frac{(\lambda_2 y)^{l-k+1}}{(l-k+1)!} \exp[-\lambda_2 y] G(dy) & \text{for } k \geqslant 1,\ l \geqslant k-1, \\ 0 & \text{for } l < k-1. \end{cases}$$

By summation we obtain

$$F_0(t) = \int_{y=0}^{t} \left(1-\exp\left[-\lambda_1(t-y)\right]\right) F(dy),$$

$$F_i(t) = \int_{y=0}^{t} F(dy) = F(t) \quad \text{for } i \geqslant 1$$

and

$$G_0(t) = \int_{y=0}^{t} \left(1-\exp\left[-\lambda_2(t-y)\right]\right) F(dy),$$

$$G_k(t) = \int_{y=0}^{t} G(dy) = G(t) \quad \text{for } k \geqslant 1.$$

We now superpose the sequence of departure times $\{{}^1T_n, n \geqslant 0\}$ and $\{{}^2T_m, m \geqslant 0\}$ to obtain a single monotone sequence $\{U_k, k \geqslant 0\}$ and define

${}^1Z_k$: the number of customers in system 1 at time U_k^+,

${}^2Z_k$: the number of customers in system 2 at time U_k^+,

I_k: the index of the server which had a departure at epoch U_k,

V_k: the time elapsed since the last departure from the server which did not complete a service at U_k.

PROPOSITION 4.1. *The superposition or merger of the departure processes from two independent $M/G/1$ queues is a Markov renewal process defined on the state space*

$$(J \times J \times \{1, 2\} \times R^+, 2^J \otimes 2^J \otimes 2^{\{1,2\}} \otimes \boldsymbol{R}^+),$$

where J is the set of non-negative integers.

The proposition follows directly from Theorem 3.1.

Note that the assumption of non-singular probability distribution functions ensures that the superposition of the two output processes is regular, that is, the probability of simultaneous transitions in the two departure processes is zero. The chief significance of Proposition 4.1 is structural, that is, it establishes the structure of the superposition or merging of the two output processes and suggests that, conceptually, the merged process is not difficult to deal with. The transition probabilities for the process can be determined by Corollary 3.1. Unfortunately, examination of this corollary suggests that computationally the merged process is very unwieldy.

From Theorem 3.3 we may infer that the event

$$\{{}^1Z_n = i, {}^2Z_n = k, I_n = q, V_n \in (v-\varepsilon, v+\varepsilon)\}$$

occurs infinitely often with probability one regardless of initial state. Moreover, the assumption that $\lambda_1\mu_1$ and $\lambda_2\mu_2$ are strictly less than one implies that stationary distributions $\{a_i\}_{i\in J}$ and $\{\beta_k\}_{k\in J}$ exist for the underlying Markov chains of the two independent departure processes. These distributions have generating functions [12] given by

$$\sum_{i=0}^{\infty} a_i z^i = \frac{(1-\lambda_1\mu_1)(z-1)f^*(\lambda_1-\lambda_1 z)}{z-f^*(\lambda_1-\lambda_1 z)}$$

and

$$\sum_{k=0}^{\infty} \beta_k z^k = \frac{(1-\lambda_2\mu_2)(z-1)g^*(\lambda_2-\lambda_2 z)}{z-g^*(\lambda_2-\lambda_2 z)},$$

where $f^*(s)$ and $g^*(s)$ are the Laplace-Stieltjes transforms of $F(x)$ and $G(y)$, the service time probability distribution functions for server 1 and server 2, respectively. If the stationary distributions $\{a_i\}_{i\in I}$ and $\{\beta_k\}_{k\in J}$ exist, then Theorem 3.4 implies that a stationary distribution exists for the underlying Markov chain $\{{}^1Z_n, {}^2Z_n, I_n, V_n;\ n \geqslant 0\}$ and is given by (3.4) and (3.5). For simplification we evaluate

$$\begin{aligned}\sum_{i=0}^{\infty} a_i m_i &= a_0\left(\frac{1}{\lambda_1}+\mu_1\right)+(1-a_0)\mu_1 \\ &= (1-\lambda_1\mu_1)\left(\frac{1}{\lambda_1}+\mu_1\right)+\lambda_1\mu_1^2 = \frac{1}{\lambda_1}\end{aligned}$$

and, similarly,

$$\sum_{k=0}^{\infty} \beta_k n_k = \frac{1}{\lambda_2}.$$

Thus, from (3.4) and (3.5) we obtain

$$\Pr[{}^1Z_0 = i, {}^2Z_0 = k, I_0 = 1, V_0 \in [v_0, v_0+dv_0)] = \lambda_1\lambda_2 \frac{a_i\beta_k}{\lambda_1+\lambda_2}\bar{G}_k(v_0)\,dv_0$$

and

$$\Pr[{}^1Z_0 = i, {}^2Z_0 = k, I_0 = 2, V_0 \in [v_0, v_0+dv_0)] = \frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2} a_i\beta_k \bar{F}_i(v_0)\,dv_0.$$

For the superposition of two $M/M/1$ queues, with service time distributions $1-\exp[-\nu_1 t]$ and $1-\exp[-\nu_2 t]$, respectively, we define ϱ_1

and ϱ_2 to be λ_1/ν_1 and λ_2/ν_2 and obtain, for the stationary distribution with $I_0 = 1$,

$$\Pr[{}^1Z_0 = i, {}^2Z_0 = 0, I_0 = 1, V_0 \in [v_0, v_0 + dv_0)]$$
$$= \frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}(1-\varrho_1)\varrho_1^i(1-\varrho_2)\left(\frac{\nu_2}{\nu_2-\lambda_2}\exp[-\lambda_2 v_0] - \right.$$
$$\left. - \frac{\lambda_2}{\nu_2-\lambda_2}\exp[-\nu_2 v_0]\right)dv_0, \qquad i \geqslant 0,$$

and

$$\Pr[{}^1Z_0 = i, {}^2Z_0 = k, I_0 = 1, V_0 \in [v_0, v_0 + dv_0)]$$
$$= \frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}(1-\varrho_1)\varrho_1^i(1-\varrho_2)(\varrho_2)^k\exp[-\nu_2 v_0]dv_0, \qquad k \geqslant 1,\ i \geqslant 1.$$

Similar results follow for $I_0 = 2$.

Now we may infer that the limiting probability

$$\lim_{t\to\infty}\Pr[{}^1Z(t) = i, {}^2Z(t) = k, I(t) = q, V(t) \in (v-\varepsilon, v+\varepsilon)]$$

exists by Theorem 3.5 since the departure processes are normal. The limits are given by

$$\lim_{t\to\infty}\Pr[{}^1Z(t) = i, {}^2Z(t) = k, I(t) = 1, V(t) \in (v-\varepsilon, v+\varepsilon)]$$
$$= \int_{z=0}^{\infty}\int_{u=v-\varepsilon}^{v+\varepsilon}\bar{F}_i(z)\bar{G}_k(z+u)\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\alpha_i\beta_k\,dz\,du$$

and

$$\lim_{t\to\infty}\Pr[{}^1Z(t) = i, {}^2Z(t) = k, I(t) = 2, V(t) \in (v-\varepsilon, v+\varepsilon)]$$
$$= \int_{z=0}^{\infty}\int_{u=v-\varepsilon}^{v+\varepsilon}\bar{F}_i(z+u)\bar{G}_k(z)\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\alpha_i\beta_k\,dz\,du.$$

Thus, with $\varrho_1 = \lambda_1/\nu_1$ and $\varrho_2 = \lambda_2/\nu_2$ we obtain from Proposition 3.3 for an arbitrary (j, l, q, v_0):

$$\lim_{t\to\infty}\Pr[{}^1Z(t) = 0, {}^2Z(t) = 0, I(t) = 1, V(t) \geqslant v \mid {}^1Z_0 = j, {}^2Z_0 = l,$$
$$I_0 = q, V_0 = v_0] = \frac{\lambda_1\lambda_2}{(\nu_1-\lambda_1)(\nu_2-\lambda_2)}(1-\varrho_1)(1-\varrho_2)\times$$
$$\times\left[\frac{\exp[-\lambda_2 v]}{\varrho_2}\left(\frac{\nu_1}{\lambda_1+\lambda_2} - \frac{\lambda_1}{\lambda_2+\nu_1}\right) - \varrho_2\exp[-\nu_2 v]\left(\frac{\nu_1}{\lambda_1+\lambda_2} - \frac{\lambda_1}{\lambda_1+\nu_2}\right)\right],$$

$$\lim_{t\to\infty} \Pr[{}^1Z(t) = 0, {}^2Z(t) = k, I(t) = 1, V(t) \geqslant v \mid {}^1Z_0 = j, {}^2Z_0 = l, I_0 = q, V_0 = v_0]$$

$$= \frac{\lambda_1\lambda_2}{\nu_1 - \lambda_1}(1-\varrho_1)(1-\varrho_2)\varrho_2^k \frac{\exp[-\nu_2 v]}{\nu_2}\left(\frac{\nu_1}{\lambda_1+\nu_2} - \frac{\lambda_1}{\nu_1+\nu_2}\right), \quad k \geqslant 1,$$

$$\lim_{t\to\infty} \Pr[{}^1Z(t) = i, {}^2Z(t) = 0, I(t) = 1, V(t) \geqslant v \mid {}^1Z_0 = j, {}^2Z_0 = l, I_0 = q, V_0 = v_0]$$

$$= \frac{\lambda_1\lambda_2}{\nu_2 - \lambda_2}(1-\varrho_1)\varrho_1^i(1-\varrho_2)\left(\frac{\exp[-\lambda_2 v]}{\varrho_2(\lambda_2+\nu_1)} - \varrho_2\frac{\exp[-\nu_2 v]}{\nu_2+\nu_1}\right), \quad i \geqslant 1,$$

$$\lim_{t\to\infty} \Pr[{}^1Z(t) = i, {}^2Z(t) = k, I(t) = 1, V(t) \geqslant v \mid {}^1Z_0 = j, {}^2Z_0 = l, I_0 = q, V_0 = v_0]$$

$$= \lambda_1\lambda_2(1-\varrho_1)\varrho_1^i(1-\varrho_2)\varrho_2^k \frac{\exp[-\nu_2 v]}{\nu_2}\frac{1}{\nu_1+\nu_2}, \quad i \geqslant 1,\ k \geqslant 1.$$

Similar results hold for the case $I(t) = 2$.

It is interesting to observe the form of the marginal distribution of the backward recurrence time in the limiting results. Summing over the indices i and k we obtain

$$\lim_{t\to\infty} \Pr[I(t) = 1, V(t) \geqslant v] = \frac{\lambda_1}{\lambda_1+\lambda_2}\exp[-\lambda_2 v]$$

and

$$\lim_{t\to\infty} \Pr[I(t) = 2, V(t) \geqslant v] = \frac{\lambda_2}{\lambda_1+\lambda_2}\exp[-\lambda_1 v].$$

The result implies that in equilibrium the superposed system behaves like a Poisson process with parameter $\lambda_1+\lambda_2$ with backward recurrence time distribution functions $1-\exp[-\lambda_2 v]$ and $1-\exp[-\lambda_1 v]$ chosen if the last event was from process 1 or process 2, respectively. The service time parameters ν_1 and ν_2 do not appear. Obviously, this result follows directly from [2] and the independence of the two queues.

4.2. A reliability and maintenance model. In this section we consider a system consisting of two independent components in parallel. One might think of these components as generators supplying power; each can meet system requirements alone, but a backup is provided which

runs continuously between breakdowns. We suppose that each component operates for a random amount of time and then requires minor repair or major repair. Minor repair may lead to major repair, but following a major repair the component is again serviceable. We assume that the operating-repair cycle of the component can be modelled as a Markov renewal process with the following states: 1 — operating, 2 — undergoing minor repair, and 3 — undergoing major repair. We denote the transition probabilities for the two components by the matrices $(A_{ij}(t))$ and $(B_{kl}(t))$, where

$$(A_{ij}(t)) = \begin{bmatrix} 0 & A_{12}(t) & A_{13}(t) \\ A_{21}(t) & 0 & A_{23}(t) \\ A_{31}(t) & 0 & 0 \end{bmatrix}$$

and

$$(B_{kl}(t)) = \begin{bmatrix} 0 & B_{12}(t) & B_{13}(t) \\ B_{21}(t) & 0 & B_{23}(t) \\ B_{31}(t) & 0 & 0 \end{bmatrix}.$$

The Markov renewal processes $\{{}^1Z_n, {}^1T_n;\ n \geqslant 0\}$ and $\{{}^2Z_m, {}^2T_m;\ m \geqslant 0\}$ describing the operating-repair cycles of components 1 and 2, respectively, are assumed to be aperiodic, normal, and conservative. Furthermore, the conditional probability distribution functions of sojourn times are assumed to be non-singular with $A_{ij}(0) = B_{kl}(0) = 0$ for all i, j, k, and l. The process to be modelled is illustrated in Fig. 4.2.

Superposing the sequences $\{{}^1T_n;\ n \geqslant 0\}$ and $\{{}^2T_m;\ m \geqslant 0\}$ and re-ordering the indices we obtain the sequence of epoch times $\{U_k;\ k \geqslant 0\}$. We then define

${}^1Z_k$ to be the state of component 1 at time U_k^+,

${}^2Z_k$ to be the state of component 2 at time U_k^+,

I_k to be the index of the component whose status changed at U_k,

V_k to be the time elapsed since the last change of status of the component which does not change at epoch U_k.

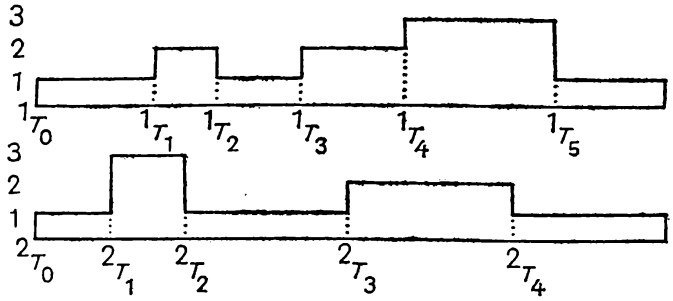

Fig. 4.2. The superposition of the operating-repair cycles of two independent components with states 1, 2, 3

Then by Theorem 3.1 the stochastic process (3.1) is a Markov renewal process defined on

$$[\{1,2,3\}\times\{1,2,3\}\times\{1,2\}\times R^+, 2^{\{1,2,3\}}\otimes 2^{\{1,2,3\}}\otimes 2^{\{1,2\}}\otimes \boldsymbol{R}^+).$$

From Corollary 3.1 we obtain the transition probabilities (3.2) and (3.3).

Similar results follow for the cases $I_{n+1}=2, I_n=2$ and $I_{n+1}=1$, $I_n=2$.

The assumed aperiodicity of $\{{}^1Z_n, {}^1T_n;\ n\geqslant 0\}$ and $\{{}^2Z_m, {}^2T_m;\ m\geqslant 0\}$ implies that the transition probability matrices for the underlying Markov chains of the component processes take the forms

$$\big(A_{ij}(\infty)\big)=\begin{bmatrix}0 & a_1 & 1-a_1\\ a_2 & 0 & 1-a_2\\ 1 & 0 & 0\end{bmatrix}$$

and

$$\big(B_{kl}(\infty)\big)=\begin{bmatrix}0 & b_1 & 1-b_1\\ b_2 & 0 & 1-b_2\\ 1 & 0 & 0\end{bmatrix}.$$

Hence both underlying Markov chains have stationary probability vectors, say $\{\alpha_i\}_{i=1,2,3}$ and $\{\beta_k\}_{k=1,2,3}$, where

$$\alpha_1=\frac{1}{2+a_1-a_2a_1},\qquad \alpha_2=\frac{a_1}{2+a_1-a_2a_1},\qquad \alpha_3=\frac{1-a_2a_1}{2+a_1-a_2a_1}$$

and

$$\beta_1=\frac{1}{2+b_1-b_2b_1},\qquad \beta_2=\frac{b_1}{2+b_1-b_2b_1},\qquad \beta_3=\frac{1-b_2b_1}{2+b_1-b_2b_1}.$$

Thus, provided that both component processes are normal, we obtain from Theorem 3.5, regardless of the initial state (j, l, q, v_0), the limiting probabilities (3.6) and (3.7), where $i, k\in\{1,2,3\}$, and

$$F_i(t)=\sum_{j=1}^{3}A_{ij}(t),\qquad G_k(t)=\sum_{l=1}^{3}B_{kl}(t),$$

$$m_i=\int_0^\infty \bar{F}_i(t)\,dt,\qquad n_j=\int_0^\infty \bar{G}_k(t)\,dt.$$

Again the limiting probability that both components are undergoing repair is the product of the limiting probabilities that each component is undergoing repair, i.e.,

$$\lim_{t\to\infty}\Pr[{}^1Z(t)\in\{2,3\}, {}^2Z(t)\in\{2,3\}]=\frac{a_2m_2+a_3m_3}{\sum a_im_i}\,\frac{\beta_2n_2+\beta_3n_3}{\sum\beta_jn_j}.$$

That the limiting probability has the product form is not surprising since the components are independent. However, this form does not justify the following model which has been used in the past for investigations of systems such as that described above. In the model, a Markov renewal process on a countable state space is used under the assumption that backward recurrence times can be neglected. To illustrate this model, suppose that the two components are identical. The state space becomes $\{1, 2, 3\} \times \{1, 2, 3\}$ and the probability transition mechanism is given by

$$\Pr[({}^1Z_n, {}^2Z_n) = (j, l),\ W_n \leqslant w \mid ({}^1Z_{n-1}, {}^2Z_{n-1}) = (i, k)]$$

$$= \begin{cases} \int_{y=0}^{w} a_{ij}(y)\, \bar{G}_k(y)\, dy & \text{for } k = l, \\ \int_{y=0}^{w} b_{kl}(y)\, \bar{F}_i(y)\, dy & \text{for } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

This model neglects the backward recurrence time or, equivalently, assumes that an event in one process resets to zero the clock in the second process without a state transition. Clearly, if the sojourn times in the component processes have increasing failure rates, i.e.,

$$\Pr[X_{n+1} > x + x_{n+1} \mid X_{n+1} \geqslant x_{n+1}] < \Pr[X_{n+1} > x],$$

the use of this model is not without danger. In particular, limiting results derived from this structure may not agree with those obtained from the superposition model. Consider, e.g., two identical processes with Markov renewal transition matrix

$$\begin{bmatrix} 0 & \frac{3}{4}(1-\exp[-x^2/4]) & \frac{1}{4}(1-\exp[-x^2/64]) \\ \frac{7}{8}(1-\exp[-x^2]) & 0 & \frac{1}{8}(1-\exp[-16x^2]) \\ 1-\exp[-x^2/16] & 0 & 0 \end{bmatrix}.$$

Note that the sojourn times have Weibull distributions and increasing failure rates. The two processes satisfy the assumptions of Theorem 3.1 and we may, therefore, assert that the process (3.1) is a Markov renewal process where the random variables ${}^1Z_n$, ${}^2Z_n$, I_n, V_n, and U_n have their usual meanings.

The component processes are clearly normal. Thus, by Proposition 3.3, regardless of initial state,

$$\lim_{t \to \infty} \Pr[{}^1Z(t) = i, {}^2Z(t) = k] = \frac{a_i m_i \beta_k m_k}{(\sum a_i m_i)^2}.$$

As before $\{a_i\}_{i\in\{1,2,3\}}$ is a stationary distribution for the underlying Markov chain of the component process and m_i, $i \in \{1, 2, 3\}$, is the mean sojourn time in state i.

Approximating the process with the model outlined above we obtain a nine-state Markov renewal process with limiting probabilities

$$\lim_{t\to\infty} \Pr\left[\left({}^1Z(t), {}^2Z(t)\right) = i, k\right] = \frac{\beta_{ik} n_{ik}}{\sum_j \sum_l \beta_{jl} n_{jl}},$$

where $\{\beta_{ik}\}_{i,k\in\{1,2,3\}\times\{1,2,3\}}$ is a stationary distribution of the underlying Markov chain of the approximation and n_{ik} are the appropriate sojourn times.

For the numerical example above the true model of superposition and the approximate model have the limiting probabilities as given in Table 4.1.

TABLE 4.1. Comparison of limiting probabilities of two models

State	Superposition model	Approximate model
11	0.397024	0.383957
13	0.155974	0.147734
31	0.155974	0.147734
33	0.061275	0.049079
23	0.030290	0.037973
32	0.030290	0.037973
12	0.077101	0.089302
21	0.077101	0.089302
22	0.014973	0.016948

5. SUMMARY, COMMENTS AND TWO PROBLEMS

5.0. Summary and comments. In this paper we have been concerned with the interval process occurring in the superposition of two independent Markov renewal processes. A complete structure has been given for this process as well as transition functions. With these results one can now, in principle, explore the superposed process in detail using adaptations of the well-known results of Markov renewal theory. To proceed with that in practice, however, seems to be a formidable task simply due to the dimensionality of the problem. Thus, while it is clear how to proceed to the study of the superposition of any finite number of independent Markov renewal processes, detailed investigations appear to be facing a monumental problem of computation. This is unfortunate since in the study

of most flow processes (e.g., road traffic processes, queueing network flow) one is concerned with properties of the stream of events obtained by superposing a few streams. Whether the study of such processes is made simpler by examining the related counting processes, as has been done in most previous studies of superposition, will have to be seen.

5.1. Two problems. There are two problems that occur to one in pursuing this superposition topic. To the best of our knowledge all known superposition results assume the processes to be superposed are independent. Yet in studies of flow in queueing networks it appears that flows along alternate routes may not be independent. In the simplest case that we have encountered, one takes the output of a server and splits it into two processes using a Bernoulli process to decide which subprocess each output joins. One of these two substreams departs from the system. The other returns units to the tail of the queue for more service. In this process the returning stream and the arrival process are superposed to form the stream of inputs to the server. This superposed stream of inputs is not a renewal process even in the case of $M/M/1$ queues. Very little else is known about that process. Further research could be useful.

A corollary problem to that above occurs in many practical situations when streams of events are merged. While the various streams may be independent, the merging is not a superposition problem as normally defined. For example, the merging of side street traffic into traffic on a limited access highway is not a superposition problem. We know of no results on the properties of the merged stream.

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SUPERPOZYCJA DWÓCH NIEZALEŻNYCH PROCESÓW ODNOWY TYPU MARKOWA

STRESZCZENIE

Przy badaniu superpozycji dwóch lub więcej strumieni stochastycznych dotychczas zwracano głównie uwagę na superpozycję procesów odnowy, traktowanych jako procesy licznikowe (*counting processes*). Niniejsza praca podaje strukturę procesu przedziałowego (*interval process*), będącego wynikiem superpozycji dwóch niezależnych markowowskich procesów odnowy, które mają przeliczalnie wiele stanów. Rozdział 4 przytacza zastosowania takiej charakterystyki do teorii obsługi masowej i do zagadnień niezawodności.
