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APPROXIMATION BY CIRCULAR SPLINES FOR SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

In the paper a procedure for obtaining a circular spline approximating the solution of the initial value problem in ordinary differential equations is presented. The function approximating the exact solution is of class C^1. The proposed method is a one-step method of second order.

1. Introduction and description of the method. Loscalzo and Talbot presented in [2] and [3] a method for the construction of a spline function of second or third degree being the approximate solution of the initial value problem in ordinary differential equations. Their method relates to the well-known linear difference formulas of closed type: the trapezoidal rule and the Milne-Simpson method.

Splines and circular splines are applied to numerical control of machines connected with a computer [1]. So there is the question: Can we find similarly the approximate solution of the initial value problem in ordinary differential equations in the form of a circular spline? The purpose of this paper is to give an answer to this question. The method proposed leads to a non-linear difference formula of closed type.

Definition 1. Let $[a, b]$ be a closed and bounded interval on the real axis. The function $s \in C^1[a, b]$ is said to be a *circular spline* if there exist an integer number N and a partition of the interval $[a, b]$,

$$a = x_0 < x_1 < \ldots < x_N = b,$$

such that s is a circular arc in each of the subintervals $[x_n, x_{n+1}]$, $n = 0, 1, \ldots, N-1$ (we assume that the straight line is a circle with radius $r = \infty$).

Definition 2. The points $x_1, x_2, \ldots, x_{N-1}$ are called *knots* or *joints*.

Let the differential equation be of the form

$$y' = f(x, y), \quad a \leqslant x \leqslant b, \tag{1}$$

with the initial condition

$$y(a) = y_0. \tag{2}$$

We assume that f is continuous on some domain G,

$$G = \{(x, y):\ a \leqslant x \leqslant b\},$$

and that on G it satisfies the Lipschitz condition

$$|f(x, y) - f(x, y^*)| \leqslant L|y - y^*|, \quad x \in [a, b]. \tag{3}$$

Thus the existence and uniqueness of the exact solution to (1) and (2) are guaranteed.

We shall construct the approximate solution s in the form of a circular spline for equidistant knots, i.e., we assume that

$$x_n = a + nh, \quad n = 0, 1, \ldots, N, \tag{4}$$

where $h = (b-a)/N$. It is possible to omit this assumption and employ a variable stepsize. We require also that $s(x_0) = y(x_0)$ and $s'(x_0) = y'(x_0)$, where y is the exact solution to (1) and (2). Starting with the known values $s_0 = s(x_0)$ and $s_0' = s'(x_0)$, it is proposed to calculate successively the values

$$s_n = s(x_n), \quad s_n' = s'(x_n), \quad n = 1, 2, \ldots, N,$$

by requiring that s satisfies (1) for $x = x_1, x_2, \ldots, x_N$. In this manner we obtain a circular spline s satisfying the equation

$$s_n' = f(x_n, s_n), \quad n = 0, 1, \ldots, N. \tag{5}$$

Note that any three values from among $s_n, s_n', s_{n+1}, s_{n+1}'$ are sufficient to determine the circular arc being the component of s in the subinterval $[x_n, x_{n+1}]$.

2. A certain relation for a circular spline. Let (p, q) be the centre of a circle of radius r, and let (x_n, s_n) and (x_{n+1}, s_{n+1}) be two points both lying either in the upper or in the lower part of this circle. Moreover, let s_n' and s_{n+1}' be the values of the first derivative of the circle at these points, respectively, and let $z = -1$ if the points are situated in the upper part of the circle, and $z = 1$ if they are in the lower part. Then for $i = n, n+1$ the following equations hold:

$$p = x_i - zr\frac{s_i'}{\sqrt{1+s_i'^2}}, \quad q = s_i + zr\frac{1}{\sqrt{1+s_i'^2}}.$$

Hence

$$x_{n+1} - x_n = zr\left(\frac{s_{n+1}'}{\sqrt{1+s_{n+1}'^2}} - \frac{s_n'}{\sqrt{1+s_n'^2}}\right) \tag{6}$$

and

$$s_{n+1} - s_n = zr\left(\frac{1}{\sqrt{1+s_n'^2}} - \frac{1}{\sqrt{1+s_{n+1}'^2}}\right).$$

Eliminating zr and writing $h = x_{n+1}-x_n$, by simple transformations we obtain the relation

$$s_{n+1} = s_n + h[s'_n + A(s'_n, s'_{n+1})], \tag{7}$$

where, for any fixed $c \in (-\infty, \infty)$,

$$A(c, x) = \sqrt{1+c^2}\,\frac{x-c}{\sqrt{1+x^2}+\sqrt{1+c^2}}, \quad x \in (-\infty, \infty). \tag{8}$$

Thus we have proved the following

THEOREM 1. *If s is a circular spline with knots* (4), *then any two consecutive pairs of quantities from among the pairs*

$$s_n = s(x_n), \quad s'_n = s'(x_n), \quad n = 0, 1, \ldots, N,$$

satisfy relation (7).

3. Auxiliary considerations. We shall need in the sequel the estimates of the function A and its first and second derivatives with respect to x.

THEOREM 2. *For any $c, x \in (-\infty, \infty)$ the following inequalities hold:*

$$|A(c, x)| < \sqrt{1+c^2}, \tag{9}$$

$$0 < A'(c, x) < 2, \tag{10}$$

$$|A''(c, x)| < 2. \tag{11}$$

Proof. Differentiating the function A with respect to x, we have

$$A'(c, x) = \frac{\sqrt{1+c^2}}{\sqrt{1+x^2}}\,\frac{\sqrt{1+x^2}\sqrt{1+c^2}+cx+1}{(\sqrt{1+x^2}+\sqrt{1+c^2})^2}, \quad c, x \in (-\infty, \infty). \tag{12}$$

Hence, in view of the inequality

$$|cx|+1 \leqslant \sqrt{1+x^2}\sqrt{1+c^2}, \tag{13}$$

we obtain

$$0 < \frac{\sqrt{1+c^2}}{\sqrt{1+x^2}}\,\frac{|cx|+1+cx+1}{(\sqrt{1+x^2}+\sqrt{1+c^2})^2} \leqslant A'(c, x)$$

$$\leqslant \frac{\sqrt{1+c^2}}{\sqrt{1+x^2}}\,\frac{2\sqrt{1+x^2}\sqrt{1+c^2}}{(\sqrt{1+x^2}+\sqrt{1+c^2})^2} = \frac{2(1+c^2)}{(\sqrt{1+x^2}+\sqrt{1+c^2})^2} < \frac{2(1+c^2)}{(\sqrt{1+c^2})^2} = 2.$$

This gives the desired estimate (10).

The left-hand side implies that A is a strictly increasing function of x for any fixed c. In that case, from the identity

$$\lim_{x\to\pm\infty} A(c, x) = \pm\sqrt{1+c^2}$$

we get (9).

Differentiating (12) with respect to x, we have

$$A''(c,x) = \frac{\sqrt{1+c^2}}{\sqrt{1+x^2}} \frac{(c-x)(\sqrt{1+x^2}+\sqrt{1+c^2})-2x\sqrt{1+x^2}(1+cx+\sqrt{1+x^2}\sqrt{1+c^2})}{(1+x^2)(\sqrt{1+x^2}+\sqrt{1+c^2})^3},$$

$$c, x \in (-\infty, \infty).$$

Hence, by (13), we obtain

$$|A''(c,x)| \leqslant \frac{\sqrt{1+c^2}}{\sqrt{1+x^2}} \frac{(|c|+|x|)(\sqrt{1+x^2}+\sqrt{1+c^2})+4|x|(1+x^2)\sqrt{1+c^2}}{(1+x^2)(\sqrt{1+x^2}+\sqrt{1+c^2})^3}$$

$$\leqslant \sqrt{1+c^2}\frac{(\sqrt{1+x^2}+\sqrt{1+c^2})^2+4(1+x^2)\sqrt{1+x^2}\sqrt{1+c^2}}{(1+x^2)(\sqrt{1+x^2}+\sqrt{1+c^2})^3}$$

$$\leqslant \sqrt{1+c^2}\frac{(\sqrt{1+x^2}+\sqrt{1+c^2})^2+4\sqrt{1+x^2}\sqrt{1+c^2}}{(\sqrt{1+x^2}+\sqrt{1+c^2})^3}.$$

From the inequality

$$4\sqrt{1+x^2}\sqrt{1+c^2} \leqslant (\sqrt{1+x^2}+\sqrt{1+c^2})^2$$

we finally obtain

$$|A''(c,x)| \leqslant \frac{2\sqrt{1+c^2}}{\sqrt{1+x^2}+\sqrt{1+c^2}} < \frac{2\sqrt{1+c^2}}{\sqrt{1+c^2}} = 2.$$

This completes the proof.

4. Existence and uniqueness of the approximate solution. The following theorem shows that the construction of a circular spline (described in Section 1) approximating the exact solution to (1) and (2) is realizable and univocal.

THEOREM 3. *If $h < 1/2L$, then for partition* (4) *there exists a unique circular spline s satisfying identities* (5).

Proof. Let s_n and s'_n be the values of the circular spline s and its first derivative at a point x_n satisfying the identity $s'_n = f(x_n, s_n)$ (initially, by the assumptions of the construction of s, at x_0 we have $s_0 = y_0$ and $s'_0 = f(x_0, y_0)$). We want to determine in the interval $[x_n, x_{n+1}]$ a circular arc being the consecutive component of the function s. Thus the values $s_n, s'_n, s_{n+1}, s'_{n+1}$ should satisfy relation (7). Moreover, in accordance with our construction of the approximate solution s, we require that

$$s'_{n+1} = f(x_{n+1}, s_{n+1}). \tag{14}$$

This identity is satisfied if and only if

$$s_{n+1} = s_n + h\left[s'_n + A\left(s'_n, f(x_{n+1}, s_{n+1})\right)\right]. \tag{15}$$

Denoting the right-hand member of this identity by $g_h(s_{n+1})$, in view of (3) and (10) we have

$$\begin{aligned}|g_h(u) - g_h(v)| &= h\left|A\left(s'_n, f(x_{n+1}, u)\right) - A\left(s'_n, f(x_{n+1}, v)\right)\right| \\ &= hA'\left(s'_n, \theta_n(u, v)\right)|f(x_{n+1}, u) - f(x_{n+1}, v)| \leqslant 2Lh|u-v|\end{aligned}$$

for any u, v. Thus, if $h < 1/2L$, then g_h is a strong contraction mapping. Applying Banach's fixed-point theorem, we deduce that equation (15) has a unique solution s_{n+1}, which may be found by iteration. Next, having given the quantities s_n, s'_n, s_{n+1}, we can determine the parameters of the sought circular arc and, by (14), the quantity s'_{n+1}. So the theorem is established.

The method of successive computation of the ordinates s_n is described by the following iterative formula:

$$s_{n+1}^{(i+1)} = s_n + h\left[s'_n + A\left(s'_n, f(x_{n+1}, s_{n+1}^{(i)})\right)\right], \quad i = 0, 1, \ldots$$

It is best to find the initial prediction $s_{n+1}^{(0)}$ by using the Euler method or the Runge-Kutta method of second order. The use of methods of order greater than two is not recommended, since it is shown that the considered method is of second order.

5. Error estimates. First we estimate the error of the approximate solution s at the points x_n.

THEOREM 4. *If $f \in C^2$ and there exist constants M, N_x, and N_y such that*

$$|f(x, y)| \leqslant M, \quad \left|\frac{\partial f(x, y)}{\partial x}\right| \leqslant N_x, \quad \left|\frac{\partial f(x, y)}{\partial y}\right| \leqslant N_y$$

and a constant Y_3 such that for the third derivative of the exact solution y the inequality

$$|y'''(x)| \leqslant Y_3 \quad \text{for } x \in [a, b]$$

holds, then, for $h < 1/2L$,

$$|y(x_n) - s(x_n)| \leqslant Kh^2, \quad n = 0, 1, \ldots, N, \tag{16}$$

where

$$K = \frac{(1/12)Y_3 + [N_x + N_y(1+2M)]^2}{L}\left[e^{(4/3)L(b-a)} - 1\right].$$

Proof. Let

$$\varepsilon_n = y(x_n) - s_n, \quad n = 0, 1, \ldots, N.$$

By (8) and (12), this implies $A(c, c) = 0$ and $A'(c, c) = \frac{1}{2}$ for any c. Therefore, we write

$$\begin{aligned}
\varepsilon_{n+1} &= y(x_{n+1}) - s_{n+1} = y(x_{n+1}) - s_n - h[s'_n + A(s'_n, s'_{n+1})] \\
&= y(x_n) - s_n + y(x_{n+1}) - y(x_n) - \\
&\quad - h\left[s'_n + A(s'_n, s'_n) + A'(s'_n, s'_n)(s'_{n+1} - s'_n) + A''\big(s'_n, s'(\eta_n)\big)\frac{(s'_{n+1} - s'_n)^2}{2}\right] \\
&= \varepsilon_n + y(x_{n+1}) - y(x_n) - \frac{h}{2}(s'_{n+1} + s'_n) - hA''\big(s'_n, s'(\eta_n)\big)\frac{(s'_{n+1} - s'_n)^2}{2} \\
&= \varepsilon_n + \frac{h}{2}[y'(x_{n+1}) - s'_{n+1} + y'(x_n) - s'_n] + y(x_{n+1}) - y(x_n) - \\
&\quad - \frac{h}{2}[y'(x_{n+1}) + y'(x_n)] - hA''\big(s'_n, s'(\eta_n)\big)\frac{(s'_{n+1} - s'_n)^2}{2}.
\end{aligned}$$

We now investigate consecutive elements of the above expression.

Applying the Lipschitz condition (3), we get

$$(17) \qquad |y'(x_{n+1}) - s'_{n+1}| \leqslant L|\varepsilon_{n+1}| \quad \text{and} \quad |y'(x_n) - s'_n| \leqslant L|\varepsilon_n|.$$

Next, we have

$$(18) \qquad \left| y(x_{n+1}) - y(x_n) - \frac{h}{2}[y'(x_{n+1}) + y'(x_n)] \right| \leqslant \frac{1}{12} h^3 Y_3,$$

since the expression within the symbols of the modulus represents exactly the error which is produced by the trapezoidal rule integrating the function y' within the limits x_n, x_{n+1}, and which equals $-(1/12)h^3 y'''(\xi_n)$ (see [4], p. 127-128).

At last we have also

$$\begin{aligned}
s'_{n+1} - s'_n &= f(x_{n+1}, s_{n+1}) - f(x_n, s_n) \\
&= \frac{\partial f(\gamma_n, s(\zeta_n))}{\partial x} h + \frac{\partial f(\gamma_n, s(\zeta_n))}{\partial y}(s_{n+1} - s_n) \\
&= h\left\{\frac{\partial f(\gamma_n, s(\zeta_n))}{\partial x} + \frac{\partial f(\gamma_n, s(\zeta_n))}{\partial y}[s'_n + A(s'_n, s'_{n+1})]\right\}.
\end{aligned}$$

From (9) we get

$$|s'_n + A(s'_n, s'_{n+1})| \leqslant |s'_n| + \sqrt{1 + s'^2_n} \leqslant 1 + 2|s'_n| = 1 + 2|f(x_n, s_n)|.$$

Hence

$$(19) \qquad |s'_{n+1} - s'_n| \leqslant h[N_x + N_y(1 + 2M)].$$

Introducing

$$T = \frac{1}{12} Y_3 + [N_x + N_y(1+2M)]^2,$$

in view of (17), (18), (19), and (11) we obtain

$$|\varepsilon_{n+1}| \leqslant |\varepsilon_n| + \frac{Lh}{2}(|\varepsilon_{n+1}| + |\varepsilon_n|) + Th^3,$$

and hence

$$|\varepsilon_{n+1}|\left(1 - \frac{Lh}{2}\right) \leqslant |\varepsilon_n|\left(1 + \frac{Lh}{2}\right) + Th^3.$$

Since $0 < h < 1/2L$, we have

(20) $$\frac{3}{4} < 1 - \frac{Lh}{2} < 1.$$

Thereby, we obtain the recurrent estimate of the error at a point x_{n+1} by the error at x_n:

$$|\varepsilon_{n+1}| \leqslant |\varepsilon_n| \frac{1+Lh/2}{1-Lh/2} + \frac{T}{1-Lh/2} h^3.$$

By this inequality, it is easy to express the estimate of the error ε_n by the known quantities. Indeed, introducing

$$C = \frac{1+Lh/2}{1-Lh/2} \quad \text{and} \quad D = \frac{T}{1-Lh/2},$$

we have

$$|\varepsilon_{n+1}| \leqslant C|\varepsilon_n| + Dh^3.$$

Then, since $\varepsilon_0 = 0$, we obtain

$$|\varepsilon_1| \leqslant Dh^3,$$
$$|\varepsilon_2| \leqslant Dh^3(1+C),$$
$$|\varepsilon_3| \leqslant Dh^3(1+C+C^2),$$
. . . .
$$|\varepsilon_n| \leqslant Dh^3(1+C+C^2+\ldots+C^{n-1}) = Dh^3\frac{C^n-1}{C-1}.$$

Replacing C and D by suitable expressions, we get

$$|\varepsilon_n| \leqslant \frac{Th^3}{1-Lh/2} \frac{\left(\frac{1+Lh/2}{1-Lh/2}\right)^n - 1}{\frac{1+Lh/2}{1-Lh/2} - 1} = h^2 \frac{T}{L}\left[\left(\frac{1+Lh/2}{1-Lh/2}\right)^n - 1\right].$$

Therefore, in view of (20) and the fact that $1+u < e^u$ for $u > 0$, we can write

$$|\varepsilon_n| \leqslant h^2 \frac{T}{L}\left[\left(1+\frac{Lh}{1-Lh/2)}\right)^n - 1\right] \leqslant h^2 \frac{T}{L}\left[\left(1+\frac{4}{3}Lh\right)^n - 1\right]$$
$$= h^2 \frac{T}{L}\left[\left(1+\frac{(4/3)L(x_n-x_0)}{n}\right)^n - 1\right] \leqslant h^2 \frac{T}{L}\left[\exp\left\{\frac{4}{3}L(x_n-x_0)\right\} - 1\right].$$

This inequality proves the theorem.

The function s'' is undetermined at the knots x_n, $n = 1, 2, \ldots, N-1$. We now define it at these points as follows:

$$(21)\quad s''(x_n) = \frac{1}{2}[\lim_{x\to x_n-} s''(x) + \lim_{x\to x_n+} s''(x)], \qquad n = 1, 2, \ldots, N-1.$$

THEOREM 5. *If the assumptions of Theorem 4 are satisfied, then for all* $h < 1/2L$ *the inequalities*

$$|y(x)-s(x)| \leqslant K_0 h^2, \qquad |y'(x)-s'(x)| \leqslant K_1 h^2,$$
$$|y''(x)-s''(x)| \leqslant K_2 h$$

hold for $x \in [a, b]$, *where*

$$K_2 = Y_3 + 2LK + 3M(1+M^2)[N_x + N_y(1+2M)]^2,$$
$$K_0 = \frac{6K + K_2/L}{4}, \qquad K_1 = LK + K_2.$$

Proof. Let $n \in \{0, 1, \ldots, N-1\}$. The function s is formed in the interval $[x_n, x_{n+1}]$ by an arc lying either in the upper ($z = -1$) or in the lower ($z = 1$) part of a circle. Let r be the radius of this circle. Then, by (6) and (13), we obtain

$$\frac{1}{r} = \frac{1}{h}\left|\frac{s'_{n+1}}{\sqrt{1+s'^2_{n+1}}} - \frac{s'_n}{\sqrt{1+s'^2_n}}\right|$$
$$= \frac{1}{h}\,\frac{|s'_{n+1}\sqrt{1+s'^2_n} - s'_n\sqrt{1+s'^2_{n+1}}|}{\sqrt{1+s'^2_n}\sqrt{1+s'^2_{n+1}}}$$
$$= \frac{1}{h}\,\frac{|s'_{n+1}\sqrt{1+s'^2_n} - s'_n\sqrt{1+s'^2_{n+1}}|\left(\sqrt{1+s'^2_n} + \sqrt{1+s'^2_{n+1}}\right)}{\sqrt{1+s'^2_n}\sqrt{1+s'^2_{n+1}}\left(\sqrt{1+s'^2_n} + \sqrt{1+s'^2_{n+1}}\right)}$$
$$\leqslant \frac{1}{h}\,\frac{|s'_{n+1}(1+s'^2_n) + s'_{n+1}\sqrt{1+s'^2_n}\sqrt{1+s'^2_{n+1}} - s'_n\sqrt{1+s'^2_n}\sqrt{1+s'^2_{n+1}} - s'_n(1+s'^2_{n+1})|}{2\sqrt{1+s'^2_n}\sqrt{1+s'^2_{n+1}}}$$
$$\leqslant \frac{1}{h}\,\frac{|s'_{n+1}-s'_n|\left(1+|s'_n s'_{n+1}| + \sqrt{1+s'^2_n}\sqrt{1+s'^2_{n+1}}\right)}{2\sqrt{1+s'^2_n}\sqrt{1+s'^2_{n+1}}} \leqslant \frac{|s'_{n+1}-s'_n|}{h}.$$

Therefore, if $x, x^* \in (x_n, x_{n+1})$, then, in view of the known relation

$$s''(x) = \frac{1}{zr}\{\sqrt{1+[s'(x)]^2}\}^3$$

and the inequalities

$$|s'(x)| \leqslant \max\{|s'_n|, |s'_{n+1}|\} \leqslant M, \qquad |s'(x)-s'(x^*)| \leqslant |s'_{n+1}-s'_n|,$$

arising from the convexity of a circle, we can write

$$\begin{aligned}|s''(x)-s''(x^*)| &= \frac{1}{r}\left|\{\sqrt{1+[s'(x)]^2}\}^3-\{\sqrt{1+[s'(x^*)]^2}\}^3\right|\\ &\leqslant \frac{|s'_{n+1}-s'_n|}{h}\left|\sqrt{1+[s'(x)]^2}-\sqrt{1+[s'(x^*)]^2}\right|\{1+[s'(x)]^2+\\ &\quad+\sqrt{1+[s'(x)]^2}\sqrt{1+[s'(x^*)]^2}+1+[s'(x^*)]^2\}\\ &\leqslant 3(1+M^2)\frac{|s'_{n+1}-s'_n|}{h}\frac{|[s'(x)]^2-[s'(x^*)]^2|}{\sqrt{1+[s'(x)]^2}+\sqrt{1+[s'(x^*)]^2}}\\ &\leqslant 3(1+M^2)\frac{|s'_{n+1}-s'_n|}{2h}|s'(x)-s'(x^*)|\,|s'(x)+s'(x^*)|\\ &\leqslant 3M(1+M^2)\frac{|s'_{n+1}-s'_n|^2}{h}.\end{aligned}$$

Hence, by (19),

$$|s''(x)-s''(x^*)| \leqslant 3M(1+M^2)[N_x+N_y(1+2M)]^2h. \tag{22}$$

From the Lipschitz condition (3) and estimate (16) we obtain

$$|y'(x_n)-s'(x_n)| = |f(x_n, y(x_n))-f(x_n, s(x_n))| \leqslant LKh^2, \qquad n = 0, 1, \ldots, N. \tag{23}$$

By the mean-value theorem, we have

$$y'(x_{n+1}) = y'(x_n)+hy''(\xi_{1,n}), \qquad x_n < \xi_{1,n} < x_{n+1},$$
$$s'(x_{n+1}) = s'(x_n)+hs''(\xi_{2,n}), \qquad x_n < \xi_{2,n} < x_{n+1}.$$

Combining these identities and (23), we get

$$|y''(\xi_{1,n})-s''(\xi_{2,n})| \leqslant \frac{1}{h}[|y'(x_n)-s'(x_n)|+|y'(x_{n+1})-s'(x_{n+1})|] \leqslant 2LKh.$$

Therefore, by (22),

$$\begin{aligned}|y''(x)-s''(x)| &\leqslant |y''(x)-y''(\xi_{1,n})|+|y''(\xi_{1,n})-s''(\xi_{2,n})|+|s''(\xi_{2,n})-s''(x)|\\ &\leqslant \{Y_3+2LK+3M(1+M^2)[N_x+N_y(1+2M)]^2\}h = K_2h\end{aligned} \tag{24}$$

for $x \in (x_n, x_{n+1})$.

By (21) this inequality holds also at the points x_n, $n = 0, 1, \ldots, N$. This establishes the third part of the thesis.

If $x \in [x_n, x_{n+1}]$, then, by Taylor's theorem,

$$|y(x)-s(x)| \leqslant |y(x_n)-s(x_n)|+h|y'(x_n)-s'(x_n)|+\frac{h^2}{2!}|y''(\eta_{1,n})-s''(\eta_{1,n})|$$

and

$$|y'(x)-s'(x)| \leqslant |y'(x_n)-s'(x_n)|+h|y''(\eta_{2,n})-s''(\eta_{2,n})|,$$

where $\eta_{1,n}, \eta_{2,n} \in (x_n, x_{n+1})$. Therefore, by (16), (23), (24), and the inequality $h < 1/2L$, we finally obtain

$$|y(x)-s(x)| \leqslant Kh^2+LKh^3+\frac{1}{2}K_2h^3 \leqslant \frac{1}{4}\left(6K+\frac{K_2}{L}\right)h^2 = K_0h^2$$

and

$$|y'(x)-s'(x)| \leqslant LKh^2+K_2h^2 = (LK+K_2)h^2 = K_1h^2.$$

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APROKSYMACJA FUNKCJAMI SKLEJANYMI Z ŁUKÓW KOŁOWYCH W ROZWIĄZYWANIU RÓWNAŃ RÓŻNICZKOWYCH ZWYCZAJNYCH

STRESZCZENIE

W pracy przedstawiono metodę konstrukcji funkcji sklejanej z łuków kołowych, aproksymującej rozwiązanie zagadnienia początkowego dla równań różniczkowych zwyczajnych. Otrzymane rozwiązanie przybliżone jest klasy C^1. Metoda opisana jest za pomocą zamkniętego nieliniowego schematu różnicowego. Jest to metoda jednokrokowa rzędu drugiego.
