ZASTOSOWANIA MATEMATYKI
APPLICATIONES MATHEMATICAE
XIII, 1 (1972)

W. SOBIESZEK (Gliwice)

ON THE STRUCTURE OF THE SOLUTION OF THE ALLOCATION PROBLEM

0. Introduction. This paper grew out of some economical problem which we shall explain further on. It is known that the total production costs of x units of some good consist of two main parts: the constant part (fixed costs $= K_s$) which is constant independently from the amount of production and which exists even if the production equals zero (e.g. expenses for the amortization of the factory buildings) and a variable part (variable costs $= K_z(x)$) depending upon the volume of production (e.g. expenses for raw materials, wages, etc.). The function of variable costs (and, consequently, the function of total costs) is an increasing function with one point of inflection (the optimum of production). Fig. 1 illustrates the shape of the total costs curve. In the interval $\langle 0, a\rangle$ the function $K(x)$ is strictly concave and in the interval $\langle a, +\infty)$ it is strictly

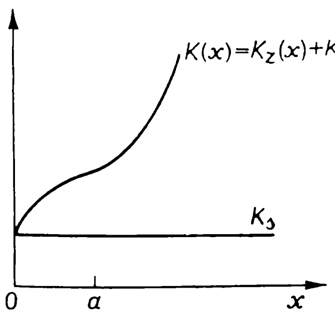

Fig. 1

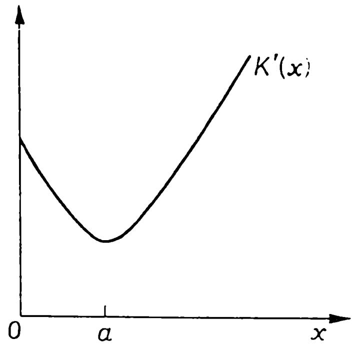

Fig. 2

convex. From the structure of the function $K(x)$ it follows the structure of the function of marginal costs $K'(x)$. In Fig. 2 the shape of the curve of marginal costs is illustrated. Let us suppose that some good may be produced in two factories F_1 and F_2. Then it is necessary to consider the following three cases as possible decisions:

(a) To produce only in the factory F_1 with total costs $C_1+f_1(x)$.

(b) To produce only in the factory F_2 with total costs $C_2+f_2(x)$.

(c) To produce in factories F_1 and F_2 with total costs

$$C_1+C_2+m_2(x), \qquad \text{where } m_2(x) = \min_{0\leqslant y\leqslant x}[f_1(x-y)+f_2(y)].$$

An optimal decision will be that one which realizes

$$\min[C_1+f_1(x),\ C_2+f_2(x),\ C_1+C_2+m_2(x)].$$

In connection with the considered problem of production costs J. Łoś has raised the question: is the function $m_2(x)$ of the same type as functions $f_1(x)$ and $f_2(x)$? This question requires the research of the structure of the function

$$m_n(x) = \min_{\substack{x_1+\ldots+x_n=x\\ x_i\geqslant 0}} [f_1(x_1)+f_2(x_2)+\ldots+f_n(x_n)] \tag{0.1}$$

under the assumption that the functions $f_i(x)$ are continuous and any of them has one point of inflection. The knowledge of the function (0.1) is useful in solving many problems in economy and in operations research (see e.g. [1]-[4] and [6]). Problems of this type are generally named problems of *allocation*.

Therefore we call the function (0.1) *solution* of the problem of allocation. The function (0.1) may be found from the following recurrence formulas:

$$\begin{aligned} m_1(x) &= f_1(x), \\ m_i(x) &= \min_{0\leqslant y\leqslant x}[f_i(y)+m_{i-1}(x-y)], \qquad i = 2, 3, \ldots, n. \end{aligned} \tag{0.2}$$

I wish to express thanks to Professor Jerzy Łoś who encouraged me to undertake this problem and helped me in writing.

1. Notation and definitions. Given are the functions $g(x)$ and $h(x)$, defined and continuous for $x \geqslant 0$. We assume that each of these functions has one point of inflection. Let $a > 0$ be the point of inflection of the function $g(x)$ and let $b > 0$ be the point of inflection of the function $h(x)$.

Let us write shortly

$$G(x, y) = h(y)+g(x-y).$$

We define the functions

$$M_i(x) = \min_{a_i(x)\leqslant y\leqslant b_i(x)} G(x, y) \text{ for } x \in I_i \quad (i = 0, 1, 2, 3, 4), \tag{1.1}$$

where

$$\begin{aligned}
a_0(x) &= 0, & b_0(x) &= x, & I_0 &= \langle 0, +\infty), \\
a_1(x) &= \max(0, x-a), & b_1(x) &= \min(b, x), & I_1 &= \langle 0, a+b\rangle, \\
a_2(x) &= b, & b_2(x) &= x-a, & I_2 &= \langle a+b, +\infty), \\
a_3(x) &= 0, & b_3(x) &= \min(b, x-a), & I_3 &= \langle a, +\infty), \\
a_4(x) &= \max(b, x-a), & b_4(x) &= x, & I_4 &= \langle b, +\infty).
\end{aligned}$$

In Fig. 3 we sketched the domains, the boundaries of which are the curves $a_i(x)$ and $b_i(x)$ with the same numeration as the numeration of the domains. More exactly, $a_i(x)$ is the lower and $b_i(x)$ the upper boundary

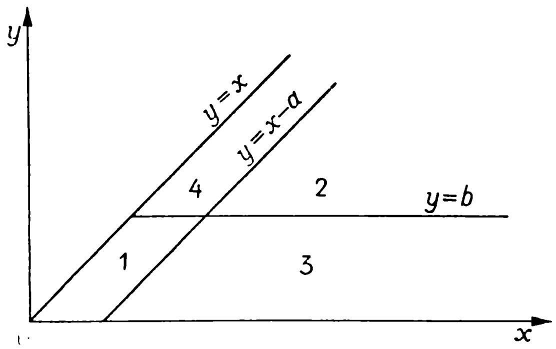

Fig. 3

or the i-th domain. The functions $M_i(x)$ are continuous in the intervals in which they are defined. It is easy to see that

$$M_0(x) = \begin{cases} M_1(x) & \text{for } 0 \leqslant x \leqslant a \\ \min[M_1(x), M_3(x)] & \text{for } a \leqslant x \leqslant b \\ \min[M_1(x), M_3(x), M_4(x)] & \text{for } b \leqslant x \leqslant a+b, \text{ if } a \leqslant b. \\ \min[M_2(x), M_3(x), M_4(x)] & \text{for } a+b \leqslant x \end{cases}$$

Analogously we construct the function $M_0(x)$ for the case $a > b$.

2. The structure of the solution in the case where the function is convexo-concave. In this section we assume that the functions $g(x)$ and $h(x)$ are *convexo-concave*. More exactly, the functions $g(x)$ and $h(x)$ are strictly convex in the intervals $\langle 0, a\rangle$ and $\langle 0, b\rangle$, respectively, whereas they are strictly concave in the intervals $\langle a, +\infty)$ and $\langle b, +\infty)$, respectively. The main result of this section is the following

THEOREM. *If* $g'_-(a) \geqslant g'_+(a)$ *and* $h'_-(b) \geqslant h'_+(b)$(¹), *then there exist numbers* $\bar{x}$ *and* $\bar{\bar{x}}$ *such that:*

(a) *the function* $M_0(x)$ *is strictly convex in the interval* $\langle 0, x_0\rangle$ *and strictly concave in the interval* $\langle x_0, +\infty)$,

(b) $M'_{0-}(x_0) \geqslant M'_{0+}(x_0)$, *where* $x_0 = \min(\bar{x}, \bar{\bar{x}})$.

The number $\bar{x}$ *is the unique root of the equation* $M_1(x) = M_3(x)$ *in the interval* $\langle a, a+b\rangle$ *and the number* $\bar{\bar{x}}$ *is the unique root of the equation* $M_4(x) = M_1(x)$ *in the interval* $\langle b, a+b\rangle$.

(¹) Since from the convexity or concavity of a function in a closed interval it does not follow that one-side derivatives are finite at the ends of the interval, in our case may be in particular:

$$g'_+(0) = -\infty, \quad h'_+(0) = -\infty, \quad g'_-(a) = +\infty,$$
$$g'_+(a) = +\infty, \quad h'_-(b) = +\infty, \quad h'_+(b) = +\infty.$$

The proof will require the following lemmas:

Lemma 2.1. *The function*

$$M(x) = \min_{0\leqslant y\leqslant y_0} G(x, y), \quad \textit{where } a\leqslant x_0\leqslant a+b,\ 0\leqslant y_0\leqslant x_0-a,$$

is strictly concave in the interval $\langle x_0, +\infty)$.

Proof. Let x_1 and x_2 be any two points satisfying the inequality $x_0\leqslant x_1\leqslant x_2$ and let us denote by y^* an arbitrary point satisfying the conditions

$$M\left(\frac{x_1+x_2}{2}\right) = \min_{0\leqslant y\leqslant y_0} G\left(\frac{x_1+x_2}{2}, y\right) = G\left(\frac{x_1+x_2}{2}, y^*\right), \qquad 0\leqslant y^*\leqslant y_0.$$

It is easy to see that the inequality $a\leqslant x_1-y^*<x_2-y^*$ holds; then, from the strict concavity of the function $g(x)$ in the interval $\langle a, +\infty)$, we have

$$g\left(\frac{x_1-y^*+x_2-y^*}{2}\right) > \frac{1}{2}[g(x_1-y^*)+g(x_2-y^*)].$$

Hence

$$M\left(\frac{x_1+x_2}{2}\right)$$

$$= g\left(\frac{x_1+x_2}{2}-y^*\right)+h(y^*) > \frac{1}{2}[g(x_1-y^*)+h(y^*)]+\frac{1}{2}[g(x_2-y^*)+h(y^*)]$$

$$\geqslant \frac{1}{2}[\min_{0\leqslant y\leqslant y_0} G(x_1, y)+\min_{0\leqslant y\leqslant y_0} G(x_2, y)] = \frac{1}{2}[M(x_1)+M(x_2)].$$

Lemma 2.2. *If* $\min[M_1(x_0), M_3(x_0)] = M_3(x_0)$, *where* $a\leqslant x_0<a+b$, *then* $M_1(x)>M_3(x)$ *for* $x_0<x\leqslant a+b$.

Proof. Let

$$(2.1)\qquad M_3(x_0) = \min_{0\leqslant y\leqslant x_0-a} G(x_0, y) = G(x_0, y_0), \qquad \text{where } 0\leqslant y_0\leqslant x_0-a.$$

Hence and by the assumption the inequality

$$(2.2)\qquad G(x_0, y_0)\leqslant G(x_0, y) \quad \text{for } 0\leqslant y\leqslant \min(b, x_0)$$

follows.

Now we shall prove two important inequalities by indirect proof:

$$(2.3)\qquad G(x, y_0)<G(x, y) \quad \text{for } x>x_0 \text{ and } y_0<y\leqslant \min(x-x_0+y_0, b),$$

$$(2.4)\qquad G(x, x-x_0+y_0)<G(x, y)$$

$$\text{for } x_0<x<b+x_0-y_0 \text{ and } x-x_0+y_0<y\leqslant \min(b, x).$$

Let us suppose the contrary, i.e. that inequality (2.3) is not true. Then there exists a point $x^* > x_0$ such that the function $G(x^*, y)$ has a minimum in the interval $\langle y_0, \min(x^*, x_0+y_0, b)\rangle$ in a point $y^* > y_0$.

Hence the left-hand derivative of the function $G(x^*, y)$ in a point y^* must be non-positive:

$$h'_-(y^*) \leqslant g'_+(x^*-y^*). \tag{2.5}$$

Since from the inequalities

$$0 \leqslant y_0 \leqslant x_0 - a \quad \text{and} \quad y_0 < y^* \leqslant \min(x^* - x_0 + y_0, b)$$

the inequalities

$$0 \leqslant y_0 < y^* \leqslant b \quad \text{and} \quad a \leqslant x_0 - y_0 \leqslant x^* - y^*$$

follow, from (2.5) and from the strict convexity of the function $h(x)$ in the interval $\langle 0, b\rangle$ and from the strict concavity of the function $g(x)$ in the interval $\langle a, +\infty)$ we have

$$h'_+(y_0) < h'_-(y^*) \leqslant g'_+(x^*-y^*) \leqslant g'_-(x_0-y_0) \quad \text{if } x_0 - y_0 > a$$

or

$$h'_+(y_0) < h'_-(y^*) \leqslant g'_+(x^*-y^*) \leqslant g'_+(a) \quad \text{if } x_0 - y_0 = a.$$

Hence, and since by assumption $g'_+(a) \leqslant g'_-(a)$, we have in both cases the inequality

$$h'_+(y_0) < g'_-(x_0 - y_0).$$

The inequality means that the right-hand derivative of the function $G(x_0, y)$ at the point y_0 is non-positive. This fact contradicts that point y_0 is an optimal one (see (2.2)).

Now let us suppose that inequality (2.4) is not true. Then for some point (x^*, y^*) we have

$$G(x^*, x^* - x_0 + y_0) \geqslant G(x^*, y^*), \tag{2.6}$$
$$x_0 < x^* < b + x_0 - y_0, \quad x^* - x_0 + y_0 < y^* \leqslant \min(x^*, b).$$

Let us consider the auxiliary function

$$U(x) = h(x - x_0 + y_0) + g(x_0 - y_0) - h(x - x^* + y^*) - g(x^* - y^*)$$
$$\text{for } x_0 \leqslant x \leqslant x^*.$$

By transformation of the inequality $x^* - x_0 + y_0 < y^* \leqslant \min(x^*, b)$ we get the inequality $0 \leqslant x - x_0 + y_0 < x - x^* + y^* \leqslant b$, where $x_0 \leqslant x \leqslant x^*$. Hence and from the strict convexity of the function $h(x)$ in the interval $\langle 0, b\rangle$, it follows the existence of the derivative

$$U'(x) = h'(x - x_0 + y_0) - h'(x - x^* + y^*) < 0$$

almost everywhere in the interval $\langle x_0, x^*\rangle$. Then the function $U(x)$ is decreasing in the interval $\langle x_0, x^*\rangle$. Since it is easy to verify that $U(x_0) = G(x_0, y_0) - G(x_0, x_0 - x^* + y^*)$, and $U(x^*) = G(x^*, x^* - x_0 + y_0) - G(x^*, y^*)$, from the monotonicity of the function $U(x)$ and by (2.6) we obtain the inequality

$$G(x_0, y_0) > G(x_0, x_0 - x^* + y^*).$$

This inequality is contradictory to (2.2), because as is easy to see $0 \leqslant y_0 < x_0 - x^* + y^* \leqslant \min(x_0, b)$.

Let us remark that

$$\min(x - x_0 + y_0, b) = x - x_0 + y_0, \quad \text{when } x_0 < x \leqslant a + b.$$

Hence and by (2.3) we get

$$G(x, y_0) < G(x, x - x_0 + y_0) \quad \text{for } x_0 < x \leqslant a + b. \tag{2.7}$$

Since $a + b \leqslant b + x_0 - y_0$ and $x - x_0 + y_0 \leqslant x - a$, inequality (2.4) holds also for every $y \in \langle x - a, \min(b, x)\rangle$ for any fixed $x \in (x_0, a + b\rangle$. Hence, in particular, we have (see definition (1.1)):

$$G(x, x - x_0 + y_0) \leqslant \min_{x - a \leqslant y \leqslant \min(b, x)} G(x, y) = M_1(x) \quad \text{for } x_0 < x \leqslant a + b. \tag{2.8}$$

Obviously, for $x \geqslant x_0$, we have $0 \leqslant y_0 \leqslant x - a$, then (see also (1.1))

$$M_3(x) = \min_{0 \leqslant y \leqslant x - a} G(x, y) \leqslant G(x, y_0) \quad \text{for } x_0 \leqslant x \leqslant a + b. \tag{2.9}$$

By (2.7), (2.8) and (2.9) we obtain $M_1(x) > M_3(x)$ for $x_0 < x \leqslant a + b$. This completes the proof.

COROLLARY 2.1. *If the point x_0 fulfills the assumptions of lemma 2.2, then the function $M_3(x)$ is strictly concave in the interval $\langle x_0, +\infty)$.*

Proof. Let the point (x_0, y_0) fulfil conditions (2.1). If y varies in the interval $\langle x - x_0 + y_0, \min(b, x - a)\rangle$ then, according to (2.4), for arbitrary $x \in \langle x_0, b + x_0 - y_0\rangle$ we have

$$\min_{x - x_0 + y_0 \leqslant y \leqslant \min(b, x - a)} G(x, y) = G(x, x - x_0 + y_0) \quad \text{for } x_0 \leqslant x \leqslant b + x_0 - y_0. \tag{2.10}$$

On the other hand, from (2.3) it follows

$$\min_{y_0 \leqslant y \leqslant \min(b, x - x_0 + y_0)} G(x, y_0) = G(x, y_0) \quad \text{for } x \geqslant x_0. \tag{2.11}$$

Thus, by (2.10) and (2.11), we have

$$\min_{y_0 \leqslant y \leqslant \min(b, x - a)} G(x, y_0) = G(x, y_0) \quad \text{for } x \geqslant x_0. \tag{2.12}$$

According to the definition (1.1) and by (2.12) we obtain

$$M_3(x) = \min_{0\leqslant y\leqslant \min(b,x-a)} G(x,y) = \min_{0\leqslant y\leqslant y_0} G(x,y) \quad \text{for } x\geqslant x_0.$$

Hence and from lemma 2.1 it follows the strict concavity of the function $M_3(x)$ in the interval $\langle x_0, +\infty)$.

COROLLARY 2.2. *The equation* $M_1(x) = M_3(x)$ *has exactly one root* $\bar{x}$ *in the interval* $\langle a, a+b\rangle$ *such that*

$$\min[M_1(x), M_3(x)] = \begin{cases} M_1(x) & \text{for } a\leqslant x\leqslant \bar{x},\\ M_3(x) & \text{for } \bar{x}\leqslant x\leqslant a+b\end{cases}$$

and the function $M_3(x)$ *is strictly concave in the interval* $\langle \bar{x}, +\infty)$.

Proof. According to the definition (1.1) of the functions $M_1(x)$ and $M_3(x)$ we have

$$M_1(a) \quad = \min_{0\leqslant y\leqslant a} G(a,y)\leqslant G(a,0) = M_3(a),$$

$$M_3(a+b) = \min_{0\leqslant y\leqslant b} G(a+b,y)\leqslant G(a+b,b) = M_1(a+b).$$

From these inequalities and from the continuity of functions $M_1(x)$ and $M_3(x)$ the existence of the root $\bar{x}$ follows. The rest of corollary 2.2 follows, as it is easy to see, directly from lemma 2.2 and from corollary 2.1 except of the strict concavity of the function $M_3(x)$ for $x\geqslant \bar{x} = a+b$. This last follows in this case from lemma 2.1.

Let us remark that according to the definition (1.1) of the functions $M_3(x)$ and $M_4(x)$ we have

$$M_3(x) = \min_{\substack{0\leqslant y\leqslant b\\ x-y\geqslant a}} [h(y)+g(x-y)] \quad \text{and} \quad M_4(x) = \min_{\substack{0\leqslant y\leqslant a\\ x-y\geqslant b}} [g(y)+h(x-y)].$$

It is easily seen that from this and by the assumptions which are satisfied by the functions g and h (it is sufficient to exchange the functions g and h and the numbers a and b) from corollary 2.2 we obtain the following

COROLLARY 2.3. *The equation* $M_1(x) = M_4(x)$ *has exactly one root* $\bar{\bar{x}}$ *in the interval* $\langle b, a+b\rangle$ *such that*

$$\min[M_1(x), M_4(x)] = \begin{cases} M_1(x) & \text{for } b\leqslant x\leqslant \bar{\bar{x}},\\ M_4(x) & \text{for } \bar{\bar{x}}\leqslant x\leqslant a+b\end{cases}$$

and function $M_4(x)$ *is strictly concave in the interval* $\langle \bar{\bar{x}}, +\infty)$.

Proof of the theorem. Let us establish that $a<b$ and $\tilde{x} = \min(\bar{x}, \bar{\bar{x}})$, where $\bar{x}, \bar{\bar{x}}$ are numbers defined in corollaries 2.2 and 2.3. From the strict concavity of function $g(x)$ in the interval $\langle a, +\infty)$ and

from the strict concavity of function $h(x)$ in the interval $\langle b, +\infty)$ it follows the strict concavity of function $G(x, y)$ with respect to variable $y \in \langle b, x-a\rangle$ and for any arbitrary fixed $x \geqslant a+b$. Hence, according to the definition (1.1) of the function $M_2(x)$ we obtain

$$M_2(x) = \min_{b\leqslant y\leqslant x-a} G(x, y) = \min[G(x, b), G(x, x-a)] \quad \text{for } x \geqslant a+b.$$

On the other hand, for $x \geqslant a+b$ we have

$$M_3(x) = \min_{0\leqslant y\leqslant b} G(x, y) \leqslant G(x, b) \text{ and } M_4(x) = \min_{x-a\leqslant y\leqslant x} G(x,y) \leqslant G(x, x-a).$$

Applying the above given equality to the function $M_2(x)$ we get

$$\min[M_2(x), M_3(x), M_4(x)] = \min[M_3(x), M_4(x)] \quad \text{for } x \geqslant a+b. \tag{2.13}$$

By (2.13) and by corollaries (2.2) and (2.3), according to (1.2) we have

$$M_0(x) = \begin{cases} M_1(x) & \text{for } 0 \leqslant x \leqslant \bar{x}, \\ M_3(x) & \text{for } \bar{x} \leqslant x \leqslant \bar{\bar{x}}, \\ \min[M_3(x), M_4(x)] & \text{for } x \geqslant \bar{\bar{x}}. \end{cases}$$

From the strict concavity of the function $M_3(x)$ in the interval $\langle \bar{x}, +\infty)$ (see corollary (2.2)), from the strict concavity of the function $M_4(x)$ in the interval $\langle \bar{\bar{x}}, +\infty)$ (see corollary (2.3)) and by the continuity of functions $M_0(x)$, $M_3(x)$ and $M_4(x)$ it follows the strict concavity of the function

$$M_0(x) = \begin{cases} M_3(x) & \text{for } \bar{x} \leqslant x \leqslant \bar{\bar{x}}, \\ \min[M_3(x), M_4(x)] & \text{for } x \geqslant \bar{\bar{x}} \end{cases}$$

in the interval $\langle \bar{x}, +\infty)$. From the strict convexity of the function $h(x)$ in the interval $\langle 0, b\rangle$ and from the strict convexity of the function $g(x)$ in the interval $\langle 0, a\rangle$ it follows the strict convexity of the function $M_1(x)$ in the interval $\langle 0, a+b\rangle$ (see [4]). It is easy to see that this completes the proof of part (a) of our theorem.

To prove point (b) of the theorem, let us remark that $M_0(x_0) = M_1(x_0) = M_3(x_0)$ for $x_0 = \bar{x}$.

Let

$$M_1(x_0) = \min_{x_0-a\leqslant y\leqslant \min(b, x_0)} [h(y)+g(x_0-y)] = h(y_0)+g(x_0-y_0),$$

$$x_0-a \leqslant y_0 \leqslant \min(b, x_0),$$

$$M_3(x_0) = \min_{0\leqslant y\leqslant x_0-a} [h(y)+g(x_0-y)] = h(y_0')+g(x_0-y_0'),$$

$$0 \leqslant y_0' \leqslant x_0-a.$$

With this notation it is obvious that

$$(2.14)\qquad M_0(x_0) = h(y_0)+g(x_0-y_0) = h(y_0')+g(x_0-y_0').$$

In the case of $y_0 > 0$ for sufficiently small $\Delta x > 0$ we have the inequalities $0 < y_0'+\Delta x < x_0+\Delta x$ and $0 < y_0-\Delta x \leqslant x_0-\Delta x$, whence

$$M_0(x_0+\Delta x) = \min_{0\leqslant y\leqslant x_0+\Delta x} [h(y)+g(x_0+\Delta x-y)] \leqslant h(y_0'+\Delta x)+g(x_0-y_0'),$$

$$M_0(x_0-\Delta x) = \min_{0\leqslant y\leqslant x_0-\Delta x} [h(y)+g(x_0-\Delta x-y)] \leqslant h(y_0-\Delta x)+g(x_0-y_0).$$

In virtue of the last two inequalities we have by (2.14)

$$\frac{M_0(x_0+\Delta x)-M_0(x_0)}{\Delta x} \leqslant \frac{h(y_0'+\Delta x)-h(y_0')}{\Delta x},$$

$$\frac{M_0(x_0-\Delta x)-M_0(x_0)}{-\Delta x} \geqslant \frac{h(y_0-\Delta x)-h(y_0)}{-\Delta x}.$$

Passing to the limit when $\Delta x \to 0$, we get the inequalities

$$M_{0+}'(x_0) \leqslant h_+'(y_0') \quad\text{and}\quad M_{0-}'(x_0) \geqslant h_-'(y_0).$$

Hence and from the strict convexity of function $h(x)$ in the interval $\langle 0, b\rangle$ we infer that $M_{0+}'(x_0) \leqslant h_+'(y_0') \leqslant h_-'(y_0) \leqslant M_{0-}'(x_0)$ because $0 \leqslant y_0' \leqslant y_0 \leqslant b$.

Let us remark that in the case $y_0 = 0$ it must be $y_0' = 0$ and $x_0 = a$. Thus, by (2.14) and by the inequalities

$$M_0(a-\Delta x) = \min_{0\leqslant y\leqslant a-x} [h(y)+g(a-\Delta x-y)] \leqslant h(0)+g(a-\Delta x)$$

and

$$M_0(a+\Delta x) = \min_{0\leqslant y\leqslant a+\Delta x} [h(y)+g(a+\Delta x-y)] \leqslant h(0)+g(a+\Delta x)$$

we obtain the inequalities

$$\frac{M_0(a-\Delta x)-M_0(a)}{-\Delta x} \geqslant \frac{g(a-\Delta x)-g(a)}{-\Delta x},$$

$$\frac{M_0(a+\Delta x)-M_0(a)}{\Delta x} \leqslant \frac{g(a+\Delta x)-g(a)}{\Delta x}.$$

Passing to the limit when $\Delta x \to 0$ we get the inequalities $M_{0-}'(a) \geqslant g_-'(a)$ and $M_{0+}'(a) \leqslant g_+'(a)$. Hence and from the assumption $g_-'(a) \geqslant g_+'(a)$ we infer that $M_{0-}'(a) \geqslant M_{0+}'(a)$, which completes the proof of the theorem.

It is easy to remark (see formulae (0.2)) that the theorem we have proved can be generalized for the function (0.1) for $n > 2$.

3. The structure of the solution if the function is concavo-convex. In this section we assume that

(a) $g(x)$ and $h(x)$ are increasing functions of the class C^1 for $x \geqslant 0$;

(b) $g(x)$ and $h(x)$ are functions of the type concavo-convex, i.e. they are strictly concave in intervals $\langle 0, a\rangle$ and $\langle 0, b\rangle$, respectively, and they are strictly convex in intervals $\langle a, +\infty)$ and $\langle b, +\infty)$, respectively;

(c) $g(0) = h(0) = 0$, $g(\infty) = h(\infty) = \infty$, $h'(b) \geqslant g'(a)$ (2).

The shapes of functions $g(x)$ and $h(x)$ and the shapes of their derivatives are illustrated in Figs. 4 and 5.

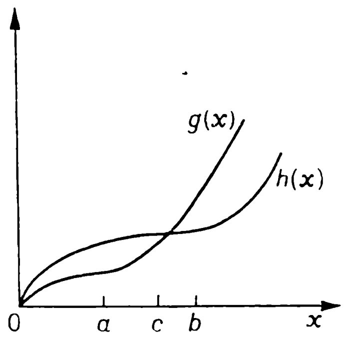

Fig. 4

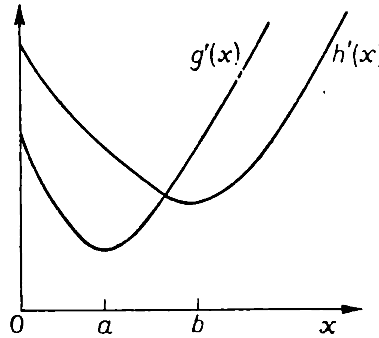

Fig. 5

For simplicity of notation let us put

$$\varphi(x) = G[x, a_4(x)], \tag{3.1}$$

$$\psi(x) = G[x, b_3(x)]. \tag{3.2}$$

Let us define the sets

$$B_3 = \{x \in I_3 \colon M_3(x) < \min(G[x, a_3(x)], G[x, b_3(x)])\}, \tag{3.3}$$

$$B_4 = \{x \in I_4 \colon M_4(x) < \min(G[x, a_4(x)], G[x, b_4(x)])\}, \tag{3.4}$$

$$A_x^i = \{y \in \langle a_i(x), b_i(x)\rangle \colon M_i(x) = G(x, y), G_y'(x, y) = 0\}, \quad i = 3, 4, \tag{3.5}$$

$$B_i' = \{x \in I_i \colon A_x^i \neq 0\} \quad (i = 3, 4). \tag{3.6}$$

Since $A_x^i \neq 0$ for every $x \in B_i$ $(i = 3, 4)$, we have

$$B_i \subset B_i' \quad (i = 3, 4). \tag{3.7}$$

Let

$$y_i(x) = \inf A_x^i, \quad Y_i(x) = \sup A_x^i, \quad x \in B_i' \ (i = 3, 4). \tag{3.8}$$

Since the functions $g(x)$ and $h(x)$ are of the class C^1, it is easy to see that

$$y_i(x), \ Y_i(x) \in A_x^i \quad \text{for every } x \in B_i' \quad (i = 3, 4). \tag{3.9}$$

---

(2) The assumption $h'(b) \geqslant g'(a)$ is not restrictive because a change of the names of the functions g and h does not change the structure of function (1.1).

3.1. The structure of the function $M_4(x)$. Let us observe that in virtue of definition (1.1) we have

$$M_4(x) = \min_{a_4(x)\leqslant y\leqslant b_4(x)} G(x, y) = \min_{\max(b, x-a)\leqslant y\leqslant x} G(x, y) \quad \text{for } x \geqslant b,$$

where, obviously, $G(x, y) = h(y)+g(x-y)$.

The properties of the function $M_4(x)$ are in closed in the following theorems:

(I_4) *If $h'(b) \geqslant g'(0)$, then $M_4(x) = \varphi(x)$ for $x \geqslant b$.*

(II_4) *If $h'(\infty) \leqslant g(a)/a$, then $M_4(x) = h(x)$ for $x \geqslant b$.*

(III_4) *If $h'(b) < g'(0)$ and $h'(\infty) > g(a)/a$, then there exists exactly one root $x_4^0 > b$ of equation $h(x) = \varphi(x)$ and*

(a) *if $M_4(x_4^0) = h(x_4^0)$, then*

$$M_4(x) = \begin{cases} h(x) & \text{for } b \leqslant x \leqslant x_4^0, \\ \varphi(x) & \text{for } x \geqslant x_4^0; \end{cases}$$

(b) *if $M_4(x_4^0) < h(x_4^0)$, then*

1) $B_4' = \langle \underline{x}_4, \bar{x}_4 \rangle$,

2) $B_4 = (\underline{x}_4, \bar{x}_4)$,

3) $0 < \bar{x}_4 - \underline{x}_4 \leqslant a$,

4) $M_4(x) = \begin{cases} h(x) & \text{for } b \leqslant x \leqslant \underline{x}_4, \\ \varphi(x) & \text{for } x \geqslant \bar{x}_4, \end{cases}$

5) *$M_4(x)$ is a strictly concave function in the interval B_4',*

6) *$y_4(x)$ is decreasing and a right-hand continuous function in B_4,*

7) *$Y_4(x)$ is decreasing and a left-hand continuous function in B_4,*

8) *$y_4(x) = Y_4(x)$ almost everywhere in the interval B_4'.*

LEMMA 3.1. *If $M_4(x_0) = G(x_0, y_0)$, where $x_0 > b$ and* $\max(b, x_0 - a) \leqslant y_0 < x_0$, *then $G(x, x-x_0+y_0) < G(x, y)$ for $x > x_0$ and $x-x_0+y_0 < y \leqslant x$.*

Proof. From the assumption it follows that

$$G(x_0, y_0) \leqslant G(x_0, y) \quad \text{for } y_0 \leqslant y \leqslant x_0. \tag{3.10}$$

Suppose, on the contrary, that the lemma is not true. Then there exist points $x^* > x_0$ and $y^* \in (x^*-x_0+y_0, x^*\rangle$ fro which we have

$$G(x^*, x^*-x_0+y_0) \geqslant G(x^*, y^*). \tag{3.11}$$

Let us introduce the auxiliary function

$$U(x) = h(x-x_0+y_0)+g(x_0-y_0)-h(x-x^*+y^*)-g(x^*-y^*) \quad \text{for } x_0 \leqslant x \leqslant x^*.$$

Transforming inequality $x^*-x_0+y_0 < y^* \leqslant x^*$, we obtain

$$b \leqslant x-x_0+y_0 < x-x^*+y^* \quad \text{for } x_0 \leqslant x \leqslant x^*.$$

Hence and by the strict convexity of the function $h(x)$ in the interval $\langle b, +\infty)$ we have

$$U'(x) = h'(x-x_0+y_0)-h'(x-x^*+y^*) < 0;$$

thus the function $U(x)$ is decreasing in the interval $\langle x_0, x^*\rangle$. It is easy to see that

$$U(x_0) = G(x_0, y_0)-G(x_0, x_0-x^*+y^*)$$

and

$$U(x^*) = G(x^*, x^*-x_0+y_0)-G(x^*, y^*).$$

Hence from the monotonicity of the function $U(x)$ and by (3.11) it follows that $G(x_0, y_0) > G(x_0, x_0-x^*+y^*)$, contrary to (3.10), because $y_0 \leqslant x_0-x^*+y^* \leqslant x_0$.

Analogously, we may prove

LEMMA 3.2. *If* $M_4(x_0) = G(x_0, y_0)$, *where* $x_0 > b$ *and* $\max(b, x_0-a) < y_0 \leqslant x_0$, *then* $G(x, x-x_0+y_0) < G(x, y)$ *for* $b < x < x_0$ *and* $\max(b, x-a) \leqslant y < x-x_0+y_0$.

LEMMA 3.3. *If* $h'(y_0) \geqslant g'(x_0-y_0)$, *where* $b \leqslant x_0 < a+b$ *and* $b \leqslant y_0 \leqslant x_0$ *or* $x_0 \geqslant a+b$ *and* $x_0-a < y_0 \leqslant x_0$, *then* $G[x, \max(y_0, x-a)] < G(x, y)$ *for* $x > x_0$ *and* $\max(y_0, x-a) < y \leqslant x-x_0+y_0$.

Proof. From the assumption and from the strict convexity of the function $h(x)$ in the interval $\langle b, +\infty)$ and from the strict concavity of the function $g(x)$ in the interval $\langle 0, a\rangle$ it follows that $h'(y) > h'(y_0) \geqslant g'(x_0-y_0) \geqslant g'(x-y)$ for $b \leqslant y_0 < y$, $0 \leqslant x_0-y_0 \leqslant x-y \leqslant a$. From this inequality it follows that $G'_y(x, y) > 0$ if $\max(y_0, x-a) < y \leqslant x-x_0+y_0$. Hence, as it is easy to see, the thesis of the lemma follows. Analogously, one may prove

LEMMA 3.4. *If* $h'(y_0) \leqslant g'(x_0-y_0)$, *where* $b < x_0 \leqslant a+b$ *and* $b < y_0 < x_0$ *or* $x_0 > a+b$ *and* $x_0-a \leqslant y_0 < x_0$, *then*

$$G[x, \min(y_0, x)] < G(x, y)$$

for $b < x < x_0$ *and* $\max(b, x-x_0+y_0) \leqslant y < \min(y_0, x)$.

COROLLARY 3.1. *If* $M_4(x_0) = G(x_0, y_0)$ *and* $G'_y(x_0, y_0) = 0$, *where* $x_0 > b$ *and* $\max(b, x_0-a) < y_0 \leqslant x_0$, *then*

$$G[x, \max(y_0, x-a)] < G(x, y) \quad \textit{for } x > x_0 \textit{ and } \max(y_0, x-a) < y \leqslant x.$$

Proof. Let us observe that at the point (x_0, y_0) the assumptions of lemma 3.3 hold. If $y_0 = x_0$, then corollary 3.1 and lemma 3.3 are equivalent. If $y_0 < x_0$, then at the point (x_0, y_0) the assumptions of lemma 3.1 also hold. In this case corollary 3.1 follows directly from lemmas (3.1) and (3.3).

COROLLARY 3.2. *If* $M_4(x_0) = G(x_0, y_0)$ *and* $G'_y(x_0, y_0) = 0$, *where* $x_0 > b$, $y_0 > b$, $\max(b, x_0 - a) \leqslant y_0 < x_0$, *then*

$$G[x, \min(y_0, x)] < G(x, y)$$

for $b < x < x_0$ *and* $\max(b, x-a) \leqslant y < \min(y_0, x)$.

Corollary 3.2 follows from lemma 3.4 if $y_0 = \max(b, x_0 - a) = x_0 - a$ or from lemmas 3.4 and 3.2 if $y_0 > \max(b, x_0 - a)$.

LEMMA 3.5. *The set* B_4 *is open.*

Proof. Let us suppose the contrary, i.e. that the set B_4 is not open. Then there exists a point $x_0 \in B_4$ and a sequence x_n the terms of which do not belong to the set B_4 and which is convergent to the point x_0. Hence, by (3.4) and in view of (3.1) and of the assumption that $g(0) = 0$, we have

$$M_4(x_n) = \min[\varphi(x_n), h(x_n)].$$

Since the functions $M_4(x)$ and $\min[\varphi(x), h(x)]$ are continuous, passing to the limit when $n \to \infty$ yields the equality

$$M_4(x_0) = \min[\varphi(x_0), h(x_0)],$$

contrary to the assumption $x_0 \in B_4$.

LEMMA 3.6. *If* $h'(\infty) > g(a)/a$ *and* $h'(b) < g'(0)$, *then the equation* $h(x) = \varphi(x)$ *has a root* $x_0 > b$ *such that* $h(x) < \varphi(x)$ *for* $b < x < x_0$ *and* $h(x) > \varphi(x)$ *for* $x > x_0$ *and*

1. $b < x_0 < a+b$, *when* $h(a+b) > h(b) + g(a)$;
2. $x_0 = a+b$, *when* $h(a+b) = h(b) + g(a)$;
3. $x_0 > a+b$, *when* $h(a+b) < h(b) + g(a)$.

Proof. To prove this lemma it will be convenient to make the following notation (see def. (3.1)):

$$U(x) = h(x) - \varphi(x) = \begin{cases} h(x) - h(b) - g(x-b) & \text{for } b \leqslant x \leqslant a+b, \\ h(x) - h(x-a) - g(a) & \text{for } x \geqslant a+b. \end{cases}$$

The function $U(x)$ has the following properties:

(a) $U(b) = 0$,

(b) $U(\infty) > 0$,

(c) $U(x)$ is increasing for $x \geqslant a+b$,

(d) $U(x)$ is negative in some right-hand neighbourhood of the point b.

Property (a) immediately follows from the definition of function $U(x)$ and from the assumption $g(0) = 0$.

For the proof of property (b) let us present the function $U(x)$ for $x \geqslant a+b$ in the following form:

$$U(x) = h'(\xi)a - g(a) = a\left[h'(\xi) - \frac{g(a)}{a}\right], \quad \text{where } x - a < \xi < x.$$

Passing to the limit when $x \to \infty$ and, consequently, $\xi \to \infty$, we obtain

$$U(\infty) = a\left[h'(\infty) - \frac{g(a)}{a}\right].$$

Hence and by the assumption $h'(\infty) > g(a)/a$ we obtain property (b).

From the strict convexity of the function $h(x)$ in the interval $\langle b, +\infty)$ we get the inequality $U'(x) = h'(x) - h'(x-a) > 0$ for $x \geqslant a+b$, because $x > x-a \geqslant b$. This completes the proof of property (c).

From the continuity of the derivatives g' and h' and in view of the assumption $h'(b) < g'(0)$ it follows that $U'(x) < 0$ for x from some right-hand neighbourhood of the point b. Hence and in view of property (a) we get property (d).

In the case $h(a+b) > h(b)+g(a)$ it is obvious that $U(a+b) > 0$. Hence and from property (d) it follows the existence of the root $x_0 \in (a, a+b)$ of the equation $U(x) = 0$. Let us remark that the function $\varphi(x)$ is strictly concave in the interval $\langle b, a+b\rangle$, and the function $h(x)$ is there strictly convex. Hence, in view of $h(b) = \varphi(b)$ and $h(x_0) = \varphi(x_0)$, it follows the uniqueness of the root x_0 in the interval $(b, a+b)$. Since $U(a+b) > 0$ and since the function $U(x)$ is increasing for $x \geqslant a+b$, we have $U(x) > 0$ for $x \geqslant a+b$. This completes the proof of the lemma in the first case. The proof of the remaining two cases of this lemma is analogous.

LEMMA 3.7. *The function*

$$M(x) = \min_{b_1 \leqslant y \leqslant b_2} [h(y) + g(x-y)],$$

where $b \leqslant b_1 \leqslant b_2$ *and* $b_2 \leqslant c_1 \leqslant c_2 \leqslant b_1 + a$,

is strictly concave in the interval $\langle c_1, c_2\rangle$.

Proof. Let x_1 and x_2 be any two points fulfilling the inequality $c_1 \leqslant x_1 < x_2 \leqslant c_2$. Let us denote by y^* any arbitrary point fulfilling the conditions

$$M\left(\frac{x_1+x_2}{2}\right) = h(y^*) + g\left(\frac{x_1+x_2}{2} - y^*\right), \quad b_1 \leqslant y^* \leqslant b_2.$$

In view of the inequalities $c_1 \leqslant x_1 < x_2 \leqslant c_2$, $b_1 \leqslant y^* \leqslant b_2$ and $c_2 \leqslant b_1 + a$ it follows that $0 \leqslant c_1 - b_2 \leqslant x_1$ and $y^* < x_2 - y^* \leqslant c_2 - b_1 \leqslant a$. Hence and from the strict concavity of the function $g(x)$ in the interval $\langle 0, a\rangle$ we obtain the inequality

$$g\left(\frac{x_1+x_2}{2} - y^*\right) = g\left(\frac{x_1 - y^* + x_2 - y^*}{2}\right) > \frac{1}{2}[g(x_1 - y^*) + g(x_2 - y^*)].$$

By this inequality and by the definition of the function $M(x)$ it follows that

$$M\left(\frac{x_1+x_2}{2}\right) > \frac{1}{2}[h(y^*)+g(x_1-y^*)]+\frac{1}{2}[h(y^*)+g(x_2-y^*)]$$

$$\geqslant \frac{1}{2}[M(x_1)+M(x_2)].$$

This completes the proof of the lemma.

LEMMA 3.8. *If $x_n \in B_4$ and $\lim_{n\to\infty} x_n = x_0$, then $x_0 \in B_4'$.*

Proof. Since $x_n \in B_4$ and $B_4 \subset B_4'$, we can attach the sequence $y_4(x_n)$ to the sequence x_n (see (3.8)).

In view of (3.5) and (3.9) we have $a_4(x_n) \leqslant y_4(x_n) \leqslant b_4(x_n)$. Then from the continuity of the functions $a_4(x)$ and $b_4(x)$ it follows the boundedness of the sequence $y_4(x_n)$. Thus, let $y_4(x_{n_k})$ be a convergent subsequence of $y_4(x_n)$ and let $\lim_{k\to\infty} y_4(x_{n_k}) = y_0$. In virtue of (3.5) and in view of (3.9) the sequences x_{n_k} and $y_4(x_{n_k})$ are related by the conditions

$$a_4(x_{n_k}) \leqslant y_4(x_{n_k}) \leqslant b_4(x_{n_k}), \qquad M_4(x_{n_k}) = G[x_{n_k}, y_4(x_{n_k})],$$
$$G_y'[x_{n_k}, y_4(x_{n_k})] = 0.$$

Hence, in view of the continuity of functions $a_4(x)$, $b_4(x)$, $M_4(x)$, $h(x)$, $g(x)$, $h'(x)$ and $g'(x)$, passing to the limit when $k \to \infty$ we infer that $a_4(x_0) \leqslant y_0 \leqslant b_4(x_0)$, $M_4(x_0) = G(x_0, y_0)$ and $G_y'(x_0, y_0) = 0$. From these conditions it follows that the set $A_{x_0}^4$ is not empty, i.e. (see (3.6)) $x_0 \in B_4'$.

LEMMA 3.9. *If the function $M_4(x)$ has one-sided derivatives at the point $x_0 > b$, then $M_{4+}'(x_0) \leqslant h'(y_0) \leqslant M_{4-}'(x_0)$, where y_0 is any point fulfilling the conditions $M_4(x_0) = G(x_0, y_0)$, $\max(b, x_0-a) < y_0 < x_0$.*

Proof. From the inequality $\max(b, x_0-a) < y_0 < x_0$ for a sufficiently small increment $\Delta x > 0$ we obtain the inequalities

$$\max(b, x_0+\Delta x-a) < y_0+\Delta x < x_0+\Delta x,$$

$$\max(b, x_0-\Delta x-a) < y_0-\Delta x < x_0-\Delta x.$$

Hence, in accordance with the definition (1.1) of the function $M_4(x)$, it follow the inequalities

$$M_4(x_0+\Delta x) \leqslant G(x_0+\Delta x, y_0+\Delta x),$$

$$M_4(x_0-\Delta x) \leqslant G(x_0-\Delta x, y_0-\Delta x).$$

Using the last two inequalities and in view of the equality $M_4(x_0) = G(x_0, y_0)$ we have

$$\frac{M_4(x_0+\Delta x)-M_4(x_0)}{\Delta x} \leqslant \frac{h(y_0+\Delta x)-h(y_0)}{\Delta x},$$

$$\frac{M_4(x_0-\Delta x)-M_4(x_0)}{-\Delta x} \geqslant \frac{h(y_0-\Delta x)-h(y_0)}{-\Delta x}.$$

Hence, passing to the limit when $\Delta x \to 0$, we get the thesis of the lemma.

Proof of (I_4). Applying lemma 3.3 to the point (b, b) we have

$$G[x, \max(b, x-a)] < G(x, y) \quad \text{for } x \geqslant b \text{ and } \max(b, x-a) < y \leqslant x.$$

Hence

$$M_4(x) = \min_{\max(b,x-a)\leqslant y\leqslant x} G(x, y) = G[x, \max(b, x-a)].$$

This completes, in view of the proof of theorem (I_4).

Proof of (II_4). Let us introduce the auxiliary function

$$U(z) = h(z)-h(z-x+y)-g(x-y) \quad \text{for } z \geqslant b+x-y.$$

Under the assumption $y < x$ we have $b \leqslant z-x+y < z$. Hence and from the strict convexity of the function $h(x)$ in the interval $\langle b, +\infty)$ it follows that

$$U'(z) = h'(z)-h'(z-x+y) > 0 \quad \text{for } z \geqslant b+x-y \text{ when } y < x.$$

Then, under the additional assumption $y < x$, the function $U(z)$ is increasing for $z \geqslant b+x-y$. Applying the mean-value theorem to the function h we can write the function $U(z)$ in the following form:

$$U(z) = h'(\xi)(x-y)-g(x-y) = (x-y)\cdot\left[h'(\xi)-\frac{g(x-y)}{x-y}\right],$$

where $z-x+y < \xi < z$.

Passing to the limit when $z \to \infty$ and, consequently, $\xi \to \infty$, we obtain

$$U(\infty) = (x-y)\cdot\left[h'(\infty)-\frac{g(x-y)}{x-y}\right].$$

From the strict concavity of the function $g(x)$ in the interval $\langle 0, a\rangle$ it follows the inequality

$$\frac{g(x-y)}{x-y} \geqslant \frac{g(a)}{a} \quad \text{for } 0 < x-y \leqslant a.$$

Hence and by the assumption $h'(\infty) \leqslant g(a)/a$ we get the inequality $U(\infty) \leqslant 0$. From this inequality and from the monotonicity of the function $U(z)$ we infer that $U(z) < 0$ for $z \geqslant b+x-y$, when $0 < x-y \leqslant a$.

Under the additional assumption $y \geqslant b$, we have $b+x-y \leqslant y+x-y = x$. Finally, if $y \geqslant b$ and $0 < x-y \leqslant a$, then the point $z = x$ belongs to the interval $\langle b+x-y, +\infty)$ in which the function $U(z)$ is negative. Hence

$$U(x) = h(x)-h(y)-g(x-y) < 0 \quad \text{for } \max(b, x-a) \leqslant y < x.$$

This means that

$$M(x) = \min_{\max(b,x-a)\leqslant y\leqslant x} [h(y)+g(x-y)] = h(x)+g(x-x) \quad \text{for } x \geqslant b.$$

Since $g(0) = 0$, we have $M_4(x) = h(x)$ for $x \geqslant b$.

Proof of (III_4)-point (a). Let us observe that the point $x_4^0 > b$ exists and is unique (see lemma 3.6). Moreover, let us observe that in virtue of the assumption we have $M_4(x_4^0) = G(x_4^0, x_4^0) = G[x_4^0, \max(b, x_4^0-a)]$, because $G(x_4^0, x_4^0) = h(x_4^0)$ and $G[x_4^0, \max(b, x_4^0-a)] = \varphi(x_4^0)$. Applying lemma 3.3 to the point (x_4^0, x_4^0) we obtain the inequality $G(x, x) < G(x, y)$ for $b < x < x_4^0$ and $\max(b, x-a) \leqslant y < x$. Hence it follows that

$$M_4(x) = G(x, x) = h(x) \quad \text{for } b \leqslant x \leqslant x_4^0.$$

In the case where $\max(b, x_4^0-a) = b$ applying lemmas 3.1 and 3.3 to the point (x_4^0, b) we have the inequalities $G(x, x-x_4^0+b) < G(x, y)$ for $x > x_4^0$ and $x-x_4^0+b < y \leqslant x$ and $G[x, \max(b, x-a)] < G(x, y)$ for $x > x_4^0$ and $\max(b, x-a) < y \leqslant x-x_4^0+b$, respectively. Hence it follows that

$$G[x, \max(b, x-a)] < G(x, y) \quad \text{for } x > x_4^0 \text{ and } \max(b, x-a) < y \leqslant x.$$

From this inequality it follows that

$$M_4(x) = G[x, \max(b, x-a)] = h[\max(b, x-a)]+g[x-\max(b, x-a)] \quad \text{for } x \geqslant x_4^0.$$

Thus, in accordance with (3.1), we have

$$M_4(x) = \varphi(x) \quad \text{for } x \geqslant x_4^0.$$

If $\max(b, x_4^0-a) = x_4^0-a$, applying lemma 3.1 to the point (x_4^0, x_4^0-a) we have $G(x, x-a) < G(x, y)$ for $x > x_4^0$ and $x-a < y \leqslant x$. Hence

$$M_4(x) = G(x, x-a) = h(x-a)+g(a) \quad \text{for } x \geqslant x_4^0.$$

Since in this case $x_4^0-a \geqslant b$, we have $x_4^0 \geqslant a+b$. Hence, according to the definition (3.1),

$$h(x-a)+g(a) = \varphi(x) \quad \text{for } x \geqslant x_4^0.$$

This completes the proof of the theorem.

Proof of (III_4)-point (b). Since $G[x_4^0, a_4(x_4^0)] = \varphi(x_4^0)$ (see (3.1)), $\varphi(x_4^0) = h(x_4^0)$ (see lemma 3.6) and $G[x_4^0, b_4(x_4^0)] = G(x_4^0, x_4^0) = h(x_4^0)+g(0) = h(x_4^0)$ (because $g(0) = 0$), by the assumption $M_4(x_4^0) < h(x_4^0)$ we get the inequality

(3.12) $$M_4(x_4^0) < \min(G[x_4^0, a_4(x_4^0)],\ G[x_4^0, b_4(x_4^0)]).$$

It follows that there exists a point y_0 such that

(3.13) $$\max(b, x_4^0-a) < y_0 < x_4^0, \qquad M_4(x_4^0) = G(x_4^0, y_0), \qquad G_y'(x_4^0, y_0) = 0.$$

Applying corollaries (3.1) and (3.2) to the point (x_4^0, y_0) we have

(3.14) $$G[x, \max(y_0, x-a)] < G(x, y)$$
$$\text{for } x > x_4^0 \text{ and } \max(y_0, x-a) < y \leqslant x,$$

(3.15) $$G[x, \min(y_0, x)] < G(x, y)$$
$$\text{for } b < x < x_4^0 \text{ and } \max(b, x-a) \leqslant y < \min(y_0, x),$$

respectively.

Since $\max(y_0, x-a) = x-a$ for $x \geqslant y_0+a$ and $\min(y_0, x) = x$ for $b < x \leqslant y_0$, by (3.14) and (3.15), we have

(3.16) $$G(x, x-a) < G(x, y) \quad \text{for } x \geqslant y_0+a \text{ and } x-a < y \leqslant x,$$

(3.17) $$G(x, x) < G(x, y) \quad \text{for } b < x \leqslant y_0 \text{ and } \max(b, x-a) \leqslant y < x,$$

respectively.

In view of (3.16) and (3.17), and according to the definition (1.1) of the function $M_4(x)$, we have

(3.18) $$M_4(x) = \begin{cases} G[x, a_4(x)] & \text{for } x \geqslant y_0+a, \\ G[x, b_4(x)] & \text{for } b \leqslant x \leqslant y_0. \end{cases}$$

From (3.18) it follows (see (3.4)) that the set B_4 is bounded from above by the number y_0+a and it is bounded from be low by the number y_0. From (3.12), in view of (3.4), it follows that the set B_4 is not empty ($x_4^0 \in B_4$). From the fact that B_4 is not empty, bounded and open (see lemma 3.5), it follows the existence of the numbers

(3.19) $$\underline{x}_4 = \inf B_4 \quad \text{and} \quad \overline{x}_4 = \sup B_4$$

fulfilling the conditions

(3.20) $$\underline{x}_4, \overline{x}_4 \in \overline{B}_4,$$

(3.21) $$b < y_0 \leqslant \underline{x}_4 < x_4^0 < \overline{x}_4 \leqslant y_0+a.$$

From (3.20) it follows the existence of the sequences $\underline{x}_n, \overline{x}_n \in B_4$ such that $\lim_{n\to\infty}\underline{x}_n = \underline{x}_4$ and $\lim_{n\to\infty}\overline{x}_n = \overline{x}_4$, i.e. (see lemma 3.8)

$$\underline{x}_4, \overline{x}_4 \in B_4'. \tag{3.22}$$

Applying lemma 3.6 to the point x_4^0 we have the inequalities

$$\varphi(x) > h(x) \quad \text{for} \quad b < x < x_4^0,$$
$$\varphi(x) < h(x) \quad \text{for} \quad x > x_4^0.$$

Hence, in view of the equalities $\varphi(x) = G[x, a_4(x)]$ (see (3.1)) and $h(x) = G(x, x) = G[x, b_4(x)]$ $(g(0) = 0)$, and by inequality (3.21), we infer that

$$\begin{aligned} G[x, a_4(x)] &> G[x, b_4(x)] \quad \text{for } b < x \leqslant \underline{x}_4, \\ G[x, a_4(x)] &< G[x, b_4(x)] \quad \text{for } x \geqslant \overline{x}_4. \end{aligned} \tag{3.23}$$

From the definition (3.4), of the set B_4 and from the definition (3.19) of the numbers $\underline{x}_4, \overline{x}_4$ and by (3.20) it follows that

$$M_4(x) = \min(G[x, a_4(x)], G[x, b_4(x)]) \quad \text{for } x \notin (\underline{x}_4, \overline{x}_4).$$

Hence, by (3.23), we have

$$M_4(x) = \begin{cases} G[x, b_4(x)] & \text{for } b \leqslant x \leqslant \underline{x}_4, \\ G[x, a_4(x)] & \text{for } x \geqslant \overline{x}_4. \end{cases} \tag{3.24}$$

Now we show that

$$\underline{x}_4 = \inf B_4' \quad \text{and} \quad \overline{x}_4 = \sup B_4'. \tag{3.25}$$

Let us suppose that the first of inequalities (3.25) is not true. Then there exists a point $x^* \in B_4'$ fulfilling the inequality $b \leqslant x^* < \underline{x}_4$. Since $x^* \in B_4'$, according to the definitions (3.6) and (3.5), there exists some point y^* fulfilling the conditions $a_4(x^*) \leqslant y^* \leqslant b_4(x^*)$, $M_4(x^*) = G(x^*, y^*)$ and $G_y'(x^*, y^*) = 0$. Hence, by (3.24) and (3.23), it follows that y^* must be equal to $b_4(x^*)$ and $G_y'[x^*, b_4(x^*)] = 0$. Since $G_y'[x^*, b_4(x^*)] = G_y'(x^*, x^*) = h'(x^*) - g'(0) = 0$, we have $h'(x^*) = g'(0)$.

From the strict convexity of the function $h(x)$ in the interval $\langle b, +\infty)$ it follows the inequality $h'(x) > h'(x^*) = g'(0)$ for $x > x^*$. Since $x^* < \underline{x}_4$, we have in particular the inequality $h(\underline{x}_4) > g'(0)$, i.e. the inequality $G_y'[\underline{x}_4, b_4(\underline{x}_4)] > 0$ (because $b_4(\underline{x}_4) = \underline{x}_4$), contrary to the equality $M_4(\underline{x}_4) = G[x_4, b_4(\underline{x}_4)]$ (see (3.24)).

Analogously, we can prove the equality $\overline{x}_4 = \sup B_4'$. From (3.24), (3.23) and (3.25) we infer, in particular (see (3.8), (3.9) and (3.5)), that

$$a_4(\underline{x}_4) < y_4(\underline{x}_4) \leqslant Y_4(\underline{x}_4) \leqslant b_4(\underline{x}_4), \quad a_4(\overline{x}_4) \leqslant y_4(\overline{x}_4) \leqslant Y_4(\overline{x}_4) < b_4(\overline{x}_4). \tag{3.26}$$

Applying corollary 3.1 to the point $(\underline{x}_4, Y_4(\underline{x}_4))$ (see (3.9), (3.5) and (3.26)) we have

$$G[x, \max(Y_4(\underline{x}_4), x-a)] < G(x, y) \quad \text{for } x > \underline{x}_4 \text{ and } \max(Y_4(\underline{x}_4), x-a) < y \leqslant x \tag{3.27}$$

or

$$G[x, Y_4(\underline{x}_4)] < G(x, y) \quad \text{for } \underline{x}_4 < x \leqslant Y_4(\underline{x}_4)+a \text{ and } Y_4(\underline{x}_4) < y \leqslant x, \tag{3.27'}$$

$$G(x, x-a) < G(x, y) \quad \text{for } x \geqslant Y_4(\underline{x}_4)+a \text{ and } x-a < y \leqslant x. \tag{3.27''}$$

Since $a_4(x) = x-a$ for $x \geqslant Y_4(\underline{x}_4)+a$ because $Y_4(\underline{x}_4)+a > a+b$, by (3.27'') we have the equality $M_4(x) = G[x, a_4(x)]$ for $x \geqslant Y_4(\underline{x}_4)+a$, which means (see (3.4)) that $x \notin B_4$ when $x \geqslant Y_4(\underline{x}_4)+a$. Hence and from the definition (3.19) we get the inequality

$$\overline{x}_4 \leqslant Y_4(\underline{x}_4)+a. \tag{3.28}$$

Putting in (3.27') $x = \overline{x}_4$ we obtain $G[\overline{x}_4, Y_4(\underline{x}_4)] < G(\overline{x}_4, y)$ for $Y_4(\underline{x}_4) < y \leqslant \overline{x}_4$. Hence, in view of

$$G[\overline{x}_4, y_4(\overline{x}_4)] = \min_{\max(b, \overline{x}_4-a) \leqslant y \leqslant \overline{x}_4} G(\overline{x}_4, y)$$

(see (3.9) and (3.5)) we have

$$y_4(\overline{x}_4) \leqslant Y_4(\underline{x}_4). \tag{3.29}$$

Since $Y_4(\underline{x}_4) \leqslant \underline{x}_4$, by (3.29) it follows

$$Y_4(\overline{x}_4) \leqslant \underline{x}_4. \tag{3.30}$$

Under the additional assumption $y_4(\overline{x}_4) > b$, applying corollary 3.2 to the point $(\overline{x}_4, y_4(\overline{x}_4))$ (see (3.9), (3.5) and (3.26)) we get the inequality

$$G[x, \min(y_4(\overline{x}_4), x)] < G(x, y) \quad \text{for } b < x < \overline{x}_4 \text{ and } \max(b, x-a) \leqslant a < \min(y_4(\overline{x}_4), x). \tag{3.31}$$

Since $\min(y_4(\overline{x}_4), x) = y_4(\overline{x}_4)$ for $y_4(\overline{x}_4) \leqslant x < \overline{x}_4$, in view of (3.31) we have

$$G[x, y_4(\overline{x}_4)] < G(x, y) \quad \text{for } y_4(\overline{x}_4) \leqslant x < \overline{x}_4 \text{ and } \max(b, x-a) \leqslant y < y_4(\overline{x}_4). \tag{3.31'}$$

Recapitulating, we state: if x_0 is an arbitrary fixed point in $(\underline{x}_4, \overline{x}_4)$ and if y_0 is an arbitrary point in which the function $G(x_0, y)$ attains its minimum in the interval $\langle \max(b, x_0-a), x_0 \rangle$, then $y_0 \notin (Y_4(\underline{x}_4), x_0\rangle$ (see inequalities (3.27') and (3.28)) and $y_0 \notin \langle \max(b, x_0-a), y_4(\overline{x}_4))$ when

$y_4(\bar{x}_4) > b$ (see (3.31′) and (3.30)). Hence, in account of (3.29) and in virtue of the definition (1.1) of the function $M_4(x)$, we have

$$(3.32) \qquad M_4(x) = \min_{y_4(\bar{x}_4) \leqslant y \leqslant Y_4(\underline{x}_4)} G(x, y) \quad \text{for } \underline{x}_4 \leqslant x \leqslant \bar{x}_4$$

and

(3.33)

$$M_4(x) < \min\big(G(x, x), G[x, \max(b, x-a)]\big) = \min\big(G[x, a_4(x)], G[x, b_4(x)]\big)$$
$$\text{for } \underline{x}_4 < x < \bar{x}_4.$$

From (3.4), (3.19), (3.20) and (3.33) it follows $B_4 = (\underline{x}_4, \bar{x}_4)$. Hence, and by (3.22), (3.25) and (3.7), we obtain the equality $B_4' = \langle \underline{x}_4, \bar{x}_4 \rangle$.

Point 3) of theorem follows from (3.21), and point 4) from (3.24).

Since $b \leqslant y_4(\bar{x}_4) \leqslant Y_4(\underline{x}_4)$ and $Y_4(\underline{x}_4) \leqslant \underline{x}_4 < \bar{x}_4 \leqslant y_4(\bar{x}_4) + a$ (see (3.26), (3.29) and (3.21)), in view of (3.32) and from lemma 3.7, it follows the strict concavity of the function $M_4(x)$ in the interval $\langle \underline{x}_4, \bar{x}_4 \rangle$, i.e. point 5) of the theorem. Let us observe that from the strict concavity of the function $M_4(x)$ in the interval $\langle \underline{x}_4, \bar{x}_4 \rangle$ it follows the existence of finite derivatives $M'_{4+}(x)$ and $M'_{4-}(x)$ in the interval $(\underline{x}_4, \bar{x}_4)$; further, from the equality $B_4 = (\underline{x}_4, \bar{x}_4)$ (see (3.4)) it follows the inequality $\max(b, x-a) < y_4(x) \leqslant Y_4(x) < x$ for $\underline{x}_4 < x < \bar{x}_4$.

Applying lemma 3.9 to points $\big(x_1, y_4(x_1)\big)$ and $\big(x_2, y_4(x_2)\big)$, where $\underline{x}_4 < x_1 < x_2 < \bar{x}_4$, we have $M'_{4+}(x_1) \leqslant h'[y_4(x_1)]$ and $M'_{4-}(x_2) \geqslant h'[y_4(x_2)]$. Thus, since the strict concavity of the function $M_4(x)$ and the inequality $x_1 < x_2$ imply $M'_{4+}(x_1) > M'_{4-}(x_2)$, we have $h'[y_4(x_1)] > h'[y_4(x_2)]$. Hence $y_4(x_1) > y_4(x_2)$ follows, because the function $h(x)$ is strictly convex in the interval $\langle b, +\infty)$.

Thus we have proved that the function $y_4(x)$ is decreasing in the interval B_4.

Let x_0 be an arbitrary fixed point in the interval B_4. The monotonicity of the function $y_4(x)$ implies the existence of the limit

$$\lim_{x \to x_0+} y_4(x) \leqslant y_4(x_0).$$

On the other hand, it is easy to see (see the end of proof of lemma 3.8) that

$$\lim_{x \to x_0+} y_4(x) \in A^4_{x_0};$$

thus we have (see (3.8))

$$\lim_{x \to x_0+} y_4(x) \geqslant y_4(x_0).$$

Finally, we get

$$\lim_{x \to x_0+} y_4(x) = y_4(x_0).$$

This proves the continuity from the right of the function $y_4(x)$ at point x_0.

The proof of point 7) of this theorem is analogous. If there exists the derivative of the function $M_4(x)$ at the point $x_0 \in (\underline{x}_4, \overline{x}_4)$, then, by lemma 3.9, we have $h'[y_4(x_0)] = h'[Y_4(x_0)]$. Hence, the strict convexity of the function $h(x)$ in the interval $\langle b, +\infty)$ implies the equality $y_4(x_0) = Y_4(x_0)$.

Since the function $M_4(x)$ is strictly concave in the interval $\langle \underline{x}_4, \overline{x}_4 \rangle$, it has a derivative in this interval almost everywhere. Thus equality $y_4(x) = Y_4(x)$ holds for almost every $x \in \langle \underline{x}_4, \overline{x}_4 \rangle$. The theorem is thus proved.

3.2. The structure of the function $M_3(x)$. The full results on the structure of the function $M_3(x)$ are the following:

(I_3) If $g'(\infty) \leqslant h(b)/b$, then $M_3(x) = g(x)$ for $x \geqslant a$.

(II_3) If $g'(\infty) > h(b)/b$, then there exists the unique root $x_3^0 > a$ of the equation $g(x) = \psi(x)$ and

(a) if $M_3(x_3^0) = g(x_3^0)$, then

$$M_3(x) = \begin{cases} g(x) & \text{for } a \leqslant x \leqslant x_3^0, \\ \psi(x) & \text{for } x \geqslant x_3^0; \end{cases}$$

(b) if $M_3(x_3^0) < g(x_3^0)$, then

1) $B_3' = \langle \underline{x}_3, \overline{x}_3 \rangle$,

2) $B_3 = (\underline{x}_3, \overline{x}_3)$,

3) $0 < \overline{x}_3 - \underline{x}_3 \leqslant b$,

4) $M_3(x) = \begin{cases} g(x) & \text{for } a \leqslant x \leqslant \underline{x}_3, \\ \psi(x) & \text{for } x \geqslant \overline{x}_3, \end{cases}$

5) $y_3(x)$ is an increasing function and continuous from the left in B_3,

6) $Y_3(x)$ is a function increasing and continuous from the right in B_3,

7) $y_3(x) = Y_3(x)$ almost everywhere in the interval B_3',

8) $M_3(x)$ is a strictly concave function in the interval B_3'.

We omit the proofs of theorems (I_3) and (II_3), because they follow immediately from theorems (II_4) and (III_4) (see the remark before corollary 2.3).

3.3. The structure of the function $M_2(x)$. According to the definition (1.1) of the function $M_2(x)$ we have

$$M_2(x) = \min_{b \leqslant y \leqslant x-a} [h(y) + g(x-y)] \quad \text{for } x \geqslant a+b.$$

In account of the inequality $h'(b) \geqslant g'(a)$ the full results on the structure of the function $M_2(x)$ are the following:

(I_2) If $h'(b) \geqslant g'(\infty)$, then $M_2(x) = h(b) + g(x-b)$ for $x \geqslant a+b$.

(II_2) If $h'(b) < g'(\infty)$, then

$$M_2(x) = \begin{cases} h(b) + g(x-b) & \text{for } a+b \leqslant x \leqslant \underline{x}_2, \\ h[y_2(x)] + g[x - y_2(x)] & \text{for } x \geqslant \underline{x}_2, \end{cases}$$

where $b < y_2(x) < x-a$, $y_2(\underline{x}_2) = b$ and $h'(b) = g'(\underline{x}_2 - b)$. The above given theorems and also the strict convexity of the function $M_2(x)$ in the interval $\langle a+b, +\infty)$ follow from more general theorems from [4] and [5].

3.4. The structure of the function $M_1(x)$. According to the definition (1.1) of the function $M_1(x)$ we have

$$M_1(x) = \begin{cases} \min\limits_{0\leqslant y\leqslant x} G(x,y) & \text{for } 0 \leqslant x \leqslant a, \\ \min\limits_{x-a\leqslant y\leqslant x} G(x,y) & \text{for } a \leqslant x \leqslant b, \\ \min\limits_{x-a\leqslant y\leqslant b} G(x,y) & \text{for } b \leqslant x \leqslant a+b. \end{cases}$$

The strict concavity of the functions $g(x)$ and $h(x)$ in the intervals $\langle 0, a\rangle$ and $\langle 0, b\rangle$, respectively, imply the strict concavity of the function $G(x,y)$ with respect to the variable $y \in \langle a_1(x), b_1(x)\rangle$ for any arbitrary fixed $x \in \langle 0, a+b\rangle$. Hence

$$M_1(x) = \begin{cases} \min[G(x,0), G(x,x)] & \text{for } 0 \leqslant x \leqslant a, \\ \min[G(x,x-a), G(x,x)] & \text{for } a \leqslant x \leqslant b, \\ \min[G(x,x-a), G(x,b)] & \text{for } b \leqslant x \leqslant a+b. \end{cases}$$

Since $G(x,0) = g(x)$ (because $h(0) = 0$), $G(x,x) = h(x)$ (because $g(0) = 0$) and $G(x,x-a) \geqslant \min\limits_{0\leqslant y\leqslant x-a} G(x,y) = M_3(x)$ for $a \leqslant x \leqslant a+b$ (see definition (1.1) of the function $M_3(x)$) and $G(x,b) \geqslant \min\limits_{b\leqslant y\leqslant x} G(x,y) = M_4(x)$ for $b \leqslant x \leqslant a+b$ (see definition (1.1) of the function $M_4(x)$), the full results relating to the structure of the function $M_1(x)$ are the following:

$$(\mathrm{I}_1)\qquad \begin{cases} M_1(x) = \min[g(x), h(x)] & \text{for } 0 \leqslant x \leqslant a, \\ M_1(x) \geqslant \min[M_3(x), h(x)] & \text{for } a \leqslant x \leqslant b, \\ M_1(x) \geqslant \min[M_3(x), M_4(x)] & \text{for } b \leqslant x \leqslant a+b. \end{cases}$$

Before discussing the structure of the function $M_0(x)$, we shall prove five lemmas.

LEMMA 3.10. *If*

$$h(\infty) \leqslant \inf_{x\geqslant 0} \frac{g(x)}{x},$$

then

$$M_4(x) = h(x) < M_2(x) \quad \textit{for } x \geqslant a+b.$$

LEMMA 3.11. *If*

$$h(\infty) > \inf_{x\geqslant 0} \frac{g(x)}{x},$$

then $M_4(x) > M_2(x)$ *for sufficiently large* $x \geqslant a+b$.

LEMMA 3.12. *If*

$$g(\infty) \leqslant \inf_{x \geqslant 0} \frac{h(x)}{x},$$

then

$$M_3(x) = g(x) < M_2(x) \quad \textit{for } x \geqslant a+b.$$

LEMMA 3.13. *If*

$$g(\infty) > \inf_{x \geqslant 0} \frac{h(x)}{x},$$

then $M_3(x) > M_2(x)$ *for sufficiently large* $x \geqslant a+b$.

Proof of lemma 3.10. Let us introduce the auxiliary function

$$U(z) = h(z) - h(z-x+y) - g(x-y) \quad \text{for } z \geqslant b+x-y,$$

where $x \geqslant a+b$ and $b \leqslant y \leqslant x-a$. $U(z)$ is an increasing function and (see the proof of (II_4))

$$U(\infty) = (x-y)\left[h'(\infty) - \frac{g(x-y)}{x-y}\right]. \tag{3.34}$$

From the assumption and from (3.34) it follows that $U(z) < 0$ for $z \geqslant b+x-y$, when $x \geqslant a+b$ and $b \leqslant y \leqslant x-a$. Since $b+x-y \leqslant y+x-y = x$, for $z = x$ we have $U(x) = h(x) - h(y) - g(x-y) < 0$ for $x \geqslant a+b$ and $b \leqslant y \leqslant x-a$. Hence, in particular,

$$h(x) < \min_{b \leqslant y \leqslant x-a} [h(y) + g(x-y)] \quad \text{for } x \geqslant a+b$$

and this, in conformity with definition (1.1) of the function $M_2(x)$, leads to the inequality $h(x) < M_2(x)$ for $x \geqslant a+b$.

Since from the assumption it follows, in particular, that $h'(\infty) \leqslant g(a)/a$, by ($\mathrm{II}_4$) we infer that $M_4(x) = h(x)$ for $x \geqslant b$, which completes the proof of the lemma.

Proof of lemma 3.11. Let us introduce the auxiliary function

$$U(x) = h(x) - h(x-a_1) - g(a_1) \quad \text{for } x \geqslant b+a_1, \text{ where } a_1 > a.$$

Applying the mean value theorem to the function h, we can write

$$U(x) = a_1 h'(\xi) - g(a_1) = a_1\left[h'(\xi) - \frac{g(a_1)}{a_1}\right],$$

where $x - a_1 < \xi < x$. Passing to the limit as $x \to \infty$ and, consequently, $\xi \to \infty$, we obtain

$$U(\infty) = a_1\left[h'(\infty) - \frac{g(a_1)}{a_1}\right]. \tag{3.35}$$

It is easy to see that either there exists a number $a_1' > a$ such that

$$\frac{g(a_1')}{a_1'} = \inf_{x \geqslant 0} \frac{g(x)}{x}$$

or

$$\inf_{x \geqslant 0} \frac{g(x)}{x} = \lim_{x \to \infty} \frac{g(x)}{x} = g'(\infty).$$

In the first case the inequality $U(\infty) > 0$ follows from the assumption and from (3.35) for $a_1 = a'$. Hence and by the continuity of the function $U(x)$, we have $U(x) = h(x) - h(x - a_1') - g(a_1') > 0$ for sufficiently large $x \geqslant b + a_1'$. Since $b < x - a_1' < x - a$ for $x > b + a_1'$, we have

$$h(x) > h(x - a_1') + g(a_1') \geqslant \min_{b \leqslant y \leqslant x-a} [h(y) + g(x-y)] = M_2(x)$$

for sufficiently large $x > b + a_1'$.

In the second case the inequality $h'(\infty) > g'(\infty)$ follows from the assumptions. Passing in (3.35) to the limit as $a_1 \to \infty$ we obtain the equality $V(\infty) = \infty$. The continuity of the function $U(x)$ for some fixed $a_1 > a$ implies that $h(x) > h(x - a_1) + g(a_1)$ for sufficiently large $x \geqslant b + a_1$. Hence, analogously as above, we have $h(x) > M_2(x)$ for sufficiently large $x > b + a_1$.

On the other hand (see (I_2), (II_2) and (3.1))

$$M_2(x) = \min_{b \leqslant y \leqslant x-a} [h(y) + g(x-y)] < h(x-a) + g(a) = \varphi(x) \quad \text{for } x > a + b.$$

Since, for sufficiently large x, (see (I_4), (II_4) and (III_4)) $M_4(x) = h(x)$ or $M_4(x) = \varphi(x)$ the proof is completed.

It is obvious that lemmas (3.12) and (3.13) also are true.

LEMMA 3.14. *If there exists the number* $\underline{x}_2$, *defined in* (II_2), *and if*

$$M_2(x_0) = \min[M_3(x_0), M_4(x_0)], \quad \textit{where } x_0 > \underline{x}_2,$$

then

$$M_2(x) > \min[M_3(x), M_4(x)] \quad \textit{for } a + b \leqslant x < x_0.$$

Proof. We prove this lemma in the case

$$M_2(x_0) = \min[M_3(x_0), M_4(x_0)] = M_4(x_0),$$

because the proof is analogous in the contrary case.

Let y_0 be a point fulfilling the conditions $M_4(x_0) = G(x_0, y_0)$, $x_0 - a \leqslant y_0 \leqslant x_0$. Since

$$M_2(x_0) = G[x_0, y_2(x_0)] < G(x_0, x_0 - a) \leqslant M_4(x_0) = G(x_0, y_0),$$

equality $M_4(x_0) = M_2(x_0)$ implies that $x_0 - a < y_0 \leqslant x_0$. Hence

$$x - a < x - x_0 + y_0 \leqslant x \quad \text{for } x \geqslant a + b. \tag{3.36}$$

Let us introduce the ausciliary function

$$U(x) = M_2(x) - h(x - x_0 + y_0) - g(x_0 - y_0) \quad \text{for } x \geqslant a + b.$$

Obviously, $U(x_0) = M_2(x_0) - M_4(x_0) = 0$.

In virtue of (II_2), the function $U(x)$ may be written in the following form:

$$U(x) = \begin{cases} h(b) + g(x-b) - h(x - x_0 + y_0) - g(x_0 - y_0) & \text{for } a + b \leqslant x \leqslant \underline{x}_2, \\ h[y_2(x)] + g[x - y_2(x)] - h(x - x_0 + y_0) - g(x_0 - y_0) & \text{for } x \geqslant \underline{x}_2. \end{cases}$$

Since

$$\big(h[y_2(x)] + g[x - y_2(x)]\big)' = h'[y_2(x)]y_2'(x) + [1 - y_2'(x)]g'[x - y_2(x)]$$
$$= \big(h'[y_2(x)] - g'[x - y_2(x)]\big)y_2'(x) + g'[x - y_2(x)] = h'[y_2(x)]$$

because of $h'[y_2(x)] = g'[x - y_2(x)]$, by differentiation of the function $U(x)$ we obtain

$$U'(x) = \begin{cases} g'(x-b) - h'(x - x_0 + y_0) & \text{for } a + b \leqslant x \leqslant \underline{x}_2, \\ h'[y_2(x)] - h'(x - x_0 + y_0) & \text{for } x \geqslant \underline{x}_2. \end{cases}$$

From the strict convexity of the function $g(x)$ in the interval $\langle a, +\infty)$ and from the equality $g'(\underline{x}_2 - b) = h'(b)$ (see (II_2)) it follows that $g'(x-b) \leqslant g'(\underline{x}_2 - b) = h'(b)$ for $a + b \leqslant x \leqslant \underline{x}_2$.

On the other hand, from the strict convexity of the function $h(x)$ in the interval $\langle b, +\infty)$ and in view of $b < x - x_0 + y_0$ (see (3.36)) we get $h'(b) < h'(x - x_0 + y_0)$ for $x \geqslant a + b$.

Finally, we have $U'(x) < 0$ for $a + b \leqslant x \leqslant \underline{x}_2$.

From the strict convexity of the function $h(x)$ in the interval $\langle b, +\infty)$ we infer that $h'[y_2(x)] < h'(x - x_0 + y_0)$ for $x \geqslant \underline{x}_2$, because $y_2(x) < x - x_0 + y_0$ (see (3.36) and (II_2)). Thus the function $U(x)$ is decreasing and $U(x_0) = 0$. Therefore

$$M_2(x) > h(x - x_0 + y_0) + g(x_0 - y_0) \quad \text{for } a + b \leqslant x < x_0.$$

Since

$$h(x - x_0 + y_0) + g(x_0 - y_0) = G(x, x - x_0 + y_0) \geqslant \min_{x - a \leqslant y \leqslant x} G(x, y) = M_4(x)$$
$$\text{for } x \geqslant a + b$$

(definition (1.1) of the function $M_4(x)$ and (3.36)), we have

$$M_2(x) > M_4(x) \geqslant \min[M_3(x), M_4(x)] \quad \text{for } a + b \leqslant x < x_0.$$

Thus, the lemma is proved.

3.5. The structure of the function $M_0(x)$. Under the assumption $a \leqslant b$, we prove the following theorems:

(V_1) *If* $h'(\infty) \leqslant \inf\limits_{x \geqslant 0} \frac{g(x)}{x}$, *then*

$$M_0(x) = \begin{cases} \min[g(x), h(x)] & \textit{for } 0 \leqslant x \leqslant a, \\ \min[M_3(x), h(x)] & \textit{for } x \geqslant a. \end{cases}$$

(V_2) *If* $g'(\infty) \leqslant \inf\limits_{x \geqslant 0} \frac{h(x)}{x}$, *then*

$$M_0(x) = \begin{cases} \min[g(x), h(x)] & \textit{for } 0 \leqslant x \leqslant b, \\ \min[g(x), M_4(x)] & \textit{for } x \geqslant b. \end{cases}$$

(V_3) *If* $h'(\infty) > \inf\limits_{x \geqslant 0} \frac{g(x)}{x}$ *and* $g'(\infty) > \inf\limits_{x \geqslant 0} \frac{h(x)}{x}$, *then*

$$M_0(x) = \begin{cases} \min[g(x), h(x)] & \textit{for } 0 \leqslant x \leqslant a, \\ \min[M_3(x), h(x)] & \textit{for } a \leqslant x \leqslant b, \\ \min[M_3(x), M_4(x)] & \textit{for } b \leqslant x \leqslant w_2, \\ M_2(x) < \min[M_3(x), M_4(x)] & \textit{for } x > w_2, \end{cases}$$

where w_2 *is the unique root of equation* $M_2(x) = \min[M_3(x), M_4(x)]$ *in the interval* $\langle \underline{x}_2, +\infty)$ *and* $\underline{x}_2$ *is the number defined in* (II_2).

The proof of (V_1) follows from (1.2), (I_1) and from lemma 3.10; the proof of (V_2) follows from (1.2), (I_1) and from lemma 3.12.

Proof of (V_3). Let us observe that either there exists a number $b_1 > b$ such that

$$\inf_{x \geqslant 0} \frac{h(x)}{x} = \frac{h(b_1)}{(b_1)} = h'(b_1)$$

or

$$\inf_{x \geqslant 0} \frac{h(x)}{x} = \lim_{x \to \infty} \frac{h(x)}{x} = h'(\infty).$$

Hence, and by the assumption

$$g'(\infty) > \inf_{x \geqslant 0} \frac{h(x)}{x},$$

and from the strict convexity of function $h(x)$ in the interval $\langle b, +\infty)$, we get the inequality $g'(\infty) > h'(b)$. In account of the above-said there exists the number $\underline{x}_2$ defined in (II_2). In view of lemmas (3.11) and (3.13), we infer that $M_2(x) < \min[M_3(x), M_4(x)]$ for sufficiently large $x \geqslant \underline{x}_2$.

On the other hand, we have (see (II_2) and definition (1.1) of the function $M_3(x)$)

$$M_2(\underline{x}_2) = G(\underline{x}_2, b) \geqslant \min_{0\leqslant y\leqslant b} G(\underline{x}_2, y) = M_3(\underline{x}_2) \geqslant \min[M_3(\underline{x}_2), M_4(\underline{x}_2)].$$

From this it follows that the equation $M_2(x) = \min[M_3(x), M_4(x)]$ has a root w_2 in the interval $\langle \underline{x}_2, +\infty)$. It is easy to see that the uniqueness of this root follows from lemma 3.14. The continuation of the proof of theorem (V_3) follows from (1.2) and (I_1).

Before giving the results on the structure of the function $M_0(x)$, let us define the set

$$B_0 = \{x \geqslant 0: M_0(x) < \min[g(x), h(x)]\}.$$

Referring to the economic problem discussed in the introduction of this paper, we shall say that factories F_1 and F_2 cooperate in the production of x units of some good if $m_2(x) < \min[f_1(x), f_2(x)]$. Thus in this case B_0 is the set of cooperations of factories F_1 and F_2.

To give full results on the structure of the set B_0 and, consequently, on the structure of the function $M_0(x)$, we should consider about one hundred different cases. In connection with this we shall make some additional assumptions which, on one hand, will have no essential influence upon the change of the structure of the set B_0 and which, on the other hand, are very natural if one takes the economic interpretation of the set B_0.

Let $g(x)$ be the variable costs of the smaller factory and let $h(x)$ be the variable costs of the greater factory. The consequence of the assumption about the sizes of the factories F_1 and F_2 are the conditions

$$(\mathrm{N}) \qquad \begin{cases} 0 < a < b, & \\ g(x) < h(x) & \text{for } 0 < x < c, \\ g(x) > h(x) & \text{for } x > c, \end{cases}$$

where c is a number in the interval (a, b). Taking into account assumptions (N) we can write theorems (V_1), (V_2) and (V_3), respectively, as follows:

(V_1') *If* $h'(\infty) \leqslant \inf_{x>0} \frac{g(x)}{x}$, *then*

$$M_0(x) = \begin{cases} g(x) & \textit{for } 0 \leqslant x \leqslant a, \\ M_3(x) & \textit{for } a \leqslant x \leqslant c, \\ \min[M_3(x), h(x)] & \textit{for } x \geqslant c. \end{cases}$$

(V_2') *If* $g'(\infty) \leqslant \inf_{x\geqslant 0} \frac{h(x)}{x}$, *then*

$$M_0(x) = \begin{cases} g(x) & \text{for } 0 \leqslant x \leqslant c, \\ h(x) & \text{for } c \leqslant x \leqslant b, \\ M_4(x) & \text{for } x \geqslant b. \end{cases}$$

(V_3') *If* $h'(\infty) > \inf_{x\geqslant 0} \frac{g(x)}{x}$ *and* $g'(\infty) > \inf_{x\geqslant 0} \frac{h(x)}{x}$, *then*

$$M_0(x) = \begin{cases} g(x) & \text{for } 0 \leqslant x \leqslant a, \\ M_3(x) & \text{for } a \leqslant x \leqslant c, \\ \min[M_3(x), h(x)] < g(x) & \text{for } c \leqslant x \leqslant b, \\ \min[M_3(x), M_4(x)] < g(x) & \text{for } b \leqslant x \leqslant w_2, \\ M_2(x) < \min[M_3(x), M_4(x)] & \text{for } x > w_2. \end{cases}$$

Using (V_1'), (V_2'), (V_3') and theorems on the structure of functions $M_3(x)$ and $M_4(x)$, it is easy to prove the following theorems:

(I_0) *If* $h'(\infty) \leqslant \inf_{x\geqslant 0} \frac{g(x)}{x}$, *then*

(a) *the set* B_0 *is non-empty if and only if* $B_3 \neq 0$ *and* $\underline{x}_3 < c$;

(b) *if the set* B_0 *is non-empty, then* $B_0 = (\underline{x}_3, w_3)$, *where* w_3 *is the unique root of equation* $h(x) = M_3(x)$ *in the interval* $\langle c, \bar{x}_3)$.

(II_0) *If* $g'(\infty) \leqslant \inf_{x\geqslant 0} \frac{h(x)}{x}$, *then the set* B_0 *is non-empty.*

From theorem (V_3) it follows that under the assumptions

$$h'(\infty) > \inf_{x\geqslant 0} \frac{g(x)}{x} \quad \text{and} \quad g'(\infty) > \inf_{x\geqslant 0} \frac{h(x)}{x}$$

the necessary condition for $B_0 \cap \langle a, w_2\rangle$ being non-empty is either

$$(\alpha): [(\alpha_1): (B_4 \neq 0, \underline{x}_4 < w_2 \leqslant \bar{x}_4) \text{ or } (\alpha_2): (B_4 \neq 0, \bar{x}_4 < w_2)]$$

or

$$(\beta): [(\beta_1): (B_3 \neq 0, \underline{x}_3 < w_2 \leqslant \bar{x}_3) \text{ or } (\beta_2): (B_3 \neq 0, \bar{x}_3 < w_2)].$$

It is easy to show that if B_3 is a non-empty set, then (β) follows, i.e. condition (β) is equivalent to B_3 being non-empty. Hence, denoting by (α') the contradiction of (α) and by (β') the contradiction of (β), we can write

$$(\alpha'): [(\alpha_1'): (B_4 = 0) \text{ or } (\alpha_2'): (B_4 \neq 0, \underline{x}_4 \geqslant w_2)],$$

$$(\beta'): (B_3 = 0).$$

The sense of an alternative implies the necessity of considering the following possibilities: $[(\alpha), (\beta')]$ or $[(\alpha), (\beta)]$ or $[(\alpha'), (\beta)]$.

Writing the above possibilities in detail, we obtain the following cases:

(δ_1): $[(\alpha_1), (\beta_1)]$, (δ_2): $[(\alpha_1), (\beta_2)]$, (δ_3): $[(\alpha_1), (\beta')]$,
(δ_4): $[(\alpha_2), (\beta_1)]$, (δ_5): $[(\alpha_2), (\beta_2)]$, (δ_6): $[(\alpha_2), (\beta')]$,
(δ_7): $[(\beta_1), (\alpha_1')]$, (δ_8): $[(\beta_1), (\alpha_2')]$, (δ_9): $[(\beta_2), (\alpha_1')]$,
(δ_{10}): $[(\beta_2), (\alpha_2')]$, (δ_{11}): $[(\alpha_1'), (\beta')]$, (δ_{12}): $[(\alpha_2'), (\beta')]$.

(III_0) *If* $h'(\infty) > \inf\limits_{x\geqslant 0}\frac{g(x)}{x}$ *and* $g'(\infty) > \inf\limits_{x\geqslant 0}\frac{h(x)}{x}$, *then*:

(T_1) *The cases* (δ_5) *and* (δ_6) *are contradictory.*

(T_2) *In cases* (δ_9), (δ_{10}), (δ_{11}) *and* (δ_{12}) *we have* $B_0 = (w_2, +\infty)$.

(T_3) *In cases* (δ_2) *and* (δ_3) *we have*

$$B_0 = (\underline{x}_4, w_2) \cup \langle w_2, +\infty).$$

(T_4) *In cases* (δ_7) *and* (δ_8) *the following theorems hold*:

(a) *if* $\underline{x}_3 \geqslant c$ *and* $h(w_2) \leqslant M_3(w_2)$, *then* $B_0 = (w_2, +\infty)$;

(b) *if* $\underline{x}_3 \geqslant c$ *and* $h(w_2) > M_3(w_2)$, *then*

$$B_0 = (w_3, w_2) \cup \langle w_2, +\infty),$$

where w_3 *is the unique root of equation* $h(x) = M_3(x)$ *in the interval* $(\underline{x}_3, w_2)$;

(c) *if* $\underline{x}_3 < c$ *and the equation* $h(x) = M_3(x)$ *has no roots in the interval* $\langle c, w_2\rangle$, *then*

$$B_0 = (\underline{x}_3, w_2) \cup \langle w_2, +\infty);$$

(d) *if* $\underline{x}_3 < c$ *and the equation* $h(x) = M_3(x)$ *has a unique root* w_3 *in the interval* $\langle c, w_2\rangle$, *then*

$$B_0 = (\underline{x}_3, w_3) \cup (w_2, +\infty);$$

(e) *if* $x_3 < c$ *and the equation* $h(x) = M_3(x)$ *has exactly two roots* $w_3' < w_3''$ *in the interval* $\langle c, w_2)$, *then*

$$B_0 = (\underline{x}_3, \underline{x}_3') \cup (w_3'', w_2) \cup \langle w_2, +\infty).$$

Remark. If $\underline{x}_3 < c$, then one can prove that the equation $h(x) = M_3(x)$ cannot have more than two roots in the interval $\langle c, w_2\rangle$.

(T_5) In the case (δ_1) the following theorems hold:

(a) *If* $\underline{x}_4 \leqslant \underline{x}_3$, *then* $B_0 = (\underline{x}_4, w_2) \cup \langle w_2, +\infty)$.

(b) *If* $c \leqslant \underline{x}_3 < \underline{x}_4$ *and* $h(\underline{x}_4) \leqslant M_3(\underline{x}_4)$, *then*

$$B_0 = (\underline{x}_4, w_2) \cup \langle w_2, +\infty).$$

(c) *If* $c \leqslant \underline{x}_3 < \underline{x}_4$ *and* $h(\underline{x}_4) > M_3(\underline{x}_4)$, *then*

$$B_0 = (w_3, w_2) \cup \langle w_2, +\infty),$$

where w_3 *is the unique root of the equation* $h(x) = M_3(x)$ *in the interval* $(\underline{x}_3, \underline{x}_4)$.

(d) *If* $\underline{x}_3 < c$ *and the equation* $h(x) = M_3(x)$ *has no roots in the interval* $(\underline{x}_3, \underline{x}_4)$, *then*

$$B_0 = (\underline{x}_3, w_2) \cup \langle w_2, +\infty).$$

(e) *If* $\underline{x}_3 < c$ *and the equation* $h(x) = M(x)$ *has a unique root* w_3 *in the interval* $(\underline{x}_3, \underline{x}_4)$, *then*

$$B_0 = (\underline{x}_3, w_3) \cup (\underline{x}_4, w_2) \cup \langle w_2, +\infty).$$

(f) *If* $\underline{x}_3 < c$ *and the equation* $h(x) = M_3(x)$ *has exactly two roots* $w_3' < w_3''$ *in the interval* $(\underline{x}_3, \underline{x}_4)$, *then*

$$B_0 = (\underline{x}_3, w_3') \cup (w_3'', w_2) \cup \langle w_2, +\infty).$$

Remark. If $\underline{x}_3 < c$, then one can prove that the equation $h(x) = M_3(x)$ cannot have more than two roots in the interval $(\underline{x}_3, \underline{x}_4)$.

(T_6) In the case (δ_4) the following theorems are ture:

(a) *If* $\underline{x}_3 \geqslant c$ *and* $h(w_2) \leqslant M_3(w_2)$, *then* $B_0 = (w_2, +\infty)$.

(b) *If* $\underline{x}_3 \geqslant c$ *and* $h(w_2) > M_3(w_2)$, *then*

$$B_0 = (w_3, w_2) \cup \langle w_2, +\infty),$$

where w_3 *is the unique root of the equation* $h(x) = M_3(x)$ *in the interval* $(\underline{x}_3, w_2)$.

(c) *If* $\underline{x}_3 < c$ *and the equation* $h(x) = M_3(x)$ *has no roots in the interval* (c, w_2), *then*

$$B_0 = (\underline{x}_3, w_2) \cup \langle w_2, +\infty).$$

(d) *If* $\underline{x}_3 < c$ *and the equation* $h(x) = M_3(x)$ *has a unique root* w_3 *in the interval* (c, w_2), *then*

$$B_0 = (\underline{x}_3, w_3) \cup (w_2, +\infty).$$

(e) *If* $\underline{x}_3 < c$ *and the equation* $h(x) = M_3(x)$ *has exactly two roots* $w_3' < w_3''$ *in the interval* (c, w_2), *then*

$$B_0 = (\underline{x}_3, w_3') \cup (w_3'', w_2) \cup \langle w_2, +\infty)$$

(see remark below (T_4)).

Example. Let

$$g(x) = \begin{cases} -\dfrac{x^2}{2} + 2x & \text{for } 0 \leqslant x \leqslant 1, \\[2ex] \dfrac{13}{2}x^2 - 12x + 7 & \text{for } x \geqslant 1; \end{cases}$$

$$h(x) = \begin{cases} -\dfrac{1}{6}x^2 + \dfrac{8}{3}x & \text{for } 0 \leqslant x \leqslant 2, \\[2ex] x^2 - 2x + \dfrac{14}{3} & \text{for } x \geqslant 2. \end{cases}$$

It is easy to verify that the functions g and h satisfy the general assumptions of section 3 and also assumption (N). In this example we have

$$\inf_{x\geqslant 0}\frac{g(x)}{x} = 3\sqrt{\frac{14}{13}}-12 < h'(\infty) = \infty,$$

$$\inf_{x\geqslant 0}\frac{h(x)}{x} = 2\left(\sqrt{\frac{14}{3}}-1\right) < g'(\infty) = \infty.$$

Thus, in account of (V_3), we have

$$M_0(x) = \begin{cases} \min[g(x),\ h(x) & \text{for } 0\leqslant x\leqslant 1,\\ \min[M_3(x),\ h(x)] & \text{for } 1\leqslant x\leqslant 2,\\ \min[M_3(x),\ M_4(x)] & \text{for } 2\leqslant x\leqslant w_2,\\ M_2(x) & \text{for } x\geqslant w_2.\end{cases}$$

Using theorem (II_2), we calculate the number $\underline{x}_2 = 40/13$ and the function

$$M_2(x) = \begin{cases} \frac{13}{2}(x-2)^2-2(x-2)+\frac{35}{3} & \text{for } 3\leqslant x\leqslant \frac{40}{13},\\ \frac{13}{15}x^2-\frac{10}{3}x+\frac{25}{3} & \text{for } x\geqslant\frac{40}{13}.\end{cases}$$

In virtue of (I_4), we obtain

$$M_4(x) = \begin{cases} h(2)+g(x-2) = -\frac{(x-2)^2}{2}+2(x-2)+\frac{14}{3} & \text{for } 2\leqslant x\leqslant 3,\\ h(x-1)+g(1) = (x-1)^2-2(x-1)+\frac{37}{6} & \text{for } x\geqslant 3.\end{cases}$$

According to theorem (II_3), after calculating $\underline{x}_3 = 44/39$ and $\overline{x}_3 = 40/13$ we get

$$M_3(x) = \begin{cases} \frac{13}{2}x^2-12x+7 & \text{for } 1\leqslant x\leqslant\frac{44}{39},\\ -\frac{13}{76}x^2+\frac{174}{57}x-\frac{85}{57} & \text{for } \frac{44}{39}\leqslant x\leqslant\frac{40}{13},\\ \frac{13}{2}(x-2)^2-2(x-2)+\frac{35}{3} & \text{for } x\geqslant\frac{40}{13}.\end{cases}$$

Obviously, in our case $y_3(x) = Y_3(x) = \frac{39}{38}x-\frac{44}{38}$ and $B_3' = \langle\frac{44}{39}, \frac{40}{13}\rangle$. It is easily seen that

$$\min[g(x),\ h(x)] = g(x) \quad \text{for } 0 \leqslant x \leqslant 1,$$
$$\min[M_3(x),\ h(x)] = M_3(x) \quad \text{for } 1 \leqslant x \leqslant 2,$$
$$\min[M_3(x),\ M_4(x)] = M_3(x) \quad \text{for } 2 \leqslant x \leqslant \frac{40}{13},$$

and $w_2 = 40/13$, because $M_3(40/13) = M_2(40/13)$.

Finally, we obtain

$$M_0(x) = \begin{cases} -\dfrac{x^2}{2}+2x & \text{for } 0 \leqslant x \leqslant 1, \\ \dfrac{13}{2}x^2-12x+7 & \text{for } 1 \leqslant x \leqslant \dfrac{44}{39}, \\ -\dfrac{13}{76}x^2+\dfrac{174}{57}x-\dfrac{85}{57} & \text{for } \dfrac{44}{39} \leqslant x \leqslant \dfrac{40}{13}, \\ \dfrac{13}{15}x^2-\dfrac{10}{3}x+\dfrac{25}{3} & \text{for } x \geqslant \dfrac{40}{13}. \end{cases}$$

References

[1] Z. Galas, *Wyznaczanie optymalnego planu produkcji materiałów substytucyjnych*, Przegl. Statyst. 7 (1960), p. 27-42.
[2] J. D. Kettelle, *Least-cost allocations of reliability investment*, Operat. Res. 10 (1962), p. 249-262.
[3] M. Klein, *Some production planning problems*, Naval Res. Log. Quart. 4 (1957), p. 269-286.
[4] W. Sobieszek, *O pewnej klasie warunkowych zagadnień ekstremalnych*, Zastosow. Matem. 8 (1965), p. 169-177.
[5] — *O zagadnieniu alokacji*, Przegl. Statyst. 4 (1966), p. 353-365.
[6] A. Vazsonyi, *Scientific programming in business and industry*, New York 1956.

CHAIR OF APPLIED MATHEMATICS
SILESIAN TECHNICAL UNIVERSITY, GLIWICE

Received on 20. 6. 1968;
revised version on 10. 7. 1970

W. SOBIESZEK (Gliwice)

O STRUKTURZE ROZWIĄZANIA ZAGADNIENIA ALOKACJI

STRESZCZENIE

Tematyka pracy wyrosła z następującego problemu ekonomicznego. W fabrykach F_1 i F_2 produkuje się x jednostek pewnego dobra materialnego. Zmienne koszty produkcji są: $f_1(x)$ w F_1, $f_2(x)$ w F_2. Pytamy o strukturę funkcji

$$M_0(x) = \min_{\substack{x_1+x_2=x \\ x_1,x_2\geqslant 0}} [f_1(x_1)+f_2(x_2)],$$

przy założeniu, że $f_1(x)$ i $f_2(x)$ są funkcjami ciągłymi, rosnącymi i mają po jednym punkcie przegięcia oraz $f_1(0) = f_2(0) = 0$. Niech $a, b > 0$ będą punktami przegięcia odpowiednio dla funkcji $f_1(x)$ i $f_2(x)$. Praca składa się z dwóch zasadniczych części. W pierwszej z nich bada się strukturę funkcji $M_0(x)$ przy założeniu silnej wypukłości funkcji $f_1(x)$ i $f_2(x)$ odpowiednio w przedziałach $\langle 0, a\rangle$ i $\langle 0, b\rangle$ oraz silnej wklęsłości odpowiednio w przedziałach $\langle a, +\infty)$ i $\langle b, +\infty)$, w drugiej zaś przy założeniu silnej wklęsłości w przedziałach $\langle 0, a\rangle$ i $\langle 0, b\rangle$ oraz silnej wypukłości w przedziałach $\langle a, +\infty)$ i $\langle b, +\infty)$. W pierwszej części pracy udowodniono, że funkcja $M_0(x)$ jest również tego samego typu co funkcje $f_1(x)$ i $f_2(x)$, tzn. jest ona silnie wypukła w przedziale $\langle 0, x_0\rangle$ i silnie wklęsła w przedziale $\langle x_0, +\infty)$; punkt przegięcia x_0 wyznacza się jako jednoznaczny pierwiastek pewnego równania.

Dla zwięzłego ujęcia wyników drugiej części pracy wprowadzono następujące dodatkowe założenia

$$\text{(N)} \qquad \begin{cases} 0< a< b, \\ f_1(x)< f_2(x) & \text{dla } 0< x< c, \\ f_1(x) > f_2(x) & \text{dla } x > c, \end{cases}$$

gdzie $a< c< b$. Założenia te można uzasadnić ekonomicznie, że fabryka F_1 ma rozmiary produkcji mniejsze niż fabryka F_2.

Dla sprecyzowania wniosków, o charakterze ekonomicznym, wynikających z rezultatów uzyskanych w drugiej części pracy niech

$$B_0 = \{x\colon M_0(x) < \min [f_1(x), f_2(x)]\}.$$

Punkty zbioru B_0 nazywać będziemy *punktami kooperacji* fabryk F_1 i F_2, przy czym jeżeli w pewnym przedziale zawartym w B_0 funkcja $M_0(x)$ jest silnie wklęsła, to nazywać go będziemy przedziałem *kooperacji wklęsłej*. Analogicznie określa się przedział *kooperacji wypukłej*.

Główne wyniki drugiej części pracy są następujące:

1. Jeżeli

$$f_2'(\infty)\leqslant \inf_{x\geqslant 0} \frac{f_1(x)}{x},$$

to zbiór B_0 jest pusty albo jest przedziałem kooperacji wklęsłej. (Podano warunek konieczny i wystarczający na to, aby zbiór B_0 był niepusty.)

2. Jeżeli

$$f_1'(\infty) \leqslant \inf_{x\geqslant 0} \frac{f_2(x)}{x},$$

to zbiór B_0 jest pusty.

3. Jeżeli

$$f_1'(\infty) > \inf_{x \geqslant 0} \frac{f_2(x)}{x} \quad \text{i} \quad f_2'(\infty) > \inf_{x \geqslant 0} \frac{f_1(x)}{x},$$

to generalnie możliwe są trzy przypadki:

(a) B_0 redukuje się do przedziału kooperacji wypukłej;

(b) B_0 jest sumą przedziału kooperacji wklęsłej i przedziału kooperacji wypukłej;

(c) B_0 jest sumą dwóch przedziałów kooperacji wklęsłej i przedziału kooperacji wypukłej.

W tych trzech przypadkach przedział koperacji wypukłej jest nieograniczony. Podkreślić należy, że jeśli jedna z fabryk kooperujących, np. F_1, ma wklęsłe koszty całkowite, druga zaś, w tym wypadku F_2, wypukłe, to łączne koszty $M_0(x)$ są zawsze wklęsłe, przy czym udział w kooperacji fabryki F_1 jest rosnący wraz z x-em. Długość przedziału kooperacji nie przekracza w tym przypadku liczby a.