ZASTOSOWANIA MATEMATYKI
APPLICATIONES MATHEMATICAE
XVI, 1 (1977)

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AN INTEGRAL EQUATION TECHNIQUE FOR SOLVING MIXED BOUNDARY VALUE PROBLEMS

We discuss here the solution of n $(n > 1)$ simultaneous Fredholm integral equations of the first kind which cannot be solved directly and present an integral equation method to convert them into $4n$ Volterra integral equations of the first kind and $4n$ Fredholm integral equations of the second kind. The $4n$ Volterra integral equations have simple kernels and can therefore be easily inverted whereas the $4n$ Fredholm integral equations of the second kind can be solved approximately by iteration. Finally, we illustrate our method by considering an electrostatic potential problem involving two annular coaxial concentric spherical caps in a free space.

1. Introduction. Various methods have been developed to solve mixed boundary value problems in mechanics and mathematical physics (see [1], [2], [8], [13]-[15], [19], [20]). One of the methods was originated by Williams [19] and [20] and was later modified by Jain and Kanwal [8]. In this method the solution of a mixed boundary value problem is expressed in terms of the solution of a Fredholm integral equation of the first kind. Later, by using an interesting technique, the integral equation is converted into a set of Fredholm integral equation of the second kind and Volterra integral equations of the first kind. Using this technique, several mixed boundary value problems were solved successfully (see [4]-[7] and [9]-[11]). All these boundary value problems were associated with only one object. Vaid and Jain [16] presented an integral equation technique to consider $2n$ part boundary value problems which arise in connection with n coaxial disks or n coaxial concentric spherical caps. They have solved various boundary value problems involving two coaxial disks or two concentric coaxial spherical caps (see [17] and [18]).

The objective of this paper is to connect the integral equation techniques given in [8] and [16]. We propose to develop an integral equation method which is useful in discussing mixed boundary value problems which arise in connection with n coaxial annular disks or n annular coaxial concentric spherical caps. Thus it represents a development of Williams work [19].

By following Green's function approach [12], we express the solution of the problem in terms of the solution of n simultaneous Fredholm integral equations of the first kind which cannot be solved directly. Following the techniques of [8] and [16] and splitting the kernels in the given n Fredholm integral equations of the first kind, we readily convert their solutions to the solutions of $4n$ Fredholm integral equations of the second kind and $4n$ Volterra integral equations of the first kind. The Volterra integral equations have simple kernels and thus can be easily inverted, and the Fredholm integral equations of the second kind can be solved approximately by standard iterations in terms of small perturbation parameters.

The scheme of this paper is as follows. In Section 2 we present the method of solving the generalized $3n$ part boundary value problems. In order to illustrate the method we consider the electrostatic potential problem of two coaxial concentric annular spherical caps charged to prescribed potentials in a free space. By using this method we can solve many problems occurring in electrostatics, acoustics as well as in electromagnetic diffraction and hydrodynamics for the cases where they are associated with several annular objects and where the corresponding problem for one object is already solved. In fact, the author has already solved some of these problems and they will be the subject of future communications.

2. Integral equation method. Solutions of many mixed boundary value problems in mechanics and mathematical physics, associated with n coaxial disks, are described by (see [12]) the equation

$$\int_{b_i}^{a_i} K_{ii}(t,\varrho)g_i(t)\,dt+\sum_{\substack{j=1\\ j\neq i}}^{n}\int_{b_i}^{a_i} G_{ij}(t,\varrho)g_j(t)\,dt = f_i(\varrho), \qquad b_i<\varrho<a_i \quad (i=1,2,\ldots,n), \tag{1}$$

where b_i and a_i are the inner and the outer radii, respectively, of the i-th disk $(i=1,2,\ldots,n)$, f_i, K_{ii} and G_{ij} are known functions in the integral equations (1), and $g_i(t)$ $(i=1,2,\ldots,n)$ are the functions to be determined. All the equations in this paper are valid for $i=1,2,\ldots,n$ unless otherwise stated. These n equations cannot easily be solved for $g_i(t)$. However, we can convert (1) into $4n$ Volterra integral equations of the first kind and $4n$ Fredholm integral equations of the second kind.

Assuming (see [3])

$$f_i(\varrho)=\sum_{j=-\infty}^{\infty} a_{ij}\varrho^j, \qquad b_i<\varrho<a_i, \tag{2}$$

we introduce the $4n$ functions as

(3) $$f_{i1}(\varrho) = \sum_{j=0}^{\infty} a_{ij}\varrho^j, \quad 0 < \varrho < a_i,$$

(4) $$f_{i2}(\varrho) = \sum_{j=-\infty}^{-1} a_{ij}\varrho^j, \quad b_i < \varrho < \infty,$$

(5) $$g_{i1}(t)+g_{i2}(t) = \begin{cases} 0, & 0 < t < b_i, \\ g_i(t), & b_i < t < a_i, \\ 0, & a_i < t < \infty, \end{cases}$$

thus splitting (1) into the following $2n$ equations (6) and (7):

(6) $$\int_0^{\infty} K_{ii}(t, \varrho) g_{i1}(t)dt + \sum_{\substack{j=1 \\ j\neq i}}^{n} \int_0^{\infty} G_{ij}(t, \varrho) g_{j1}(t)dt = f_{i1}(\varrho), \quad 0 < \varrho < \infty,$$

(7) $$\int_0^{\infty} K_{ii}(t, \varrho) g_{i2}(t)dt + \sum_{\substack{j=1 \\ j\neq i}}^{n} \int_0^{\infty} G_{ij}(t, \varrho) g_{j2}(t)dt = f_{i2}(\varrho), \quad 0 < \varrho < \infty.$$

This method requires the kernels K_{ij} and G_{ij} to satisfy the following two conditions:

(a) K_{ii} can be split into the form

(8) $$K_{ii} = L_i + G_{ii},$$

where G_{ij} are in some sense smaller than the dominating part L_i of the kernel K_{ii};

(b) the kernels L_i are expressible in the following forms:

(9) $$L_i(t, \varrho) = \begin{cases} h_{i11}(\varrho)h_{i13}(t) \displaystyle\int_0^{\min(\varrho,t)} K_i(w, \varrho)K_i(w, t)[h_{i12}(w)]^2 dw, & 0 < \varrho,\ t < \infty, \\ h_{i21}(\varrho)h_{i23}(t) \displaystyle\int_{\max(\varrho,t)}^{\infty} K_i(\varrho, w)K_i(t, w)[h_{i22}(w)]^2 dw, & 0 < \varrho,\ t < \infty. \end{cases}$$

Here h_{ikj} $(k, j = 1, 2, 3)$ and K_i are known functions such that the $2n$ Volterra integral equations

(10) $$\int_0^{\varrho} K_i(t, \varrho) g(t)dt = F_1(\varrho), \quad 0 < \varrho < \infty,$$

and

(11) $$\int_{\varrho}^{\infty} K_i(\varrho, t) g(t)dt = F_2(\varrho), \quad 0 < \varrho < \infty,$$

have explicit unique solutions for $g(t)$ in terms of the arbitrary differentiable functions F_1 and F_2. It is obvious that relations (10) and (11) ensure that the kernels G_{ij} can be expressed as

$$(12)\quad G_{ij}(t,\varrho) = \begin{cases} h_{i11}(\varrho)h_{i13}(t)\int\limits_0^\varrho\int\limits_0^t K_i(w,\varrho)K_i(v,t)h_{i12}(w)h_{i12}(v)L_{ij1}(v,w)\,dv\,dw, \\ h_{i21}(\varrho)h_{i23}(t)\int\limits_\varrho^\infty\int\limits_t^\infty K_i(\varrho,w)K_i(t,v)h_{i22}(w)h_{i22}(v)L_{ij2}(v,w)\,dv\,dw. \end{cases}$$

Using (8), (9) and (12) in equations (6) and (7), and interchanging various orders of integrations, we transform these equations into

$$(13)\quad h_{i11}(\varrho)\Big[\int\limits_0^\infty K_i(w,\varrho)\{h_{i12}(w)\}^2\int\limits_w^\infty K_i(w,t)g_{i1}(t)h_{i13}(t)\,dt\,dw + \\ +\sum_{j=1}^n\int\limits_0^\varrho K_i(w,\varrho)h_{i12}(w)\int\limits_0^\infty L_{ij1}(v,w)h_{i12}(v)\int\limits_v^\infty K_i(v,t)\times \\ \times g_{j1}(t)h_{i13}(t)\,dt\,dv\,dw\Big] = f_{i1}(\varrho),\qquad 0<\varrho<a_i,$$

$$(14)\quad h_{i21}(\varrho)\Big[\int\limits_\varrho^\infty K_i(\varrho,w)\{h_{i22}(w)\}^2\int\limits_0^w K_i(t,w)g_{i2}(t)h_{i23}(t)\,dt\,dw + \\ +\sum_{j=1}^n\int\limits_\varrho^\infty K_i(\varrho,w)h_{i22}(w)\int\limits_0^\infty L_{ij2}(v,w)h_{i22}(v)\int\limits_0^v K_i(t,v)\times \\ \times g_{j2}(t)h_{i23}(t)\,dt\,dv\,dw\Big] = f_{i2}(\varrho),\qquad b_i<\varrho<\infty.$$

Finally, we define $6n$ functions S_{ik}, T_{ik}, and C_{ik} $(k=1,2)$ so that

$$(15)\quad h_{i12}(\varrho)\int\limits_\varrho^\infty K_i(\varrho,t)g_{i1}(t)h_{i13}(t)\,dt = \begin{cases} S_{i1}(\varrho), & 0<\varrho<a_i, \\ -T_{i1}(\varrho), & a_i<\varrho<\infty, \end{cases}$$

$$(16)\quad h_{i22}(\varrho)\int\limits_0^\varrho K_i(t,\varrho)g_{i2}(t)h_{i23}(t)\,dt = \begin{cases} -T_{i2}(\varrho), & 0<\varrho<b_i, \\ S_{i2}(\varrho), & b_i<\varrho<\infty, \end{cases}$$

$$(17)\quad h_{i11}(\varrho)\int\limits_0^\varrho K_i(w,\varrho)C_{i1}(w)h_{i12}(w)\,dw = f_{i1}(\varrho),\qquad 0<\varrho<a_i,$$

$$(18)\quad h_{i21}(\varrho)\int\limits_\varrho^\infty K_i(\varrho,w)C_{i2}(w)h_{i22}(w)\,dw = f_{i2}(\varrho),\qquad b_i<\varrho<\infty.$$

It is obvious that the expressions for the $2n$ functions $C_{ik}(\varrho)$ can explicitly be obtained from equations (17) and (18) in view of the assumptions that equations (10) and (11) can be inverted. Using (15)-(18)

in equations (13) and (14), we obtain

$$(19)\qquad S_{i1}(\varrho)+\sum_{j=1}^{n}\int_{0}^{a_j} L_{ij1}(v,\varrho)S_{j1}(v)\,dv = C_{i1}(\varrho)+\sum_{j=1}^{n}\int_{a_j}^{\infty} L_{ij1}(v,\varrho)T_{j1}(v)\,dv, \qquad 0<\varrho<a_i,$$

$$(20)\qquad S_{i2}(\varrho)+\sum_{j=1}^{n}\int_{b_j}^{\infty} L_{ij2}(v,\varrho)S_{j2}(v)\,dv = C_{i2}(\varrho)+\sum_{j=1}^{n}\int_{0}^{b_j} L_{ij2}(v,\varrho)T_{j2}(v)\,dv, \qquad b_i<\varrho<\infty.$$

Using equations (5), (15) and (16), we also get

$$(21)\qquad T_{i1}(\varrho)=h_{i12}(\varrho)\int_{\varrho}^{\infty} K_i(\varrho,t)g_{i2}(t)h_{i13}(t)\,dt, \qquad a_i<\varrho<\infty,$$

$$(22)\qquad T_{i2}(\varrho)=h_{i22}(\varrho)\int_{0}^{\varrho} K_i(t,\varrho)g_{i1}(t)h_{i23}(t)\,dt, \qquad 0<\varrho<b_i.$$

By equations (2), (3) and (4), we get f_{i1} and f_{i2} which enables us to determine C_{i1} and C_{i2} in view of equations (17) and (18). By inverting (15) and (16), we get g_{i1} and g_{i2} in terms of S_{ik} and T_{ik} $(k=1,2)$. Substituting these values of g_{ik} into equations (21) and (22) we obtain $2n$ relations which together with $2n$ equations (19) and (20) are enough to determine the approximate values of the $4n$ unknown functions S_{ik} and T_{ik} $(k=1,2)$. Once S_{ik} and T_{ik} are defined, we determine g_{ik} which gives the values of $g_i(t)$ in view of relation (5).

Finally, we point out that the above analysis can also be applied to n simultaneous Fredholm integral equations of the first kind,

$$\int_{\beta_i}^{a_i} K_{ii}(t,\theta)g_i(t)\,dt+\sum_{\substack{j=1\\ j\neq i}}^{n}\int_{\beta_j}^{a_j} G_{ij}(t,\theta)g_j(t)\,dt=f_i(\theta), \qquad \beta_i<\theta<a_i,$$

which embodies the solutions of many mixed boundary value problems in mechanics and mathematical physics for n concentric coaxial spherical caps, where we change $\varrho, a_i, b_i, \infty$ into $\theta, a_i, \beta_i, \pi$, respectively, a_i, β_i being the bounding angles of the i-th annular cap.

So far we have considered the solution of a generalized $3n$ part boundary value problem. For a $3n$ part boundary value problem, we have $G_{ii}=0$, and so $L_{ii}=0$, and the rest of the analysis is the same.

We illustrate the method of this section by considering one mixed boundary value problem in electrostatics involving n coaxial concentric annular spherical caps. Later we consider a special case by taking $n=2$.

3. Electrostatic potential problem for n annular coaxial concentric spherical caps. Let us consider n annular concentric coaxial spherical caps of radii a_i with inner and outer bounding angles β_i and α_i, respectively. Let the i-th annular cap be charged to an axially symmetric potential $f_i(\theta)$ in a free space. Let $V(r, \theta, \varphi)$ denote the potential of the system in spherical polar coordinates (r, θ, φ) with $\theta = 0$ as the axis of symmetry, origin being at their common centre. Then we have the boundary problem

$$\begin{aligned} \nabla^2 V(r, \theta, \varphi) &= 0 \quad \text{in } D, \\ V(a_i, \theta, \varphi) &= f_i(\theta) \quad \text{for all } \beta_i < \theta < \alpha_i, \end{aligned} \tag{23}$$

where V and $\partial V/\partial r$ are continuous across the region $r = a_i$, $0 < \theta < \beta_i$, $\alpha_i < \theta < \pi$, and D being the region exterior to all the caps.

Using Green's function approach [12], we get

$$V(r, \theta, \varphi) = \sum_{j=1}^{n} a_j^2 \int_{\beta_j}^{\alpha_j} \int_0^{2\pi} \left[\sin t \frac{\sigma_j(t)}{R(r, \theta, \varphi; a_j, t, \varphi')} \right] dt\, d\varphi',$$

where $R(r, \theta, \varphi; a_j, t, \varphi')$ is the distance between the points (r, θ, φ) and (a_j, t, φ'), and $\sigma_j(t)$ are the charge densities.

By the boundary conditions (23), we obtain the following n simultaneous Fredholm integral equations of the first kind:

$$a_i^2 \int_{\beta_i}^{\alpha_i} \int_0^{2\pi} \left[\sin t \frac{\sigma_i(t)}{R(a_i, \theta, \varphi; a_i, t, \varphi')} \right] d\varphi'\, dt +$$

$$+ \sum_{\substack{j=1 \\ j \neq i}}^{n} a_j^2 \int_{\beta_j}^{\alpha_j} \int_0^{2\pi} \left[\sin t \frac{\sigma_j(t)}{R(a_i, \theta, \varphi; a_j, t, \varphi')} \right] d\varphi'\, dt = f_i(\theta), \qquad \alpha_i < \theta < \beta_i.$$

Comparing these equations with the notation in the previous section, and using some well-known results [10], we have

$$K_{ii}(t, \theta) = L_i(t, \theta) = a_i \int_0^{2\pi} \frac{d\varphi'}{R(a_i, \theta, \varphi; a_i, t, \varphi')}, \qquad G_{ii} = 0,$$

$$G_{ij}(t, \theta) = a_j \int_0^{2\pi} \frac{d\varphi'}{R(a_i, \theta, \varphi; a_j, t, \varphi')},$$

$$a_i \sin t\, \sigma_{ik}(t) = g_{ik}(t) \qquad (k = 1, 2),$$

$$h_{i11} = h_{i21} = 2, \quad h_{i13} = h_{i12} = h_{i23} = h_{i22} = 1, \quad K_i(w, \theta) = (\cos w - \cos\theta)^{-1/2}$$

with

$$L_{ij1}(v, w) = \frac{2a_j}{\pi} \sum_{n=0}^{\infty} \frac{a_{\leqslant}^n}{a_{\geqslant}^{n+1}} \left[\cos\left(n+\frac{1}{2}\right)w\right]\left[\cos\left(n+\frac{1}{2}\right)v\right],$$

$$L_{ij2}(v, w) = \frac{2a_j}{\pi} \sum_{n=0}^{\infty} \frac{a_{\leqslant}^n}{a_{\geqslant}^{n+1}} \left[\sin\left(n+\frac{1}{2}\right)w\right]\left[\sin\left(n+\frac{1}{2}\right)v\right],$$

and

$$a_{\leqslant} = \min(a_i, a_j), \qquad a_{\geqslant} = \max(a_i, a_j).$$

The kernel $K_i(w, \theta)$ is elementary and the equations corresponding to (15)-(18) can easily be inverted [12].

We now consider the particular case where $n = 2$. Suppose that the two annular caps of radii a_1 and a_2 are kept at constant potentials V_1 and V_2, respectively. We further assume that $a_1/a_2 = \lambda$ and that $\beta_1, \beta_2, \lambda \ll 1$.

In this case, we have $f_{i1}(\theta) = V_i, f_{i2}(\theta) = 0$ $(i = 1, 2)$ for the rest of the analysis, and

$$C_{i1}(\theta) = \frac{1}{\sqrt{2}\pi} V_i \cos\frac{\theta}{2}, \; 0 < \theta < a_i, \qquad \text{and} \qquad C_{i2}(\theta) = 0, \; \beta_i < \theta < \pi.$$

To solve equations (19)-(22), we assume that $S_{i1} = X_{i1} + W_{i1}$, where, for $i, j = 1, 2, \; i \neq j$,

$$X_{i1}(\theta) + \int_0^{a_j} L_{ij1}(\theta, \varphi) X_{j1}(\varphi)\, d\varphi = C_{i1}(\theta), \qquad 0 < \theta < a_i, \tag{24}$$

$$W_{i1}(\theta) + \int_0^{a_j} L_{ij1}(\theta, \varphi) W_{j1}(\varphi)\, d\varphi = \int_{a_j}^{\pi} L_{ij1}(\theta, \varphi) T_{j1}(\varphi)\, d\varphi, \quad 0 < \theta < a_i. \tag{25}$$

Equations (24) and (25) have the solutions (see [16])

$$X_{i1}(\theta) = \frac{1}{\sqrt{2}\pi}\left[e_{i1}\cos\frac{\theta}{2} + e_{i3}\cos\frac{3\theta}{2} + \right.$$
$$\left. + e_{i5}\cos\frac{5\theta}{2} + e_{i7}\cos\frac{7\theta}{2} + O(\lambda^4)\right],$$

where

$$e_{11} = K_1\left[1 + \frac{\lambda}{\pi^2} W_0(a_1)W_0(a_2) + \frac{\lambda^2}{\pi^4}\{W_0(a_1)W_0(a_2)\}^2 + \frac{\lambda^2}{\pi^2} W_1(a_1)W_1(a_2) + \right.$$
$$\left. + \frac{\lambda^3}{\pi^2}\left\{W_2(a_1)W_2(a_2) + \frac{3}{\pi^2} W_0(a_1)W_1(a_1)W_0(a_2)W_1(a_2) + \frac{1}{\pi^4} W_0^3(a_1)W_0^3(a_2)\right\}\right] -$$

$$- V_2 \frac{\lambda^2}{\pi^3} \Big[W_1(a_2) W_0(a_2) W_1(a_1) + \lambda W_1^2(a_2) A(a_1) + \lambda W_0(a_2) W_2(a_2) W_2(a_1) + \\ + \frac{\lambda}{\pi^2} W_0(a_1) W_1(a_1) W_1(a_2) W_0^2(a_2) \Big],$$

$$e_{13} = K_1 \frac{\lambda^2}{\pi^2} \Big[W_1(a_2) W_0(a_1) + \lambda A(a_2) W_1(a_1) + \frac{\lambda}{\pi^2} W_0^2(a_1) W_0(a_2) W_1(a_2) \Big] - \\ - V_2 \frac{\lambda}{\pi} W_1(a_2) \Big[1 + \frac{\lambda^2}{\pi^2} W_1(a_2) W_1(a_1) \Big],$$

$$e_{15} = \frac{\lambda^2}{\pi} W_2(a_2) \Big[- V_2 + \frac{\lambda}{\pi} K_1 W_0(a_1) \Big],$$

$$e_{17} = - V_2 \frac{\lambda^3}{\pi} W_3(a_2),$$

$$e_{21} = K_2 \Big[1 + \frac{\lambda}{\pi^2} W_0(a_1) W_0(a_2) + \frac{\lambda^2}{\pi^2} W_1(a_1) W_1(a_2) + \frac{\lambda^3}{\pi^2} W_2(a_1) W_2(a_2) + \\ + \frac{\lambda^2}{\pi^4} W_0^2(a_1) W_0^2(a_2) + \frac{3\lambda^3}{\pi^4} W_1(a_1) W_0(a_1) W_0(a_2) W_1(a_2) + \\ + \frac{\lambda^3}{\pi^6} W_0^3(a_1) W_0^3(a_2) \Big] - V_1 \frac{\lambda^3}{\pi^3} W_1(a_1) W_0(a_1) W_1(a_2),$$

$$e_{23} = K_2 \frac{\lambda^2}{\pi^2} \Big[W_1(a_1) W_0(a_2) + \lambda A(a_1) W_1(a_2) + \frac{\lambda}{\pi^2} W_0(a_1) W_0^2(a_2) W_1(a_1) \Big] - \\ - V_1 \frac{\lambda^2}{\pi} W_1(a_1),$$

$$e_{25} = - V_1 \frac{\lambda^3}{\pi} W_2(a_1) + \lambda^3 K_2 W_2(a_1) W_0(a_2),$$

$$e_{27} = 0$$

with

$$W_n(a) = \begin{cases} \dfrac{\sin n a}{n} + \dfrac{\sin(n+1) a}{n+1}, & n \neq 0, \\ a + \sin a, & n = 0, \end{cases}$$

$$K_1 = V_1 - V_2 \frac{1}{\pi} W_0(a_2), \quad K_2 = V_2 - V_1 \frac{\lambda}{\pi} W_0(a_1)$$

and

$$A(a) = a + \frac{1}{3} \sin 3a.$$

The other functions are obtained in the following order: $X_{i1}, l_{i1}, T_{i2}, S_{i2}, l_{i2}, T_{i1}, W_{i1}, S_{i1}$. The results are

$$l_{i1} = T_{i2} = e_{i1} \frac{1}{2\sqrt{2}\pi^2}\left[a_i + 2\cot\frac{a_i}{2}\right]\theta + O(\beta_i^2), \qquad 0 < \theta < \beta_i,$$

$$S_{12} = e_{21}\frac{1}{6\sqrt{2}\pi^3}\left[a_2 + 2\cot\frac{a_2}{2}\right]\beta_2^3 \sin\left(\frac{\theta}{2}\right) + O(\beta_2^4), \qquad \beta_1 < \theta < \pi,$$

$$S_{22} = O(\beta_2^4), \qquad \beta_2 < \theta < \pi,$$

$$l_{12} = e_{21}\frac{1}{6\sqrt{2}\pi^3}\left[a_2 + 2\cot\frac{a_2}{2}\right]\beta_2^3 \cos\left(\frac{\theta}{2}\right) + O(\beta_1^4), \qquad a_1 < \theta < \pi,$$

$$l_{22} = O(\beta_1^4), \qquad a_2 < \theta < \pi,$$

$$T_{i1} = l_{i2} + e_{i1}\frac{1}{12\sqrt{2}\pi^3}\cot\left(\frac{\theta}{2}\right)\operatorname{cosec}\left(\frac{\theta}{2}\right)\beta_i^3\left[a_i + 2\cot\frac{a_i}{2}\right], \qquad a_i < \theta < \pi,$$

$$S_{11} = X_{11} + e_{21}\beta_2^3\frac{1}{6\sqrt{2}\pi^4}\left[a_2 + 2\cot\frac{a_2}{2}\right]\left[a_2 + 2\cot\left(\frac{a_2}{2}\right) - \pi\right]\cos\frac{\theta}{2},$$

$$S_{21} = X_{21} + O(\beta_1^4).$$

Finally, we calculate the total charge C_i on each of the annular caps. Indeed, following the same method as given in appendix B of [10], we get

$$C_i = 2\pi a_i^2 \int_{\beta_i}^{a_i} \sin\theta\,\sigma_i(\theta)\,d\theta$$

$$= 2\sqrt{2}\left[\int_0^{a_i}\cos\left(\frac{w}{2}\right)S_{i1}(w)\,dw - \int_{a_i}^{\pi}\cos\left(\frac{w}{2}\right)T_{i1}(w)\,dw - \right.$$

$$\left. - \int_0^{\beta_i}\sin\left(\frac{w}{2}\right)T_{i2}(w)\,dw + \int_{\beta_i}^{\pi}\sin\left(\frac{w}{2}\right)S_{i2}(w)\,dw\right].$$

These integrals can easily be evaluated.

The capacity of the condenser can be obtained from the previous results by taking $V_1 = V_2 = 1$ and by adding the two total charges thus obtained. Here we see that, if we take $\beta_1, \beta_2 \to 0$, we get the known results.

By giving special values to the bounding angles of the annular spherical caps we can consider several cases of interest most of which appear to be new.

Following this technique we can present the corresponding generalized $3n$ part boundary value problems when the caps are surrounded either by a grounded cylinder with axis of symmetry as $\theta = 0$ or by two grounded parallel planes.

The author is grateful to Prof. S. M. Farouq Ali and Prof. R. P. Kanwal for their encouragements. He is also very grateful to Prof. D. L. Jain for advice and for providing him preprints of his research.

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DEPT. OF PETROLEUM AND NATURAL GAS ENGG.
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Received on 3. 8. 1975

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ROZWIĄZYWANIE MIESZANYCH ZAGADNIEŃ BRZEGOWYCH PRZY UŻYCIU RÓWNAŃ CAŁKOWYCH

STRESZCZENIE

W pracy omawia się rozwiązywanie n ($n > 1$) równań całkowych Fredholma pierwszego rodzaju, nie dających się rozwiązać bezpośrednio, i przedstawia się metodę sprowadzenia ich do $4n$ równań całkowych Volterry pierwszego rodzaju i $4n$ równań całkowych Fredholma drugiego rodzaju. Równania całkowe Volterry mają jądra proste i mogą być łatwo odwrócone, natomiast równania całkowe Fredholma drugiego rodzaju można rozwiązać w sposób przybliżony poprzez iterację. Proponowaną metodę ilustruje się na przykładzie zagadnienia elektrostatycznego potencjału dla dwu pierścieniowatych współosiowych koncentrycznych czasz sferycznych w przestrzeni swobodnej.