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SOME MARTINGALES IN DAMS WITH POISSON INPUT AND RELEASE

1. Introduction. In relatively few storage models it is assumed that both inputs and releases are random. One can mention here the paper by Gani and Pyke [1], and then the papers by Puri and Senturia [5]-[7]. In [5] and [6] the authors consider the infinite dam in which the times of inputs and releases are the jump times of a semi-Markov process and all inputs and releases form two independent sequences of nonnegative independent random variables with common probability distributions in every sequence. For such a model they obtain, among others, the Laplace transform of the time to the first emptiness in the following two cases:

(A) the probability distributions of inputs and releases are exponential, while those of the distances between input and release moments remain arbitrary;

(B) the probability distributions of inputs and distances between input and release moments are exponential, while those of releases remain arbitrary.

In proofs of the theorems the authors apply the known technique of solution of integral equations.

In this paper we consider an infinite dam in which the total input $A(t)$ in the interval $(0, t]$ forms a process with stationary independent increments whose almost all paths are nondecreasing step functions vanishing at zero and the total release $B(t)$ in the interval $(0, t]$ is a compound Poisson process with exponential distribution of jumps. For such a model with given initial content of the dam we obtain the Laplace-Stieltjes transform of the time to the first emptiness. Next, we consider a finite dam in which both processes $\{A(t), t \geqslant 0\}$ and $\{B(t), t \geqslant 0\}$ are compound Poisson ones with exponential distribution of jumps and we get the Laplace-Stieltjes transform of the time to the first overflow under the assumption of null initial content of the dam. For the results we apply the method using martingale properties, which is described by Kennedy in [3].

2. Infinite dam. Assume that the input process $\{A(t), t \geqslant 0\}$ defined on the probability space $(\Omega, \mathscr{F}, P)$ is the process with stationary independent increments and nondecreasing step paths vanishing at zero. The Laplace-Stieltjes transform of the random variable $A(t)$, $t \geqslant 0$ fixed, is of the form (see, e.g., [2])

$$\mathrm{E}\{e^{-\theta A(t)}\} = \exp\{-t\xi(\theta)\} \tag{1}$$

for an arbitrary real $\theta > 0$, where

$$\xi(\theta) = \int_0^\infty (1-e^{-\theta x})\, dN(x), \tag{2}$$

$N(x)$ is a right-continuous real function in $(0, \infty)$, nondecreasing in $[0, \infty)$.

Assume further that

$$\varrho = \int_0^\infty x\,dN(x) < \infty \quad \text{and} \quad \sigma^2 = \int_0^\infty x^2\,dN(x) < \infty,$$

i.e. the random variable $A(t)$ has finite expectation $\mathrm{E}\{A(t)\} = \varrho t$ and variance $\mathrm{D}^2\{A(t)\} = \sigma^2 t$.

The output process $\{B(t), t \geqslant 0\}$ is the compound Poisson process with jump intensity μ and exponential, with parameter v, distribution of jump size, independent of the input process.

Let $\{C(t), t \geqslant 0\}$ denote the net input process, i.e. $C(t) = A(t)-B(t)$. Under the assumptions of the model the process $\{C(t), t \geqslant 0\}$ is infinitely divisible and for every $t \geqslant 0$ we can define the moment generating function of the form

$$\mathrm{E}\{e^{-\theta C(t)}\} = \exp\{-t\Phi(\theta)\}, \quad 0 < \theta < v, \tag{3}$$

where

$$\Phi(\theta) = \xi(\theta) - \frac{\mu\theta}{v-\theta}. \tag{4}$$

Next, let $\mathscr{F}(t)$ denote the σ-field generated by the process $\{C(u), 0 \leqslant u \leqslant t\}$, and $X(t)$, t fixed, the random variable determined by the formula

$$X(t) = \exp\{\Phi(\theta)t - \theta(c+C(t))\} \tag{5}$$

for $0 < \theta < v$ and $c \geqslant 0$.

THEOREM 1. *The stochastic process $\{X(t), \mathscr{F}(t), t \geqslant 0\}$ is a martingale.*

Proof. By (5), the measurability of $X(t)$ relative to $\mathscr{F}(t)$ as well as its integrability are obvious. Using (3) and the infinite divisibility of the process $\{C(t), t \geqslant 0\}$, for $0 \leqslant u \leqslant t$, $0 < \theta < v$ we have

$$\begin{aligned} & \mathrm{E}\{X(t) \mid \mathscr{F}(u)\} \\ & = \exp\{\Phi(\theta)u - \theta(c+C(u))\}\exp\{\Phi(\theta)(t-u)\}\mathrm{E}\{\exp(-\theta C(t-u)) \mid \mathscr{F}(u)\} \\ & = X(u)\exp\{\Phi(\theta)(t-u)\}\exp\{-\Phi(\theta)(t-u)\} = X(u), \end{aligned}$$

which completes the proof.

Let T_c be the time to the first emptiness of the dam with initial content $c \geqslant 0$, defined by the formula

$$T_c = \inf_{0 \leqslant t < \infty} \{t \colon c + C(t) \leqslant 0\}. \tag{6}$$

Using martingale properties of $\{X(t), \mathscr{F}(t), t \geqslant 0\}$, we obtain a simple formula for $\mathrm{E}\{\exp(-sT_c)\}$, $s > 0$.

THEOREM 2. *If* $\varrho < \mu/v$, *then*

$$\mathrm{E}\{\exp(-sT_c)\} = \frac{v - \theta(s)}{v} \exp\{-\theta(s)c\}, \tag{7}$$

where $\theta(s)$ *is the unique nonnegative solution of the equation*

$$-\Phi(\theta) = s \tag{8}$$

with unknown θ $(0 < \theta < v)$ *and an arbitrary* $s > 0$.

Proof. It is easy to see from (2) that equation (8) has the unique nonnegative solution $\theta(s) < v$ for every $s > 0$ and, under the assumption $\varrho < \mu/v$,

$$\lim_{s \searrow 0} \theta(s) = 0.$$

Thus for an arbitrary $s > 0$ and $\theta = \theta(s)$, by (4), (5), and Theorem 1, the random variable

$$X_0(t) = \exp\{-st - \theta(s)(c + C(t))\}$$

forms the martingale relative to $\{\mathscr{F}(t), t \geqslant 0\}$. Note that it is bounded on the set $\{t < T_c\}$, for then $c + C(t) > 0$. From martingale properties (see [4], p. 70) we obtain

$$\mathrm{E}\{X_0(T_c \wedge t)\} = \mathrm{E}\{X_0(0)\}, \quad t \geqslant 0,$$

where $a \wedge b = \min(a, b)$. Hence

$$\int_{\{t \geqslant T_c\}} X_0(T_c)\,dP + \int_{\{t < T_c\}} X_0(t)\,dP = \mathrm{E}\{X_0(0)\}, \quad t \geqslant 0.$$

Since $X_0(t)$ is bounded on the set $\{t < T_c\}$, the probability measure P is continuous, and the condition $P\{T_c < \infty\} = 1$ is satisfied for $\varrho < \mu/v$, we have

$$\mathrm{E}\{X_0(T_c)\} = \mathrm{E}\{X_0(0)\}. \tag{9}$$

Since the distribution of jump sizes of the output process has the "forgetful property", the following equality holds: $c+C(T_c) = 0-X$, where X is a nonnegative random variable with exponential distribution, independent of T_c. Hence, by (9), we get (7).

3. Finite dam. Before considering the continuous time model we deal with its analogue in discrete time. Assume that the capacity of the dam is equal to $K < \infty$ and the consecutive net inputs C_n, $n = 1, 2, \ldots$, are independent random variables with identical probability distributions defined by the formula

$$P\{C_1 \leqslant x\} = \begin{cases} \dfrac{\mu}{\lambda+\mu} e^{vx}, & x \leqslant 0, \\ \dfrac{\mu}{\lambda+\mu} + \dfrac{\lambda}{\lambda+\mu}(1-e^{-\delta x}), & x > 0, \end{cases} \tag{10}$$

where λ, μ, δ, and v are positive constants.

It is easy to verify that the moment generating function $\zeta(\theta) = \mathrm{E}\{\exp(-\theta C_1)\}$ is of the form

$$\zeta(\theta) = \frac{\lambda\delta}{(\lambda+\mu)(\theta+\delta)} + \frac{\mu v}{(\lambda+\mu)(v-\theta)}, \qquad -\delta < \theta < v. \tag{11}$$

Let us introduce the notation

$$S_0 = 0, \quad S_k = \sum_{i=1}^{k} C_i, \quad k = 1, 2, \ldots,$$

$$m_n = \min_{0\leqslant k\leqslant n} S_k, \quad \mathscr{F}_n = \sigma\{C_1, C_2, \ldots, C_n\}$$

and let Z_n denote the content of the dam at the moment n. It is known that if $Z_0 = 0$, then

$$Z_n = S_n - m_n, \quad n = 1, 2, \ldots \tag{12}$$

In this model we indicate the martingale which we use while considering the continuous time model.

THEOREM 3. *For real ω, θ, θ_1 such that $\zeta(\theta) < \infty$, $\theta \neq \theta_1$, and $\zeta(\theta) = \zeta(\theta_1)$, the sequence of random variables Y_n, $n = 1, 2, \ldots$, defined by the formula*

$$Y_n = \frac{\exp\{\omega m_n\}}{\zeta^n(\theta)}[A(\omega, \theta_1)\exp\{\theta Z_n\} - A(\omega, \theta)\exp\{\theta_1 Z_n\}], \tag{13}$$

where

$$A(\omega, \theta) = \frac{\omega - \theta}{(\omega + v)(\theta + v)}, \quad v > 0, \tag{14}$$

is the martingale relative to the family of the σ-fields $\{\mathscr{F}_n, n \geqslant 1\}$.

Proof. By (13), the measurability of Y_n relative to $\mathscr{F}_n$ as well as its integrability are evident since $0 \leqslant Z_n \leqslant K$. Using (10)-(12) and (14) we obtain

$$\begin{aligned} &\mathrm{E}\{\exp(\omega m_n + \theta Z_n) \mid \mathscr{F}_{n-1}\} \\ &= \exp\{\omega m_{n-1} + \theta Z_{n-1}\}\zeta(\theta) - \frac{\mu v}{\lambda + \mu} A(\omega, \theta)\exp\{\omega m_{n-1} - vZ_{n-1}\}. \end{aligned} \tag{15}$$

Simultaneously, by (11), it is easy to verify that

$$\{\theta \neq \theta_1 : \zeta(\theta) = \zeta(\theta_1)\} \neq \emptyset.$$

Then, from (13) and (15) we obtain the equality $\mathrm{E}\{Y_n \mid \mathscr{F}_{n-1}\} = Y_{n-1}$, which completes the proof.

Using the martingale $\{Y_n, \mathscr{F}_n, n \geqslant 1\}$ in the discrete time model, one can obtain the Laplace-Stieltjes transform of the distribution of the time to the first overflow $\bar{\tau}_K$ and the Laplace-Stieltjes transform of the joint distribution of the random variables $\bar{\tau}_K$ and $m_{\bar{\tau}_K}$, where $-m_{\bar{\tau}_K}$ denotes the deficit of water in the interval $(0, \bar{\tau}_K]$. We do not deal with this problem and examine the continuous time model.

We consider the finite dam of capacity $K < \infty$, in which both the input and output processes $\{A(t), t \geqslant 0\}$ and $\{B(t), t \geqslant 0\}$ are compound Poisson processes with intensities λ and μ, respectively, and exponential (with parameters δ and v) distributions of jump sizes. We recall the notation:

$$C(t) = A(t) - B(t), \quad \mathscr{F}(t) = \sigma\{C(u), 0 \leqslant u \leqslant t\}.$$

Let further

$$m(t) = \inf_{0 \leqslant u \leqslant t} C(u-)$$

and let $Z(t)$ be the content of the dam at the moment t. If $Z(0) = 0$, then — analogously to formula (12) — we have

$$Z(t) = C(t) - m(t), \quad t \geqslant 0 \tag{16}$$

(see [2]).

Now we prove the following analogue of Theorem 3:

THEOREM 4. *For real ω, θ, θ_1 determined in Theorem 3, the stochastic process $\{Y(t), t \geqslant 0\}$ given by the formula*

$$(17)\qquad Y(t) = \exp\{-(\lambda+\mu)(\zeta(\theta)-1)t+\omega m(t)\}[A(\omega, \theta_1)\exp\{\theta Z(t)\}- -A(\omega, \theta)\exp\{\theta_1 Z(t)\}], \quad t \geqslant 0,$$

where $A(\omega, \theta)$ is defined in (14), *is the martingale relative to the family of σ-fields $\{\mathscr{F}(t), t \geqslant 0\}$.*

Proof. Using (16) we get the equality

$$(18)\qquad \mathrm{E}\{\exp\{\omega(m(t)-C(u))+\theta Z(t)\} \mid \mathscr{F}(u)\} = \mathrm{E}\{\exp\{\theta(C(t)-C(u))+(\omega-\theta)\times \times \min(-Z(u), \inf_{u \leqslant y \leqslant t}\{C(y-)-C(u)\})\} \mid \mathscr{F}(u)\}, \quad 0 \leqslant u \leqslant t.$$

Note that $\{C(y-)-C(u), u \leqslant y \leqslant t\}$ does not depend upon $\mathscr{F}(u)$ and has the same distribution as $\{C(y-), 0 \leqslant y \leqslant t-u\}$. If in the interval $(0, t-u]$ there occur n jumps of the process $\{C(y-), 0 \leqslant y \leqslant t-u\}$, then its distribution is the same as the distribution of $\{S_k, 0 \leqslant k \leqslant n\}$, where

$$S_0 = 0, \quad S_k = \sum_{i=1}^{k} C_i,$$

and C_i are independent random variables with common distributions defined by (10). Simultaneously, from Theorem 3 we infer that $\mathrm{E}\{Y_{n+r} \mid \mathscr{F}_r\} = Y_r$ for positive integers n, r, and hence, by (13), we have

$$(19)\qquad \mathrm{E}\{A(\omega, \theta_1)\exp\{\theta S_n+(\omega-\theta)\min(z, m_n)\}- -A(\omega, \theta)\exp\{\theta_1 S_n+(\omega-\theta_1)\min(z, m_n)\}\} = \zeta^n(\theta)[A(\omega, \theta_1)\exp\{(\omega-\theta)z\}-A(\omega, \theta)\exp\{(\omega-\theta_1)z\}]$$

for an arbitrary $z \leqslant 0$. Denoting by $M(t)$ the number of jumps of the process $\{A(t), t \geqslant 0\}$ in the interval $(0, t]$ and by $N(t)$ the number of jumps of the process $\{B(t), t \geqslant 0\}$ in this interval and using (17)-(19), for $0 \leqslant u \leqslant t$ we obtain

$$\exp\{-\omega C(u)\}\mathrm{E}\{Y(t) \mid \mathscr{F}(u)\} = \exp\{-(\lambda+\mu)(\zeta(\theta)-1)t\}[A(\omega, \theta_1)\times \times\exp\{(\theta-\omega)Z(u)\}-A(\omega, \theta)\exp\{(\theta_1-\omega)Z(u)\}]\mathrm{E}\{(\zeta(\theta))^{M(t-u)+N(t-u)}\} = \exp\{-\omega C(u)\}Y(u),$$

which completes the proof.

Let τ_K denote the time to the first overflow defined by the formula

$$(20)\qquad \tau_K = \inf_{0 \leqslant t < \infty}\{t\colon Z(t) \geqslant K\}.$$

Using the martingale properties of $\{Y(t), \mathscr{F}(t), t \geqslant 0\}$, we obtain a formula for $\mathrm{E}\{\exp(-s\tau_K)\}$, $s > 0$.

THEOREM 5. *If* $\lambda/\delta \neq \mu/v$ *and* $\zeta(\theta) < \infty$, *then*

$$\begin{aligned}(21)\quad \mathrm{E}\{\exp(-s\tau_K)\} = v\big(\theta(s)-\theta_1(s)\big)\big(\delta-\theta_1(s)\big)\big(\delta-\theta(s)\big)\times \\ \times\big\{\delta\big[\big(\theta_1(s)+v\big)\big(\delta-\theta(s)\big)\theta(s)\exp\{\theta_1(s)K\} - \\ -\big(\theta(s)+v\big)\big(\delta-\theta_1(s)\big)\theta_1(s)\exp\{\theta(s)K\}\big]\big\}^{-1}, \quad s>0,\end{aligned}$$

where $\theta(s) \geqslant 0$, $\theta_1(s) < 0$ *are two real solutions of the equation*

$$(22)\qquad (\lambda+\mu)\big(\zeta(\theta)-1\big) = s$$

with θ *unknown, an arbitrary* $s > 0$, *and* $\zeta(\theta)$ *given in* (11).

Proof. It is easy to verify that for an arbitrary $s > 0$ equation (22) has two different roots with different signs in the region $\zeta(\theta) < \infty$. Let $\theta(s) > 0$ and $\theta_1(s) < 0$. Moreover, under the condition $\lambda/\delta \neq \mu/v$ we have

$$\lim_{s\searrow 0}\theta(s) = 0 \quad \text{and} \quad \lim_{s\searrow 0}\theta_1(s) \neq 0.$$

Therefore, from Theorem 4 we obtain, for $\omega = 0$, the martingale

$$Y_0(t) = e^{-st}\left[\frac{\theta(s)}{v\big(\theta(s)+v\big)}\exp\{\theta_1(s)Z(t)\} - \frac{\theta_1(s)}{v\big(\theta_1(s)+v\big)}\exp\{\theta(s)Z(t)\}\right], \quad t \geqslant 0,\ s > 0.$$

Since the inequality $Z(t) < K$ holds on the set $\{t < \tau_K\}$, the martingale is bounded on this set, and from martingale properties we get $\mathrm{E}\{Y_0(\tau_K)\} = \mathrm{E}\{Y_0(0)\}$. Hence and by the fact that the jump sizes of the input process are exponential we obtain (21).

The martingale $\{Y(t), \mathscr{F}(t), t \geqslant 0\}$ may also be used for obtaining the Laplace-Stieltjes transform of the joint distribution of the random variables τ_K and $m(\tau_K)$.

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