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A k-NOISY, n-SILENT VS. ONE-NOISY DUEL WITH EQUAL ACCURACY FUNCTIONS

0. Introduction. In this paper we consider a game in which there are two opponents, denoted by A and B. Player A has k noisy bullets and n silent ones ($k \geqslant 1$ and $n \geqslant 1$), and player B has one noisy bullet. A fires his noisy bullets before the silent ones. None of the silent shots of A is heard by B. Each player knows his opponent's noisy shot time. The probability of hitting the opponent at time t is called the *accuracy function* $P(t)$. We assume that both players have equal accuracy functions $P(t) = t$ and that $t \in [0, 1]$. Each player knows the total number of bullets, and their type, that his opponent can fire. If A hits B, not being hit himself, the pay-off of the duel is assumed to be $+1$; if B hits A, not being hit himself, the pay-off is taken to be -1; if both hit themselves each other or both survive, the pay-off equals 0. The game is over if one of the players is hit or if $t = 1$. The objective of A is to maximize the expected pay-off and the objective of B is to minimize it.

We show that the game has a value and we find an optimal strategy for player A and an ε-optimal strategy for player B. In section 1 we evaluate the pay-off function for the game. In section 2 the reader can find the description of the optimal strategies. Section 3 contains the system of equations for some constants which determine optimal strategies. At last, in section 4 we prove the optimality of the strategy for player A and the ε-optimality for player B.

In our paper we use the solution of the duel for $k = 0$ obtained by Styszyński [2].

1. Pay-off function for the game of timing. Let us denote by $\{z_i\}_{i=1}^{k}$ and $\{x_j\}_{j=1}^{n}$ the moments of shooting by player A his noisy and silent bullets, respectively. According to our supposition, we have

$$0 \leqslant z_1 \leqslant \ldots \leqslant z_k \leqslant x_1 \leqslant \ldots \leqslant x_n \leqslant 1.$$

The moment of shooting by player B his noisy bullet will be denoted by y, $0 \leqslant y \leqslant 1$.

If B shoots and misses, and player A has still at least one bullet left, he waits until $t = 1$ for a sure hit, if he follows an optimal strategy. Clearly, this wait until $t = 1$ in any optimal strategies is necessary and it will be assumed in all strategies for the rest of this paper.

Let us write $\bar{z}_k = (z_1, \ldots, z_k)$, $\bar{x}_n = (x_1, \ldots, x_n)$ and let $K(\bar{z}_k, \bar{x}_n; y)$ be the expected pay-off when players A and B fire their bullets precisely at the moments determined by $(\bar{z}_k, \bar{x}_n)$ and y, respectively. We may assume

$$K(\bar{z}_k, \bar{x}_n; y) = \Pr\{A \text{ survives alone}\} - \Pr\{B \text{ survives alone}\} \tag{1}$$

when players A and B use strategies $(\bar{z}_k, \bar{x}_n)$ and y, respectively. $K(\bar{z}_k, \bar{x}_n; y)$ is called the *pay-off function*. This function is of the form

$$K(\bar{z}_k, \bar{x}_n; y) \tag{2}$$

$$= \begin{cases} 1-2y, & y < z_1, \\ 1-(2-z_1)y, & y = z_1 \\ 1-2y\prod_{i=1}^{s}(1-z_i), & z_s < y < z_{s+1} \ (s = 1, 2, \ldots, k-1), \\ & \qquad z_{k+1} = x_1, \\ 1-(2-z_s)y\prod_{i=1}^{s-1}(1-z_i), & y = z_s \ (s = 1, 2, \ldots, k), \\ 1-2y\prod_{i=1}^{k}(1-z_i)\prod_{i=1}^{s}(1-x_i), & x_s < y < x_{s+1} \ (s = 1, 2, \ldots, n-1), \\ 1-\prod_{i=1}^{k}(1-z_i)\prod_{i=1}^{n-1}(1-x_i), & y = x_n, \\ 1-(1+y)\prod_{i=1}^{k}(1-z_i)\prod_{i=1}^{n}(1-x_i), & y > x_n. \end{cases}$$

These equations can easily be obtained directly from definition (1) of the pay-off function.

We check, for example, the formula in case $x_s < y < x_{s+1}$ for $s = 1, 2, \ldots, n-1$. A survives alone if he either hits B at any moment $z_1, z_2, \ldots, z_k, x_1, x_2, \ldots, x_s$ with probability

$$1-\prod_{i=1}^{k}(1-z_i)\prod_{i=1}^{s}(1-x_i)$$

or if he does not hit B at these moments and B misses at time y with probability

$$(1-y)\prod_{i=1}^{k}(1-z_i)\prod_{i=1}^{s}(1-x_i).$$

B survives alone if A misses at each moment $z_1, z_2, \ldots, z_k, x_1, \ldots, x_s$ and A will be hit at time y. Therefore, B survives alone with probability

$$y\prod_{i=1}^{k}(1-z_i)\prod_{i=1}^{s}1-x_i).$$

From (1) we obtain

$$K(\bar{z}_k, \bar{x}_n; y) = 1-\prod_{i=1}^{k}(1-z_i)\prod_{i=1}^{s}(1-x_i)+(1-y)\prod_{i=1}^{k}(1-z_i)\prod_{i=1}^{s}(1-x_i)-$$
$$-y\prod_{i=1}^{k}(1-z_i)\prod_{i=1}^{s}(1-x_i) = 1-2y\prod_{i=1}^{k}(1-z_i)\prod_{i=1}^{s}(1-x_i).$$

Let $K(\bar{x}_n; y)$ denote the pay-off function in the considered game provided $k = 0$. For $z_k < y$ we have

$$K(\bar{z}_k, \bar{x}_n; y) = 1-\prod_{i=1}^{k}(1-z_i)+\prod_{i=1}^{k}(1-z_i)K(\bar{x}_n; y). \tag{3}$$

2. Description of the optimal strategies. We seek an optimal strategy S_A for player A in the following class of strategies:

A shoots his noisy bullets at moments c_j $(j = 1, 2, \ldots, k)$ with probability 1 and the i-th silent bullet at the time $x_i \in [a_i, a_{i+1})$ according to the density function $f_i(x_i)$ $(i = 1, 2, \ldots, n)$, if he does not hear the shot of player B. However, if he has heard the shot of B and has still at least one bullet left, he waits until $t = 1$ for a sure hit. Here

$$0 < c_1 < c_2 < \ldots < c_k < a_1 < a_2 < \ldots < a_n < a_{n+1} = 1.$$

$c_1, c_2, \ldots, c_k, a_1, a_2, \ldots, a_n$ are constants which will be evaluated later.

Thus

$$\int_{a_i}^{a_{i+1}} f_i(x_i)\,dx_i = 1 \quad (i = 1, 2, \ldots, n). \tag{4}$$

The class of strategies, in which we seek an ε-optimal strategy for player B, will be defined by induction with respect to the number r of noisy bullets of player A. The strategy of this class will be denoted by $S_B^\varepsilon(r)$.

1° The strategy $S_B^\varepsilon(1)$ is defined as follows:

If B has heard his opponent's noisy shot until $t \leqslant c_k$, he fires his bullet at time y during the interval $[a_1, 1)$ chosen randomly according to the probability density $g(y)$ and a discrete component β at $y = 1$.

Thus

$$\int_{a_1}^{1} g(y)dy + \beta = 1. \tag{5}$$

On the other hand, if player B did not hear his opponent's noisy shot until $t = c_k$, he begins to shoot during the interval $(c_k, c_k + \varepsilon_k)$ with probability density $1/\varepsilon_k$. We suppose that $0 < \varepsilon_s < c_{s+1} - c_s$ $(s = 1, 2, \ldots, k)$. However, player B fires his bullet only to the moment of hearing the shot of A. At this moment he breaks the shooting in the interval $(c_k, c_k + \varepsilon_k)$ and makes his action in the interval $[a_1, 1]$ according to the probability distribution defined by $g(y)$ and β.

Let $\varepsilon > 0$ and let ε_j denote $\varepsilon/2^j$ $(j = 0, 1, \ldots, k)$.

2° Let us assume that the strategy $S_B^\varepsilon(r)$ for player B for some $r = p < k$ has been defined.

3° Using the assumption 2° we define the strategy $S_B^\varepsilon(r)$ for B for $r = p+1$.

If B has heard the first noisy shot of player A until $t = c_{k-p}$, then he follows the strategy from 2°, considering the second noisy shot of A as the first one. In the opposite case, if player B did not hear his opponent's first noisy shot until $t = c_{k-p}$, he begins to shoot during the interval $(c_{k-p}, c_{k-p} + \varepsilon_{k-p})$ with probability density $1/\varepsilon_{k-p}$ but he fires his bullet only to the moment of hearing the shot of A. At this time he breaks the shooting in the interval $(c_{k-p}, c_{k-p} + \varepsilon_{k-p})$ and follows the strategy from 2°, considering the second noisy shot of A as the first one. The strategy of player B obtained in this manner for $r = k$ is denoted by S_B^ε.

Let us denote by $W[S_A, y]$ the expected value of the pay-off function, when player A follows his mixed strategy S_A and B uses the pure strategy y. Similarly, we denote by $W[\bar{z}_k, \bar{x}_n; S_B]$ the expected value of the pay-off function, when player A follows his pure strategy $(\bar{z}_k, \bar{x}_n)$ and B uses the mixed strategy S_B^ε. Thus

$$W[S_A, y] = \int_{a_1}^{a_2} \int_{a_2}^{a_3} \ldots \int_{a_n}^{1} K(\bar{c}_k, \bar{x}_n; y) \prod_{i=1}^{n} f_i(x_i)dx_i \quad \text{for each } y, \tag{6}$$

where $\bar{c}_k = (c_1, c_2, \ldots, c_k)$.

In the case $z_i \leqslant c_i$ $(i = 1, 2, \ldots, k)$ we have

$$W[\bar{z}_k, \bar{x}_n; S_B^\varepsilon] = \int_{a_1}^{1} K(\bar{z}_k, \bar{x}_n; y)g(y)dy + \beta K(\bar{z}_k, \bar{x}_n; 1). \tag{7}$$

Define by $\bar{x}_n^*$ a vector $\bar{x}_n$ such that $x_i \in [a_i, a_{i+1})$ for $i = 1, \ldots, n$. We assume that the game has a value v and

$$W[S_A, y] = W[\bar{c}_k, \bar{x}_n^*; S_B^\varepsilon] = v \quad \text{for each } y \in [a_1, 1] \text{ and each } \bar{x}_n^*.$$

We justify this assumption in section 4.

From (6) and (7) we obtain

$$v = \begin{cases} \int\limits_{a_1}^{a_2} \int\limits_{a_2}^{a_3} \ldots \int\limits_{a_n}^{1} K(\bar{c}_k, \bar{x}_n; y) \prod_{i=1}^{n} f_i(x_i)\,dx_i & \text{for each } y \in [a_1, 1], \\ \int\limits_{a_1}^{1} K(\bar{c}_k, \bar{x}_n^*; y) g(y)\,dy + \beta K(\bar{c}_k, \bar{x}_n^*; 1) & \text{for each } \bar{x}_n^*. \end{cases}$$

Using equations (3) and (5) we have

$$v = 1 - \prod_{i=1}^{k} (1-c_i) + \prod_{i=1}^{k} (1-c_i) \int\limits_{a_1}^{a_2} \int\limits_{a_2}^{a_3} \ldots \int\limits_{a_n}^{1} K(\bar{x}_n; y) \prod_{i=1}^{n} f_i(x_i)\,dx_i \quad \text{for } y \in [a_1, 1], \tag{8}$$

$$v = 1 - \prod_{i=1}^{k} (1-c_i) + \prod_{i=1}^{k} (1-c_i) \Big\{ \int\limits_{a_1}^{1} K(\bar{x}_n^*; y) g(y)\,dy + \beta K(\bar{x}_n^*; 1) \Big\} \quad \text{for each } \bar{x}_n^*. \tag{9}$$

It follows from equations (8) and (9) that

$$\int\limits_{a_1}^{a_2} \int\limits_{a_2}^{a_3} \ldots \int\limits_{a_n}^{1} K(\bar{x}_n; y) \prod_{i=1}^{n} f_i(x_i)\,dx_i = v' = \text{const} \quad \text{for each } y \in [a_1, 1] \tag{10}$$

and

$$\int\limits_{a_1}^{1} K(\bar{x}_n^*; y) g(y)\,dy + \beta K(\bar{x}_n^*; 1) = v' \quad \text{for each } \bar{x}_n^*. \tag{11}$$

From relations (4), (5), (10) and (11) it is possible to find the density functions $f_i(x_i)$ $(i = 1, 2, \ldots, n)$, $g(y)$ and the parameters $a_1, a_2, \ldots, a_n$, β, v'.

This problem was solved by Styszyński [2]. We quote the results obtained in his paper:

$$v' = 1 - 2a_1, \quad f_i(x_i) = \frac{a_i}{x_i^3}, \quad x_i \in [a_i, a_{i+1}) \quad (i = 1, 2, \ldots, n-1),$$

$$f_n(x_n) = \frac{2P(a_n)}{P(x_n)^3}, \quad x_n \in [a_n, 1],\ P(x) = \sqrt{x^2 + 2x - 1},$$

$$g(y) = \begin{cases} \dfrac{l_i}{y^3}, & y \in [a_i, a_{i+1}) \ (i = 1, 2, \ldots, n-1), \\ \dfrac{l_n}{P(y)^3}, & y \in [a_n, 1), \end{cases}$$

$$l_i = a_1^2 \prod_{j=1}^{i} \frac{1}{(1-a_j)} \quad (i = 1, 2, \ldots, n-1),$$

$$l_n = \frac{\sqrt{2}\, l_{n-1}}{1-a_n}, \quad a_n = \sqrt{6} - 2,$$

$$\frac{1}{a_i^2} - \frac{1}{a_{i+1}^2} = \frac{2}{a_i} \quad (i = 1, 2, \ldots, n-1), \qquad \beta = \frac{l_n}{2\sqrt{2}}.$$

The density functions $f_i(x_i)$ $(i = 1, 2, \ldots, n)$, $g(y)$ and the parameters $a_1, a_2, \ldots, a_n, \beta$ and v' are uniquely determined by the relations specified above.

In paper [2] it was shown that

$$\int_{a_1}^{1} K(\bar{x}_n; y) g(y) dy + \beta K(\bar{x}_n; 1) \leqslant v' \quad \text{for all } \bar{x}_n. \tag{12}$$

where $v' = 1 - 2a_1$ denotes the value of the game of timing in the n-silent-vs.-one-noisy duel. Hence it follows from relations (9) and (11) that

$$v = 1 - 2a_1 \prod_{i=1}^{k} (1 - c_i). \tag{13}$$

3. System of equations for the constants c_i. We find now constants c_i $(i = 1, 2, \ldots, k)$ from the relations

$$\lim_{y \to c_i^-} W[S_A, y] = v \quad (i = 1, 2, \ldots, k).$$

By the definition of the strategy S_A, it is equivalent to the equations

$$\lim_{y \to c_i^-} \Big[1 - 2y \prod_{j=1}^{i-1} (1 - c_j)\Big] = 1 - 2a_1 \prod_{j=1}^{k} (1 - c_j) \quad (i = 1, 2, \ldots, k). \tag{14}$$

Hence, we obtain

$$c_i = a_1 \prod_{j=i}^{k} (1 - c_j) \quad (i = 1, 2, \ldots, k). \tag{15}$$

The constants c_i are determined by these equations in the form

$$c_i = \frac{a_1}{(k - i + 1) a_1 + 1} \quad (i = 1, 2, \ldots, k).$$

Then from (13) and (15) we have

$$v = 1 - 2a_1 \prod_{j=1}^{k} (1 - c_j) = 1 - 2c_1 \quad \text{or} \quad v = 1 - \frac{2a_1}{ka_1 + 1}.$$

4. Proof of optimality for S_A and ε-optimality for S_B^ε. In this section we prove that

$$\min_{0\leqslant y\leqslant 1} W[S_A, y] = v \tag{16}$$

and

$$\max_{0\leqslant z_1\leqslant \ldots \leqslant z_k\leqslant x_1\leqslant \ldots \leqslant x_n\leqslant 1} W[\bar{z}_k, \bar{x}_n; S_B^\varepsilon] \leqslant v+\varepsilon. \tag{17}$$

Let $c_{k+1} = a_1$.

First, using (13) and (14), we prove equation (16) considering the following cases:

1° If $y < c_1$, we have

$$W[S_A, y] = 1-2y > 1-2c_1 = v.$$

2° If $y \in (c_i, c_{i+1})$ $(i = 1, 2, \ldots, k)$, we have

$$W[S_A, y] = 1-2y\prod_{j=1}^{i}(1-c_j) > 1-2c_{i+1}\prod_{j=1}^{i}(1-c_j) = 1-2a_1\prod_{j=1}^{k}(1-c_j) = v.$$

3° If $y = c_i$ $(i = 1, 2, \ldots, k)$, we have

$$W[S_A, c_i] = 1-(2-c_i)c_i\prod_{j=1}^{i-1}(1-c_j) > 1-2c_i\prod_{j=1}^{i-1}(1-c_j) = v.$$

4° If $y \in [a_1, 1]$, then, by (8) and (10), we have $W[S_A, y] = v$. Hence it follows that equation (16) is valid.

Now, we prove inequality (17). We show that for $r = 1, 2, \ldots, k$ the inequality

$$W[\bar{z}_r, \bar{x}_n; S_B^\varepsilon(r)] \leqslant 1-2c_{k-r+1}+\varepsilon_{k-r} \tag{18}$$

holds for all $0 \leqslant z_1 \leqslant \ldots \leqslant z_r \leqslant x_1 \ \ldots \leqslant x_n \leqslant 1$.

We prove this by induction with respect to r.

1° For $r = 1$ we show that

$$W[\bar{z}_1, \bar{x}_n; S_B^\varepsilon(1)] \leqslant 1-2c_k+\varepsilon_{k-1}, \quad \text{where } \bar{z}_1 \equiv z_1,\ 0 \leqslant z_1 \leqslant x_1.$$

Let us consider the cases where $z_1 \leqslant c_k$, $z_1 \in (c_k, c_k+\varepsilon_k)$ and $z_1 \geqslant c_k+\varepsilon_k$.

(a) $z_1 \leqslant c_k$. In this case $S_B^\varepsilon(1)$ is the strategy defined in such a way that if player B has heard his opponent's noisy shot before time $t = c_k$, then he shoots in the interval $[a_1, 1]$ according to the probability distribution $G(y)$ determined by $g(y)$ and β. Using (3), (5), (12) and (15) successively we prove that

$$W[\bar{z}_1, \bar{x}_n; S_B^\varepsilon(1)] = \int_{a_1}^{1} K(\bar{z}_1, \bar{x}_n; y) g(y) dy + \beta K(\bar{z}_1, \bar{x}_n; 1)$$

$$= \int_{a_1}^{1} [z_1 + (1-z_1) K(\bar{x}_n; y)] g(y) dy + \beta [z_1 + (1-z_1) K(\bar{x}_n; 1)]$$

$$= z_1 + (1-z_1) \Big\{ \int_{a_1}^{1} K(\bar{x}_n; y) g(y) dy + \beta K(\bar{x}_n; 1) \Big\} \leqslant z_1 + (1-z_1)(1-2a_1)$$

$$\leqslant c_k + (1-c_k)(1-2a_1) = 1-(1-c_k)2a_1 < 1-2c_k + \varepsilon_{k-1}.$$

(b) $z_1 \in (c_k, c_k + \varepsilon_k)$. Now

$$W[\bar{z}_1, \bar{x}_n; S_B^\varepsilon(1)]$$

$$= \int_{c_k}^{z_1} (1-2y) \frac{1}{\varepsilon_k} dy + \frac{c_k + \varepsilon_k - z_1}{\varepsilon_k} \times$$

$$\times \Big\{ z_1 + (1-z_1) \Big[\int_{a_1}^{1} K(\bar{x}_n; y) g(y) dy + \beta K(\bar{x}_n; 1) \Big] \Big\}$$

$$\leqslant \frac{z_1 - c_k}{\varepsilon_k} (1-2c_k) + \frac{c_k + \varepsilon_k - z_1}{\varepsilon_k} \{c_k + (1-c_k)(1-2a_1) + \varepsilon_k - (1-2a_1)\varepsilon_k\}$$

$$= \frac{z_1 - c_k}{\varepsilon_k} (1-2c_k) + \frac{c_k + \varepsilon_k - z_1}{\varepsilon_k} \{1-2c_k + \varepsilon_k - (1-2a_1)\varepsilon_k\}$$

$$< 1-2c_k + 2a_1 \varepsilon_k < 1-2c_k + \varepsilon_{k-1}.$$

(c) $z_1 \geqslant c_k + \varepsilon_k$. In this case we have

$$W[\bar{z}_1, \bar{x}_n; S_B^\varepsilon(1)] = \int_{c_k}^{c_k+\varepsilon_k} (1-2y) \frac{1}{\varepsilon_k} dy \leqslant 1-2c_k + \varepsilon_{k-1}.$$

Thus inequality (18) holds for $r = 1$.

2° $r = p \leqslant k$. We assume that inequality (18) is valid for an $r = p < k$, i.e.

$$W[\bar{z}_p, \bar{x}_n; S_B^\varepsilon(p)] = 1-2c_{k-p+1} + \varepsilon_{k-p} \quad \text{for each } (\bar{z}_p, \bar{x}_n).$$

3° We have to show the following relation:

$$W[\bar{z}_{p+1}, \bar{x}_n; S_B^\varepsilon(p+1)] \leqslant 1-2c_{k-p} + \varepsilon_{k-p-1}.$$

Let us consider the cases for $z_1 \leqslant c_{k-p}$, $z_1 \in (c_{k-p}, c_{k-p} + \varepsilon_{k-p})$ and $z_1 \geqslant c_{k-p} + \varepsilon_{k-p}$.

(a) $z_1 \leqslant c_{k-p}$. In this case $S_B^\varepsilon(p+1)$ is the strategy defined in such a way that if player B has heard his opponent's noisy shot before $t = c_{k-p}$, then he follows the strategy $S_B^\varepsilon(p)$. Let us write $\bar{z}_p' = (z_2, \ldots, z_{p+1}')$. We have

$$W[\bar{z}_{p+1}, \bar{x}_n; S_B^\varepsilon(p+1)] = z_1 + (1-z_1)W[\bar{z}_p', \bar{x}_n; S_B^\varepsilon(p)].$$

From 2° and by (15) we have

$$W[\bar{z}_{p+1}, \bar{x}_n; S_B^\varepsilon(p+1)] \leqslant z_1 + (1-z_1)(1-2c_{k-p+1}+\varepsilon_{k-p})$$

$$\leqslant c_{k-p} + (1-c_{k-p})(1-2_{k-p+1}+\varepsilon_{k-p}) = 1-(1-c_{k-p})(2c_{k-p+1}-\varepsilon_{k-p})$$

$$< 1 - \frac{pa_1+1}{(p+1)a_1+1}\,\frac{2a_1}{pa_1+1} + \varepsilon_{k-p} = 1-2c_{k-p}+\varepsilon_{k-p} < 1-2c_{k-p}+\varepsilon_{k-p-1}.$$

(b) $z_1 \epsilon (c_{k-p}, c_{k-p}+\varepsilon_{k-p})$. In this case player B did not hear his opponent's first noisy shot before $t = c_{k-p}$. Therefore, he begins to shoot during the interval $(c_{k-p}, c_{k-p}+\varepsilon_{k-p})$ with probability density $1/\varepsilon_{k-p}$ but he fires his bullet only to the moment of hearing the shot of A. At this time he breaks the shooting in the interval $(c_{k-p}, c_{k-p}+\varepsilon_{k-p})$ and follows the strategy from 2°, considering the second noisy shot of A as the first one.

We have

$$W[\bar{z}_{p+1}, \bar{x}_n; S_B^\varepsilon(p+1)]$$

$$= \int\limits_{c_{k-p}}^{z_1} (1-2y)\frac{1}{\varepsilon_{k-p}}dy + \frac{c_{k-p}+\varepsilon_{k-p}-z_1}{\varepsilon_{k-p}}\{z_1+(1-z_1)W[\bar{z}_p', \bar{x}_n; S_B^\varepsilon(p)]\}$$

$$\leqslant \frac{z_1-c_{k-p}}{\varepsilon_{k-p}}(1-2c_{k-p}) +$$

$$+\frac{c_{k-p}-z_1+\varepsilon_{k-p}}{\varepsilon_{k-p}}\{c_{k-p}+\varepsilon_{k-p}+(1-c_{k-p}-\varepsilon_{k-p})(1-2c_{k-p+1}+\varepsilon_{k-p})\}$$

$$\leqslant \frac{z_1-c_{k-p}}{\varepsilon_{k-p}}(1-2c_{k-p}) + \frac{c_{k-p}-z_1+\varepsilon_{k-p}}{\varepsilon_{k-p}}\{c_{k-p}+(1-c_{k-p})(1-2c_{k-p+1})+$$

$$+\varepsilon_{k-p}-\varepsilon_{k-p}(1-2c_{k-p+1}+\varepsilon_{k-p})+\varepsilon_{k-p}(1-c_{k-p})\}$$

$$= \frac{z_1-c_{k-p}}{\varepsilon_{k-p}}(1-2c_{k-p}) + \frac{c_{k-p}-z_1+\varepsilon_{k-p}}{\varepsilon_{k-p}}\{1-2c_{k-p}+\varepsilon_{k-p}(1+c_{k-p}-\varepsilon_{k-p})\}$$

$$= 1-2c_{k-p}+\frac{c_{k-p}-z_1+\varepsilon_{k-p}}{\varepsilon_{k-p}}\varepsilon_{k-p}(1+c_{k-p}-\varepsilon_{k-p}) < 1-2c_{k-p}+\varepsilon_{k-p-1}.$$

(c) $z_1 \geqslant c_{k-p} + \varepsilon_{k-p}$. In this case player B fires in the interval $(c_{k-p}, c_{k-p} + \varepsilon_{k-p})$ according to the probability density $1/\varepsilon_{k-p}$. We have

$$W[\bar{z}_{p+1}, \bar{x}_n; S_B^\varepsilon(p+1)] = \int_{c_{k-p}}^{c_{k-p}+\varepsilon_{k-p}} (1-2y) \frac{1}{\varepsilon_{k-p}} dy$$
$$\leqslant 1 - 2c_{k-p} < 1 - 2c_k + \varepsilon_{k-p-1}.$$

In such a way the induction is completed and inequality (17) holds.

It follows from relations (16) and (17) that the game has the value

$$v = 1 - \frac{2a_1}{ka_1 + 1},$$

the strategy S_A for player A is optimal and the strategy S_B^ε for player B is ε-optimal. We see that

$$\lim_{k\to\infty} v = 1.$$

It is easy to show that

$$\lim_{n\to\infty} a_1 = 0.$$

This gives

$$\lim_{n\to\infty} v = 1;$$

thus if the number of noisy or silent bullets of player A increases, the value of the game tends to 1.

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POJEDYNEK GŁOŚNO-CICHY PRZECIWKO GŁOŚNEMU Z RÓWNYMI FUNKCJAMI CELNOŚCI

STRESZCZENIE

W pracy rozpatrzono grę czasową typu pojedynku, w której gracz A ma k kul głośnych i n cichych, gracz B zaś jedną kulę głośną ($k \geqslant 1$ i $n \geqslant 1$). Gracz A strzela swoje kule w kolejności głośne-ciche. *Funkcją celności* $P(t)$ nazywamy prawdopodobieństwo trafienia przeciwnika w danym momencie t. Zakładamy, że obaj gracze mają jednakowe funkcje celności $P(t) = t$ i że $t \in [0, 1]$. Informacje te znane są przeciwnikom.

Jeśli A trafi B, sam nie będąc trafionym, wypłata w grze wynosi $+1$; jeśli B trafi A, sam nie będąc trafionym, wypłata równa jest -1; jeśli obaj gracze trafią się jednocześnie lub obaj przeżyją cały pojedynek, wypłata wynosi 0. Gra się kończy, gdy jeden z przeciwników zostanie trafiony lub gdy $t = 1$. Zadaniem A jest zmaksymalizowanie średniej wypłaty, zadaniem B — zminimalizowanie jej.

W pracy pokazano, że gra ma wartość i znaleziono strategię optymalną dla gracza A i ε-optymalną dla gracza B.