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A CENTRAL LIMIT THEOREM FOR INDEPENDENT SUMMANDS

1. Introduction and summary. Trotter [6] introduced linear operators approach in order to prove theorems in weak convergence. The author [3] has used Trotter's method in order to prove the weak laws of large numbers. The latter author has also obtained certain central limit theorems for independent summands when each summand is suitably standardized (that is, each summand is divided by a different scale factor). Of course, no assumption of the finiteness of the second moments was made. In this paper*, we shall obtain central limit theorems when the sums are standardized in the usual way, namely, when each summand is centered at its expectation or at its truncated expectation and is divided by a common scale factor.

2. Notation. Let C denote the class of bounded uniformly continuous real-valued functions on $(-\infty, \infty)$. We say that a function h *belongs to* C_3 if h and its first three derivatives exist and are in C. Throughout c.d.f. and K stand for a cumulative distribution function and a finite generic constant, respectively. For $h \in C$, let

$$\|h\| = \sup_x |h(x)|.$$

Furthermore, for each $h \in C$, the transformation T_X associated with the r.v. X having $F(x)$ for its c.d.f. is defined by

$$(T_X h)(y) = \mathrm{E}h(X+y) = \int h(x+y)\,dF(x), \qquad \text{all real } y.$$

Thus T_X maps C to C and is a linear contraction operator. Moreover, if $X_1, \ldots, X_n$ are mutually independent random variables, then it is known that $T_{X_1+\ldots+X_n}$ is equal to the product of $T_{X_1}, \ldots, T_{X_n}$ which commute with one another (see [6], p. 228).

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3. Main results. In this section we shall present the required lemmas and the main results of this paper.

LEMMA 3.1 (Trotter [6] and Feller [2]). *Let T_n and T be operators associated with c.d.f.'s H_n and H, respectively. H_n converges to H if and only if $\|T_n h - Th\| \to 0$ as $n \to \infty$ for every $h \in C_3$.*

For the sufficiency see Trotter [6], and for the necessity see Feller [2], p. 245 and 251.

LEMMA 3.2 (Trotter [6]). *Let $T_1, \ldots, T_n$, $\tau_1, \ldots, \tau_n$ be contraction operators (that is, $|T_k h(y)| \leqslant \|h\|$ or $|\tau_k h(y)| \leqslant \|h\|$) which commute with one another. Then, for any $h \in C_3$,*

$$\|T_1 \ldots T_n h - \tau_1 \ldots \tau_n h\| \leqslant \sum_{k=1}^{n} \|T_k h - \tau_k h\|,$$

and hence

$$\|T^n h - \tau^n h\| < n\|Th - \tau h\|.$$

For the proof see Trotter [6], p. 229-230.

LEMMA 3.3. *Let the integral I_n be given by*

$$I_n = t_n^{-1} \int_0^{t_n} B_n(t)\,dt, \qquad \text{where } t_n \to \infty \text{ as } n \to \infty.$$

Then $I_n \to 0$ as $n \to \infty$ if

(i) *for every $\delta > 0$, there exist n_0 and t_0 such that $|B_n(t)| < \delta/4$ for $n > n_0$ and $t > t_0$, and*

(ii) *for every t_0, there exist n_0 and K $(K < \infty)$ such that $|B_n(t)| < K$ for $n > n_0$ and $t < t_0$.*

For the proof see Govindarajulu [3], p. 270.

LEMMA 3.4 (Hardy et al. [5]). *Let $1 < p_i < \infty$ $(i = 1, \ldots, c)$ be extended real numbers such that*

$$\sum_{i=1}^{c} 1/p_i = 1.$$

Moreover, let $|f_i|^{p_i}$ $(i = 1, \ldots, c)$ be integrable with respect to some measure ν.

Then, for every $c \geqslant 2$, we have

$$\int \Big| \prod_{i=1}^{c} f_i \Big| d\nu \leqslant \prod_{i=1}^{c} \|f_i\|_{p_i}$$

with equality when

$$\frac{|f_1|^{p_1}}{\int |f_1|^{p_1} d\nu} = \ldots = \frac{|f_c|^{p_c}}{\int |f_c|^{p_c} d\nu},$$

where $\|f\|_p = \left[\int |f|^p d\nu\right]^{1/p}$.

For the proof see Hardy et al. [5], p. 139-141. One can also establish the sufficient part of the lemma using the method of induction.

THEOREM 3.1. *Let $X_1, X_2, \ldots, X_n$ be independent random variables having $F_1, F_2, \ldots, F_n$ for their respective c.d.f.'s which may depend on n. Let*

$$S_n = \sum_{i=1}^{n} Z_{in},$$

where

$$Z_{in} = \big(X_i - \mu_i(b_n)\big)/\sigma_n(b_n), \qquad \mu_i(t) = \int_{|x|\le t} x dF_i(x),$$

$$U_i^2(t) = \int_{|x|\le t} [x-\mu_i(t)]^2 dF_i(x), \qquad \sigma_n^2(t) = \sum_{i=1}^{n} U_i^2(t),$$

and $b_n \to \infty$ is such that $n/b_n^2 \le K < \infty$ and $b_n/\sigma_n \le K < \infty$ as $n \to \infty$.
Then

$$\lim_{n\to\infty} \mathrm{P}(S_n \le x) = \Phi(x) = (2\pi)^{-1/2} \int_{-\infty}^{x} \exp[-t^2/2]dt$$

provided $\sigma_n(b_n) > 0$ and $B_n(t) \to 0$ as $n, t \to \infty$, where

$$B_n(t) = \frac{1}{n} \sum_{i=1}^{n} A_i(t), \qquad A_i(t) = t^2[1-F_i(t)+F_i(-t)], \; i = 1, \ldots, n.$$

Proof. Let $Y_{1n}, Y_{2n}, \ldots, Y_{nn}$ be independent normal variables having means 0 and variances $U_i^2(b_n)$ $(i = 1, \ldots, n)$. Since

$$S_n^* = \sum_{i=1}^{n} Y_{in}/\sigma_n(b_n)$$

has a standard normal distribution, it suffices to show that

$$\|T_{S_n} h(y) - T_{S_n^*} h(y)\| \to 0 \qquad \text{as } n \to \infty.$$

Further, in view of Lemma 3.2, it suffices to show that

$$\sum_{i=1}^{n} \|T_{Z_{in}} h(y) - T_{Y_{in}/\sigma_n(b_n)} h(y)\| \to 0 \qquad \text{as } n \to \infty.$$

Towards this, consider

$$T_{Z_{in}} h(y) = \int h\big(y + \{x - \mu_i(b_n)\}/\sigma(b_n)\big) dF_i(x) = \int_{|x|\le b_n} + \int_{|x|>b_n}.$$

Expanding h in a third-order Taylor series about y and integrating each term, we have

$$\int_{|x|\leqslant b_n} h\big(y+\{x-\mu_i(b_n)\}/\sigma_n(b_n)\big)\,dF_i(x)$$

$$= h(y)+\frac{h''(y)\,U_i^2(b_n)}{2\sigma_n^2(b_n)} - h(y)\int_{|x|>b_n} dF_i(x) + \frac{\mu_i(b_n)h'(y)}{\sigma_n(b_n)}\int_{|x|>b_n} dF_i(x) +$$
$$+\frac{1}{6\sigma_n^3(b_n)}\int_{|x|\leqslant b_n}\big(x-\mu_i(b_n)\big)^3 h'''(\xi)\,dF_i(x),$$

where ξ lies between y and $y+\{x-\mu_i(b_n)\}/\sigma_n(b_n)$.

Analogously, we obtain

$$T_{Y_{in}/\sigma_n(b_n)}h(y) = \int h\left(y+\frac{zU_i(b_n)}{\sigma_n(b_n)}\right)d\Phi(z)$$
$$= h(y)+\frac{U_i^2(b_n)}{2\sigma_n^2(b_n)}h''(y)+\frac{U_i^3(b_n)}{6\sigma_n^3(b_n)}\int z^3 h'''(\xi_1)\,d\Phi(z),$$

where ξ_1 lies between y and $y+zU_i(b_n)/\sigma_n(b_n)$. Hence

$$\text{(1)}\quad \|T_{Z_{in}}h - T_{Y_{in}/\sigma_n(b_n)}h\|$$
$$\leqslant 2\,\|h\|\int_{|x|>b_n} dF_i(x) + \frac{\|h'\|\,|\mu_i(b_n)|}{\sigma_n(b_n)}\int_{|x|>b_n} dF_i(x) +$$
$$+\frac{\|h'''\|}{6\sigma_n^3(b_n)}\int_{|x|\leqslant b_n}|x-\mu_i(b_n)|^3\,dF_i(x)+\frac{\|h'''\|\,|U_i^3(b_n)|}{6\sigma_n^3(b_n)}\int_{-\infty}^{\infty}|z|^3\,d\Phi(z).$$

Next we shall show that all terms on the right-hand side of (1), when summed from 1 to n, would tend to zero as n becomes large. Consider

$$\sum_{i=1}^{n}\int_{|x|>b_n} dF_i(x) \leqslant b_n^{-2}\sum_{i=1}^{n}A_i(b_n) = \frac{n}{b_n^2}B_n(b_n)\leqslant KB_n(b_n)\to 0 \quad \text{as } n\to\infty.$$

Moreover,

$$\sigma_n^{-1}(b_n)\sum_{i=1}^{n}|\mu_i(b_n)|\int_{|x|>b_n} dF_i(x)\leqslant \frac{n}{b_n\sigma_n}B_n(b_n)\leqslant KB_n(b_n)\to 0 \quad \text{as } n\to\infty.$$

Next, expanding $\big(x-\mu_i(b_n)\big)^3$ and using Jensen's or Hölder's inequality, one can easily show that

$$\int_{|x|\leqslant b_n}|x-\mu_i(b_n)|^3\,dF_i(x)\leqslant 8\int_{|x|\leqslant b_n}|x|^3\,dF_i(x).$$

Now, after performing integration by parts once, we have

$$\sigma_n^{-3}(b_n)\sum_{i=1}^{n}\int_{|x|\leqslant b_n}|x|^3 dF_i(x) = \sigma_n^{-3}(b_n)\sum_{i=1}^{n}\Big[-b_n A_i(b_n)+3\int_0^{b_n}A_i(t)\,dt\Big].$$

Thus

$$\sigma_n^{-3}(b_n)\sum_{i=1}^{n}\int_{|x|\leqslant b_n}|x|^3 dF_i(x) \leqslant \frac{b_n}{\sigma_n(b_n)}B_n(b_n)+\frac{3}{\sigma_n^3(b_n)}\sum_{i=1}^{n}\int_0^{b_n}A_i(t)\,dt.$$

Since $B_n(t)\to 0$ as $n, t\to\infty$, for every $\delta>0$ there exist n_0 and t_0 such that $|B_n(t)|<\delta$ for $t>t_0$ and $n>n_0$. Moreover,

$$\int_0^{b_n}A_i(t)\,dt = \int_0^{t_0}+\int_{t_0}^{b_n}\leqslant\frac{t_0^3}{3}+\int_{t_0}^{b_n}A_i(t)\,dt.$$

Hence

$$3\sigma_n^{-3}(b_n)\sum_{i=1}^{n}\int_0^{b_n}A_i(t)\,dt\leqslant\frac{nt_0^3}{\sigma_n^3(b_n)}+\frac{3n}{\sigma_n^3(b_n)}\int_{t_0}^{b_n}B_n(t)\,dt\leqslant\delta+\frac{K\delta n b_n}{\sigma_n^3(b_n)}\leqslant K\delta.$$

Next notice that

$$U_i^3(b_n)=\Big[\int_{|x|\leqslant b_n}\big(x-\mu_i(b_n)\big)^2 dF_i(x)\Big]^{3/2}\leqslant\int_{|x|\leqslant b_n}|x-\mu_i(b_n)|^3 dF_i(x)$$
$$\leqslant 8\int_{|x|\leqslant b_n}|x|^3 dF_i(x).$$

Consequently,

$$U_i^3(b_n)/\sigma_n^3(b_n)\leqslant K\delta.$$

COROLLARY 3.1. *If $F_1=F_2=\ldots=F_n=F$, the sum S_n, when suitably normalized, is distributed asymptotically as a standard normal variable provided b_n is chosen so that $n/b_n^2\leqslant K$ and $b_n^2/nU^2(b_n)\leqslant K$, and*

$$A(t)=t^2[1-F(t)+F(-t)]\to 0\quad \text{as } t\to\infty.$$

That is, if $b_n=n^{1/2}\varepsilon$ for some $\varepsilon>0$, then it suffices to have $1/U^2(b_n)\leqslant K$ and $A(t)\to 0$ as $t\to\infty$.

THEOREM 3.2. *Let $X_1, X_2, \ldots, X_n$ be independent random variables centered at their expectations and having $F_1, F_2, \ldots, F_n$ for their respective c.d.f.'s which may depend on n. Let*

$$S_n=\sum_{i=1}^{n}Z_{in},$$

where

$$Z_{in} = X_i/\sigma_n(b_n), \qquad U_i^2(t) = \int_{|x|\leqslant t} x^2 dF_i(x), \qquad \sigma_n^2(t) = \sum_{i=1}^{n} U_i^2(t),$$

and $b_n \to \infty$ *is such that* $n/b_n^2 \leqslant K < \infty$ *and* $b_n/\sigma_n(b_n) \leqslant K < \infty$ *as* $n \to \infty$. *Then*

$$\lim_{n\to\infty} \mathrm{P}(S_n \leqslant x) = \Phi(x)$$

provided $\sigma_n(b_n) > 0$ *and* $B_n(t) \to 0$ *as* $n, t \to \infty$, *where* $B_n(t)$ *is defined as in Theorem* 3.1.

Proof. Let $Y_{1n}, Y_{2n}, \ldots, Y_{nn}$ be independent normal variables having means 0 and variances $U_i^2(b_n)$ $(i = 1, \ldots, n)$. Since

$$S_n^* = \sum_{i=1}^{n} Y_{in}/\sigma_n(b_n)$$

has a standard normal distribution, it suffices to show that

$$\|T_{S_n} h(y) - T_{S_n^*} h(Y)\| \to 0 \quad \text{as } n \to \infty$$

or, equivalently, that

$$\sum_{i=1}^{n} \|T_{Z_{in}} h(y) - T_{Y_{in}/\sigma_n(b_n)} h(y)\| \to 0 \quad \text{as } n \to \infty.$$

Noting that $T_{Z_{in}} h(y) = \int h\big(y + x/\sigma_n(b_n)\big) dF_i(x)$ and expanding as in Theorem 3.1, we obtain

$$(2) \quad \|T_{Z_{in}} h - T_{Y_{in}/\sigma_n(b_n)}\|$$
$$= 2\|h\| \int_{|x|>b_n} dF_i(x) + \frac{\|h'\|}{\sigma_n(b_n)} \int_{|x|>b_n} |x| dF_i(x) + \frac{\|h'''\|}{6\sigma_n^3(b_n)} \int_{|x|<b_n} |x|^3 dF_i(x) +$$
$$+ \frac{U_i^3(b_n)}{6\sigma_n^3(b_n)} \int_{-\infty}^{\infty} |z|^3 d\Phi(z).$$

Next, we shall show that all terms on the right-hand side of (2), when summed from 1 to n, would tend to zero as n becomes large. First, third and fourth terms can be treated as in the proof of Theorem 3.1. Consider the second term and write

$$\int_{|x|>b_n} |x| dF_i = -\Big[\int_{b_n}^{\infty} x d(1 - F_i) + \int_{-\infty}^{-b_n} x dF_i\Big].$$

Integrating by parts once in each integral and noting that

$$\lim_{x\to\infty} x[1 - F_i(x)] = \lim_{x\to-\infty} xF_i(x) = 0,$$

since F_i has a finite expectation, we obtain

$$\sigma_n^{-1}\sum_{i=1}^{n}\int_{|x|>b_n}|x|\,dF_i = \sum_{i=1}^{n}\left[\frac{A_i(b_n)}{b_n}+\int_{b_n}^{\infty}\left\{\frac{A_i(x)}{x^2}\right\}dx\right]\sigma_n^{-1}$$
$$\leqslant \frac{nB_n(b_n)}{b_n\sigma_n(b_n)}+\frac{n}{\sigma_n(b_n)}\int_{b_n}^{\infty}\left\{\frac{B_n(x)}{x^2}\right\}dx.$$

Now, for every $\delta > 0$, there exists an n_0 such that $B_n(x) \leqslant \delta$ whenever $n > n_0$ in the second integral. Thus

$$\sigma_n^{-1}\sum_{i=1}^{n}\int_{|x|>b_n}|x|\,dF_i \leqslant \frac{n}{b_n\sigma_n(b_n)}2\delta < K\delta.$$

Remark 3.1. If the variances σ_i^2 of the component variables exist, one can replace $\sigma_n^2(b_n)$ by

$$\tilde{\sigma}_n^2 = \sum_{i=1}^{n}\sigma_i^2$$

provided $\tilde{\sigma}_n^2/\sigma_n^2 \to 1$ as $n \to \infty$ which is trivially true when the x_i are identically distributed.

4. Case of random sample size. Random sums based on a random number of variables commonly arise in statistics. In the sequel we shall extend Theorems 3.1 and 3.2 to situations where the sample size is random. We consider sums of the form $\sum_{i=1}^{N_n} Z_{i,n}$, where $Z_{i,n}$ are mutually independent and suitably standardized, and N_n is a positive integer-valued random variable satisfying a certain regularity condition. Then we have the following theorem:

THEOREM 4.1. *Let $X_1, X_2, \ldots$ be a sequence of independent random variables having $F_1, F_2, \ldots$ for their respective c.d.f.'s and let $N_1, N_2, \ldots$ be a sequence of positive integer-valued random variables such that N_n is independent of $(X_1, \ldots, X_{N_n})$. Assume that*

$$S_{N_n} = \sum_{i=1}^{N_n} Z_{i,n},$$

where $Z_{i,n} = (X_i - a_{in})/\sigma_n(b_n)$ ($X_i/\sigma_n(b_n)$ if the X_i are centered at their expectations),

$$a_{in} = \int_{|x|\leqslant b_n} x\,dF_i(x) \quad (i = 1, 2, \ldots),$$

$\sigma(b_n)$ is defined as in Theorem 3.1, and $b_n \to \infty$ is such that $n/b_n^2 \leqslant K < \infty$ and $b_n/\sigma_n \leqslant K < \infty$ as $n \to \infty$. Further, suppose that, for any $\delta > 0$, there

exists an a, $0 < a = a(\delta) < b = b(\delta) < \infty$, *such that*

$$P(a < N_n/n < b) \geqslant 1-\delta \quad \text{for } n \geqslant n_0(\delta).$$

Then $P(S_{N_n} \leqslant x) \to \Phi(x)$ *as* $n \to \infty$, *provided* $B_n(t) \to 0$ *as* $n, t \to \infty$, *where* $B_n(t)$ *is defined as in Theorem* 3.1.

Proof. It is sufficient to show that $\|T_{S_{N_n}}h - T_Y h\| \to 0$ as $n \to \infty$. Let $c_{k,n} = P(N_n = k)$ for $k = 1, 2, \ldots$ Then we can write

$$T_{S_{N_n}}h - T_Y h = \sum_{k=1}^{\infty} c_{k,n}\{T_{S_k}h - T_Y h\}.$$

Hence

$$\|T_{S_{N_n}}h - T_Y h\| \leqslant \sum_{k=[na]}^{[nb]} c_{k,n}\|T_{S_k}h - T_Y h\| + 2\|h\|\delta.$$

However,

$$\|T_{S_k}h - T_Y h\| \leqslant \sum_{i=1}^{k} \|T_{Z_{i,k}}h - T_{Y/\sqrt{k}}h\|$$

and, for every $\delta > 0$, there exists a k_0 such that $k > k_0$ implies that

$$\sum_{i=1}^{k} \|T_{Z_{i,k}}h - T_{Y/\sqrt{k}}h\| \leqslant K\delta$$

whenever the hypothesis of either Theorem 3.1 or 3.2 is satisfied with n replaced by k. Now, let

$$n_0' = 1 + [k_0/a] \quad \text{and} \quad n_0^* = \max(n_0, n_0').$$

Hence $n > n_0^*$ implies that $k \geqslant k_0$ which, in turn, implies that

$$\|T_{S_k}h - T_Y h\| \leqslant K\delta.$$

Consequently, $\|T_{S_{N_n}}h - T_Y h\| \leqslant K\delta$ for $n \geqslant n_0^*$. This completes the proof of Theorem 4.1.

Remark 4.1. If N_n is dependent upon $X_1, \ldots, X_{N_n}$ as in sequential statistical procedures, and if $N_n/n \to \varrho$ ($\varrho < \infty$) in probability, then one can appeal to Anscombe's theorem (see [1]) in order to assert the asymptotic normality of S_{N_n}.

5. Multi-variate case. If the random variables are vector-valued, the arguments employed in Sections 3 and 4 can be repeated verbatim in order to prove the multi-variate forms of Theorems 3.1, 3.2 and 4.1. In the following we state the multi-variate version of Theorem 3.1:

THEOREM 5.1. *Let* $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots$ *be independent c-variate random variables having* $F_1(\boldsymbol{x}), F_2(\boldsymbol{x}), \ldots$ *for their respective c.d.f.'s, where* $\boldsymbol{x} = (x_1, \ldots, x_c)'$.

Assume that

$$S_n = \sum_{i=1}^{n} Z_{i,n},$$

where

$$Z_{i,n} = (X_i - a_{i,n})/\sigma_n = \{(X_{i,1} - a_{i,1,n})/\sigma_{1,n}, \ldots, (X_{i,c} - a_{i,c,n})/\sigma_{c,n}\},$$

$$a_{ijn} = \int_{|x| \leqslant b_n} x_j \, dF_i(x), \quad \sigma_{ijn}^2 = \int_{|x| \leqslant b_n} (x_j - a_{ijn})^2 \, dF_i(x), \quad \sigma_{jn}^2 = \sum_{i=1}^{n} \sigma_{ijn}^2$$

$$(j = 1, \ldots, c; \; i = 1, \ldots, n),$$

$b_n = (b_n, \ldots, b_n)'$, *and* $b_n \to \infty$ *is such that* $n/b_n^2 \leqslant K < \infty$ *and* $b_n/\sigma_{jn} \leqslant K < \infty$ *for* $j = 1, \ldots, c$. *Let*

$$\Sigma_n^{(i)} = (\varrho_{jkn}^{(i)}),$$

where

$$\varrho_{jkn}^{(i)} = \int_{|x| \leqslant b_n} \{(x_j - a_{ijn})/\sigma_{jn}\}\{(x_k - a_{ikn})/\sigma_{kn}\} \, dF_i(x)$$

$$(j, k = 1, \ldots, c; \; i = 1, \ldots, n)$$

and let

$$\Sigma_n = \Sigma_n^{(1)} + \ldots + \Sigma_n^{(n)}.$$

Further, let Y_{in} *be a c-variate normal variable having mean vector zero and a variance-covariance matrix* $\Sigma_n^{(i)}$ $(i = 1, \ldots, n)$, *and let* Y_{in} $(i = 1, \ldots, n)$ *be mutually independent.*

Then S_n *converges in distribution to the normal variable having the mean* 0 *and a variance-covariance matrix* Σ_n *provided*

$$B_{n,j}(t) = \sum_{i=1}^{n} A_{i,j}(t)/n \to 0 \quad \text{as } n, t \to \infty \text{ for } j = 1, \ldots, c,$$

where

$$A_{i,j}(t) = t^2[1 - F_{ij}(t) + F_{ij}(-t)],$$

F_{ij} *denoting the marginal distribution of the j-th component of* X_i.

Proof. It suffices to show that

$$\sum_{i=1}^{n} \|T_{Z_{i,n}} h(y) - T_{Y_{i,n}} h(y)\| \to 0 \quad \text{as } n \to \infty$$

for every $h \in C_3$. Consider

$$T_{Z_{i,n}} h(y) = \int h\{y + (x - a_{i,n})/\sigma_n\} \, dF_i(x) = \int_{|x| \leqslant b_n} + \int_{|x| \geqslant b_n}.$$

Expanding h in Taylor series, we have

$$h(\boldsymbol{y}+\boldsymbol{z}) = h(\boldsymbol{y})+\boldsymbol{z}h'(\boldsymbol{y})+\frac{1}{2}\boldsymbol{z}h''(\boldsymbol{y})\boldsymbol{z}'+$$
$$+\frac{1}{6}\left\{\left(z_{i,1}\frac{\partial}{\partial w_1}+\ldots+z_{i,c}\frac{\partial}{\partial w_c}\right)^3 h(\boldsymbol{w}) \mid \boldsymbol{w} = \boldsymbol{\xi}_i\right\},$$

where $h'(\boldsymbol{y})$ is the column vector with k-th component as the partial derivative of h with respect to the k-th variable evaluated at $\boldsymbol{y}$, and $\boldsymbol{\xi}_i$ lies between $\boldsymbol{y}$ and $\boldsymbol{y}+\boldsymbol{z}$. Setting $\boldsymbol{z} = (\boldsymbol{x}-\boldsymbol{a}_{i,n})/\boldsymbol{\sigma}_n$ and using this expansion in the first integral, we have

$$T_{\boldsymbol{Z}_{i,n}}h(\boldsymbol{y})=h(\boldsymbol{y})+\frac{1}{2}\operatorname{trace}\{h''(\boldsymbol{y})\Sigma_n^{(i)}\}-\left[h(\boldsymbol{y})-\frac{\boldsymbol{a}_{in}}{\boldsymbol{\sigma}_n}h'(\boldsymbol{y})\right]\mathrm{P}[|\boldsymbol{X}_i|>\boldsymbol{b}_n]+\frac{R_i}{6},$$

where $\boldsymbol{a}_{in}/\boldsymbol{\sigma}_n = (a_{i1n}/\sigma_{1n},\ldots,a_{icn}/\sigma_{cn})$ and

$$R_i =$$
$$\sum_{j=1}^{c}\sum_{k=1}^{c}\sum_{l=1}^{c}\int_{|\boldsymbol{x}|\leqslant \boldsymbol{b}_n}\{(x_j-a_{ijn})/\sigma_{jn}\}\{(x_k-a_{ikn})/\sigma_{kn}\}\{(x_l-a_{iln})/\sigma_{ln}\}h'''(\boldsymbol{\xi}_i)\,dF_i(\boldsymbol{x})$$
$$=\sum_{j=1}^{c}\sum_{k=1}^{c}\sum_{l=1}^{c}R_{jkl}^{(i)}.$$

Analogously, we have

$$T_{\boldsymbol{Y}_{in}}h(\boldsymbol{y}) = h(\boldsymbol{y})+\tfrac{1}{2}\operatorname{trace}\{h''(\boldsymbol{y})\Sigma_n^{(i)}\}+R_i^*/6$$

with

$$R_i^* = \sum_{j}^{c}\sum_{k}^{c}\sum_{l}^{c}\int x_jx_kx_l h'''(\boldsymbol{\xi}^*)\,d\Phi_{in}(\boldsymbol{x}),$$

where $\Phi_{in}(\boldsymbol{x})$ denotes the c.d.f. of $\boldsymbol{Y}_{in}$, and $\boldsymbol{\xi}^*$ lies between $\boldsymbol{y}$ and $\boldsymbol{y}+\boldsymbol{x}$. Next, let $a_n = \min(\sigma_{1n},\ldots,\sigma_{cn})$, expand, repeatedly use Hölder's inequality (Lemma 3.4), and obtain

$$\int_{|\boldsymbol{x}|\leqslant \boldsymbol{b}_n}|(x_j-a_{ijn})(x_k-a_{ikn})(x_l-a_{iln})/\sigma_{jn}\sigma_{kn}\sigma_{ln}|\,dF_i(\boldsymbol{x})$$
$$\leqslant 8a_n^{-3}\left[\int|x_j|^3dF_i\right]^{1/3}\left[\int|x_k|^3dF_i\right]^{1/3}\left[\int|x_l|^3dF_i\right]^{1/3},$$

where the range of integration in each integral is $|\boldsymbol{x}|\leqslant \boldsymbol{b}_n$. Now, applying extended Hölder's inequality for sums (see Theorem 11 of Hardy et al. [5], p. 22), we obtain

$$\sum_{i=1}^{n}R_{jkl}^{(i)}\leqslant 8a_n^{-3}\prod_{m=j,k,l}\Big[\sum_{i=1}^{n}\int_{|\boldsymbol{x}|\leqslant \boldsymbol{b}_n}|x_m|^3dF_i(\boldsymbol{x})\Big]^{1/3}$$
$$\leqslant K\prod_{m=j,k,l}\Big[\prod_{i=1}^{n}b_n^{-3}\int_{|x|\leqslant b_n}|x|^3dF_{im}(x)\Big]^{1/3}$$

after integrating on the other variables in each integral, where $F_{im}(x)$ denotes the marginal c.d.f. of the m-th component of X_i, and using the hypothesis that $b_n/a_n \leqslant K < \infty$. Now, as in the proof of Theorem 3.1, one can show that, for each m,

$$b_n^{-3}\sum_{i=1}^{n}\int_{|x|\leqslant b_n}|x|^3 dF_{im}(x) \leqslant K\delta.$$

Thus, for sufficiently large n,

$$\sum_{i=1}^{n} R_i = \sum_{i=1}^{n}\sum_{j=1}^{c}\sum_{k=1}^{c}\sum_{l=1}^{c} R_{jkl}^{(i)} \leqslant K\delta.$$

Next, let us turn to R_i^*. Write

$$\sum_{i=1}^{n} R_i^* = \sum_{i=1}^{n}\sum_{j=1}^{c}\sum_{k=1}^{c}\sum_{l=1}^{c} R_{jkl}^{*(i)}, \qquad R_{jkl}^{*(i)} = \int x_j x_k x_l h'''(\xi^*)\, d\Phi_{in}(x).$$

After using Hölder's inequality one can readily obtain

$$|R_{jkl}^{*(i)}| \leqslant \|h'''\| \Big[\int |x_j|^3 d\Phi_{in}\Big]^{1/3}\Big[\int |x_k|^3 d\Phi_{in}\Big]^{1/3}\Big[\int |x_l|^3 d\Phi_{in}\Big]^{1/3}$$
$$= \|h'''\| \prod_{m=j,k,l}\Big[\int |x|^3 d\Phi_{imn}(x)\Big]^{1/3},$$

where Φ_{imn} denotes the marginal c.d.f. of Y_{imn} which is normal with mean 0 and variance $\sigma_{imn}^2/\sigma_{mn}^2$ $(m = j, k, l)$. Hence

$$|R_{jkl}^{*(i)}| \leqslant K \prod_{m=j,k,l}\{\sigma_{imn}/\sigma_{mn}\} \leqslant K a_n^{-3} \prod_{m=j,k,l}\sigma_{imn}.$$

Further, using Hölder's inequality for sums, we have

$$\sum_{i=1}^{n} |R_{jkl}^{*(i)}| \leqslant \frac{K}{a_n^3}\prod_{m=j,k,l}\Big(\sum_{i=1}^{n}\sigma_{imn}^3\Big)^{1/3}.$$

Now, as in the consideration of R_i, we can easily obtain

$$\sigma_{imn}^3 \leqslant 8\Big[\int_{|x|\leqslant b_n}|x|^3 dF_{im}(x)\Big] \qquad (m = j, k, l).$$

Thus

$$a_n^{-3}\sum_{i=1}^{n}\sigma_{imn}^3 \leqslant 8a_n^{-3}\sum_{i=1}^{n}\int_{|x|\leqslant b_n}|x|^3 dF_{im}(x) \leqslant K\delta.$$

Consequently,

$$\sum_{i=1}^{n}|R_i^*| \leqslant K\delta \qquad \text{for sufficiently large } n.$$

This completes the proof of Theorem 5.1.

Concluding remark. The results of this paper can be formulated in terms of a triangular array of random variables. For the sake of simplicity, they are stated here in terms of simple sequences. Further, the multi-variate case can also be reduced to the uni-variate case by considering arbitrary linear combinations of the component variables.

6. Examples.

(1) Let V denote the distribution function of the symmetric random variable W such that

$$t^2 \mathrm{P}[|W| \geqslant t] = \varrho(t),$$

where $\varrho(t)$ monotonically tends to zero as $t \to \infty$. Let us define a sequence of distribution functions $\{V_i\}$ by

$$V_i(x) = \begin{cases} V(x) & \text{if } x < -t_i, \\ V(-t_i) & \text{if } -t_i \leqslant x < 0, \\ V(t_i) & \text{if } 0 \leqslant x < t_i, \\ V(x) & \text{if } x \geqslant t_i, \end{cases}$$

and let $\Phi(x)$ be the standard normal distribution function. Let us define a random variable X_i having F_i for its distribution function, where

$$F_i(x) = (1-a_i)V_i(x) + a_i \Phi(x) \qquad (i = 1, 2, \ldots).$$

Assume that $X_1, X_2, \ldots, X_n$ are mutually independent for each n. Clearly, X_i has mean 0 for $i = 1, 2, \ldots$ Furthermore,

$$B_n(t) = n^{-1} \sum_{i=1}^{n} t^2[1 - F_i(t) + F_i(-t)] \leqslant t^2\{\mathrm{P}(|W| \geqslant t) + 2\Phi(-t)\}$$

which tends to zero as $t \to \infty$. Hence, for a given V, we have to choose b_n, t_i and a_i such that $n/b_n^2 \leqslant K < \infty$ and $b_n/\sigma_n \leqslant K < \infty$. In the sequel we shall provide a special case. Let $b_n = \sqrt{n}$, $t_i = i^{-\delta}$ for some $\delta > 0$, $a_i = (\log i)/i$, and

$$V(x) = \begin{cases} \dfrac{\log 2}{2x^2 \log 2|x|} & \text{if } x < -1, \\[2ex] \dfrac{1}{2} & \text{if } -1 \leqslant x < 1, \\[2ex] 1 - \dfrac{\log 2}{2x^2 \log 2x} & \text{if } x \geqslant 1. \end{cases}$$

Now, since $1-\{(\log x)/x\}$ is increasing for $x \geqslant e$ and decreasing for $x < e$, we have $1-a_i \geqslant 1-a_3$ for $i \geqslant 3$. Hence, for $i \geqslant 3$,

$$U_i^2(b_n) \geqslant 2(1-a_3) \int_{\max(1,i-\delta)}^{\sqrt{n}} x^2 dV(x)$$

$$= (1-a_3)\left[(\log 2) \int_1^{\sqrt{n}} d\left(\frac{1}{\log 2x}\right) + 2(\log 2) \int_1^{\sqrt{n}} \frac{dx}{x \log 2x}\right]$$

$$= (1-a_3)\left[(2\log 2)\log\log(2\sqrt{n}) + 1 - (2\log 2)(\log\log 2) - \frac{\log 2}{\log 2\sqrt{n}}\right].$$

Thus

$$\sigma_n^2(b_n) \geqslant \sum_{i=3}^{n} U_i^2(b_n)$$

$$\geqslant (1-a_3)(n-3)\left[(2\log 2)\log\log(2\sqrt{n}) + 1 - (2\log 2)(\log\log 2) - \frac{\log 2}{\log 2\sqrt{n}}\right].$$

Hence $n/\sigma_n^2(b_n)$ tends to zero for sufficiently large n.

In order to get the identically distributed case, set $V_i(x) \equiv V_1(x) \equiv V(x)$.

(2) Consider a sequence of independent random variables having $\{F_k\}$ for their c.d.f. sequence,

$$F_k(x) = \begin{cases} 0 & \text{if } x < -\sqrt{k+1}, \\ C_k\left[\dfrac{x^{-2}}{\log 2|x|} - \dfrac{(k+1)^{-1}}{\log 2\sqrt{k+1}}\right] & \text{if } -\sqrt{k+1} \leqslant x < -1, \\ \dfrac{1}{2} & \text{if } -1 \leqslant x < 1, \\ \dfrac{1}{2} + C_k\left[\dfrac{1}{\log 2} - \dfrac{x^{-2}}{\log 2x}\right] & \text{if } 1 \leqslant x < \sqrt{k+1}, \\ 1 & \text{if } x \geqslant \sqrt{k+1}, \end{cases}$$

where

$$C_k = \left\{2\left[\frac{1}{\log 2} - \frac{(k+1)^{-1}}{\log 2\sqrt{k+1}}\right]\right\}^{-1}.$$

It is easy to verify that

$$x^2(1-F_k(x)) \leqslant \frac{3}{4}(\log 2)\left[x^2 \frac{(k+1)^{-1}}{\log 2\sqrt{k+1}} - \frac{1}{\log 2x}\right]$$

and, for sufficiently large n,

$$\frac{1}{n}\sum_{k=1}^{n} x^2\big(1-F_k(x)\big) \leqslant \frac{3}{4}(\log 2)\left[\frac{x^2}{n}\int_1^{n+1}\frac{y^{-1}}{\log 2\sqrt{y}}\,dy - \frac{1}{\log 2x}\right]$$

$$= \frac{3}{4}(\log 2)\left[\frac{2x^2}{n}\big(\log\log(2\sqrt{n+1}) - \log\log 2\big) - \frac{1}{\log 2x}\right] \to 0 \quad \text{as } n, x \to \infty.$$

Now $a_{k,n} \equiv 0$ and setting $b_n = \sqrt{n+1}$ we obtain

$$U_{k,n}^2 = 2C_k\left[\frac{1}{\log 2} - \frac{1}{\log 2\sqrt{k+1}}\right] + 4C_k[\log\log(2\sqrt{k+1}) - \log\log 2].$$

However, since $C_k \geqslant \frac{1}{2}\log 2$, we have

$$\sigma_n^2(b_n) \geqslant \big(1-(2\log 2)\log\log 2\big)n - (\log 2)\frac{2n}{3\log 2} + (2\log 2)\int_2^n (\log\log 2\sqrt{x})\,dx.$$

Further,

$$\int_2^n (\log\log 2\sqrt{x})\,dx \geqslant n(\log\log 2\sqrt{n}) - 2(\log\log 2\sqrt{2}) - \frac{n}{2\log 2\sqrt{2}}.$$

Thus

$$\frac{n}{\sigma_n^2(b_n)} \to 0 \quad \text{as } n \to \infty,$$

Consequently, one can apply Theorem 3.1 or 3.2 to assert the asymptotic normality of $\sum_{i=1}^{n} X_i/\sigma_n$.

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PEWNE CENTRALNE TWIERDZENIE GRANICZNE DLA NIEZALEŻNYCH SKŁADNIKÓW

STRESZCZENIE

W pracy podaje się centralne twierdzenie graniczne dla przypadku, w którym sumy są normowane w zwykły sposób, tzn. tak, że każdy składnik ma wartość oczekiwaną lub uciętą wartość oczekiwaną równą zeru i jest podzielny przez wspólny czynnik.