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# UPPER BOUNDS FOR THE ABSCISSA OF STABILITY OF A STABLE MATRIX

**1. Introduction.** Let $A = \{a_{ij}\}$ be a real matrix of dimension $n \times n$. Denote by $\lambda_1, \lambda_2, \ldots, \lambda_n$ the eigenvalues of the matrix $A$. Assume that $A$ is a stability matrix, i.e. $\mathrm{Re}(\lambda_i) < 0$ $(i = 1, 2, \ldots, n)$. The real number

$$a = \max_{1 \leqslant i \leqslant n} \mathrm{Re}(\lambda_i)$$

is called an *abscissa of stability* of $A$.

In this note a method of finding a real number $\gamma$ such that $a \leqslant \gamma < 0$ is presented. The problem of finding upper bounds for the abscissa of a stable polynomial was investigated by Henrici [3]. Denote by $\det(X)$ the determinant of the matrix $X$, and by $|\det(X)|$ the absolute value of $\det(X)$. Put

$$C = A \otimes I + I \otimes A',$$

where $I$ is the identity matrix of dimension $n \times n$, $A'$ is the transpose of $A$, and $X \otimes Y$ is the Kronecker product [1]. Therefore, $C$ is a real matrix of dimension $n^2 \times n^2$. It is well known that $\lambda_i + \lambda_j$ $(i, j = 1, 2, \ldots, n)$ are the eigenvalues of the matrix $C$.

Let $R > 0$ be a real number such that

$$|\lambda_i| \leqslant R \quad (i = 1, 2, \ldots, n). \tag{1}$$

A particularly simple $R$ is given by the formula (see [2])

$$R = \max_{1 \leqslant i \leqslant n} \sum_{j=1}^{n} |a_{ij}|, \tag{2}$$

where $|a_{ij}|$ is the absolute value of $a_{ij}$.

**2. Main result.**

THEOREM. *If the matrix $A$ is stable, then*

$$a \leqslant -2^{-n^2} R^{1-n^2} |\det(C)|. \tag{3}$$

Proof. From the definition of the matrix $C$ we have

$$\det(C) = \prod_{i,j=1}^{n}(\lambda_i+\lambda_j) = 2^n\prod_{i=1}^{n}\lambda_i \prod_{1\leqslant i<j\leqslant n}(\lambda_i+\lambda_j)^2.$$

Let

$$d = \prod_{1\leqslant i<j\leqslant n}(\lambda_i+\lambda_j).$$

Since $A$ is a stability matrix by assumption, $d$ is a real number. From (1) we obtain $|d|\leqslant(2R)^{n(n-1)/2}$. If $a = \lambda_k$, we have

$$\det(C) = 2^n a\prod_{\substack{i=1\\ i\neq k}}^{n}\lambda_i d^2,$$

$$a = 2^{-n}\frac{\det(C)}{\prod\limits_{\substack{i=1\\ i\neq k}}^{n}\lambda_i d^2}\leqslant -\left|\frac{\det(C)}{2^nR^{n-1}(2R)^{n(n-1)}}\right| = -2^{-n^2}R^{1-n^2}|\det(C)|.$$

In the case $a = \mathrm{Re}(\lambda_k)$, $\bar{\lambda}_k = \lambda_l$, $k<l$, we have

$$\det(C) = 2^n\prod_{i=1}^{n}\lambda_i(2a)d\prod_{\substack{1\leqslant i<j\leqslant n\\ i\neq k, j\neq l}}(\lambda_i+\lambda_j)$$

and

$$a = -\left|\frac{\det(C)}{2^{n+1}\prod\limits_{i=1}^{n}\lambda_i d\prod\limits_{\substack{1\leqslant i<j\leqslant n\\ i\neq k, j\neq l}}(\lambda_i+\lambda_j)}\right|\leqslant -\left|\frac{\det(C)}{2^{n+1}R^n\cdot 2^{n^2-n-1}R^{n^2-n-1}}\right|.$$

Hence inequality (3) holds, which completes the proof.

Example 1. Let

$$A = \begin{bmatrix} -0.09 & 0 & 0.01\\ 0.008 & -0.1 & 0.0007\\ 0 & 0 & -0.02\end{bmatrix}.$$

Then, by (2) and (3), we have

$$\det(C) = -2^3\cdot 1.9602\cdot 10^{-10},$$

$$R = 0.1,\quad a\leqslant -0.0003.$$

In this example $a = -0.02$.

Example 2. Let

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -2 & 1 \\ 1 & -3/2 & -3/2 \end{bmatrix}.$$

Then, by (2) and (3) we obtain

$$\det(C) = -16820, \quad R = 4, \quad a \leqslant -0.002.$$

In this example $a = -0.5$.

**3. Conclusion.** From the proof of the Theorem, for $a = \mathrm{Re}(\lambda_k)$, where $\lambda_k = a + i\beta$, $\beta \neq 0$, we have

$$a \leqslant -\left|\frac{\det(C)}{\det(A)}\right| \cdot 2^{-n^2} R^{-n^2+n+1}. \tag{4}$$

In the case $a = \lambda_k$ we get

$$\det(A) = a \prod_{\substack{i=1 \\ i \neq k}}^{n} \lambda_i,$$

and hence

$$a \leqslant -|\det(A)| R^{1-n}. \tag{5}$$

Applying inequality (5) to Example 1, we have $a \leqslant -0.0018$. However, using (4) or (5) we have to know whether the eigenvalue of a matrix with the largest real part is real or complex. This fact restricts the application of inequalities (4) and (5).

## References

[1] R. Bellman, *Introduction to matrix analysis*, McGraw-Hill, New York 1960.

[2] F. R. Gantmacher, *The theory of matrices*, Chelsea Publishing Co., New York 1960.

[3] P. Henrici, *Upper bounds for the abscissa of stability of a stable polynomial*, SIAM J. Numer. Anal. 7 (1970), p. 538-544.

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