ZASTOSOWANIA MATEMATYKI
APPLICATIONES MATHEMATICAE
XV, 1 (1976)

R. SMARZEWSKI (Lublin)

A METHOD FOR SOLVING THE VOLTERRA INTEGRAL EQUATION OF THE FIRST KIND

1. Introduction. Let us consider the linear Volterra integral equation of the first kind

$$g(t) = \int_0^t k(t, s) y(s) ds, \quad 0 \leqslant s \leqslant t \leqslant T < \infty, \tag{1}$$

where

(i) $g(t) \in C^{(1)}\langle 0, T\rangle$,

(ii) $k(t, s)$ and $\partial k(t, s)/\partial t$ are continuous for $0 \leqslant s \leqslant t \leqslant T$,

(iii) $g(0) = 0$,

(iv) $k(t, t) \neq 0$.

For a natural number I, let us write $t_i = ih$ $(i = 0, 1, \ldots, I)$, $h = T/I$ and $t_{ij} = t_i + u_j h$ $(j = 1, 2, \ldots, n;\ i = 0, 1, \ldots, I-1)$. Moreover, let

$$L_k(t) = \frac{w(t)}{(t-u_k) w'(u_k)}, \ k = 1, 2, \ldots, n, \quad \text{where } w(t) = \prod_{i=1}^{n} (t-u_i). \tag{2}$$

Supposing that $0 < u_1 < \ldots < u_n = 1$ and using

$$g(t_{ij}) = \int_0^{t_i} k(t_{ij}, s) y(s) ds + \int_{t_i}^{t_{ij}} k(t_{ij}, s) y(s) ds, \tag{3}$$

Hoog and Weiss [4] have given two finite-difference methods to solve equation (1), namely

$$g(t_{ij}) = h\Big[\sum_{l=0}^{i-1} \sum_{k=1}^{n} a_k k(t_{ij}, t_{lk}) Y_{lk} + \sum_{k=1}^{n} a_{jk} k(t_{ij}, t_{ik}) Y_{ik}\Big], \tag{4}$$

$$g(t_{ij}) = h\Big[\sum_{l=0}^{i-1} \sum_{k=1}^{n} a_k k(t_{ij}, t_{lk}) Y_{lk} + u_j \sum_{k=1}^{n} Y_{ik} \sum_{m=1}^{n} a_m k(t_{ij}, t_i + u_j u_m h) L_k(u_j u_m)\Big], \tag{5}$$

$$j = 1, 2, \ldots, n, \ i = 0, 1, \ldots, I-1,$$

where Y_{ij} is an approximate value of $y(t_{ij})$ and

$$a_{jk} = \int_0^{u_j} L_k(s)\,ds, \qquad j, k = 1, 2, \ldots, n,$$

(6) $$a_k = \int_0^1 L_k(s)\,ds, \qquad k = 1, 2, \ldots, n.$$

Both these methods are numerically stable and their order of convergence is equal to n.

In the present paper we show that the difference methods (4) and (5) give the optimal accuracy when the points u_i, $i = 1, 2, \ldots, n-1$, are zeros of Jacobi's polynomial. Moreover, we propose a simple method to evaluate a_k and a_{jk}, guaranteeing simultaneously a high level of accuracy.

2. Numerical method. Let n be a fixed integer not less than 3. Suppose that u_i $(i = 1, 2, \ldots, n-1)$, $u_1 < u_2 < \ldots < u_{n-1}$, are the zeros of the Jacobi polynomial (orthogonal with respect to the weight function $\varrho()t = 1-t$ in the interval $\langle 0, 1\rangle$) of degree $n-1$ with the coefficient at x^{n-1} equal to 1. It is known that these polynomials are determined by the Rodrigues formula (see [3])

(7) $$P_n(t) = (-1)^n \frac{1}{(2n+1)^{(n)}(1-t)} \frac{d^n}{dt^n}[t^n(1-t)^{n+1}],$$

where

$$n^{(j)} = \begin{cases} 1, & j = 0, \\ n(n-1)\ldots(n-j+1), & j = 1, 2, \ldots, n. \end{cases}$$

Applying to (7) the Leibniz differentiation formula, we obtain

(8) $$P_n(t) = t^n + \sum_{j=1}^{n} (-1)^j \binom{n}{j} \frac{n^{(j)}}{(2n+1)^{(j)}} t^{n-j}.$$

For $P_n(t)$ the recurrence formula (see [3])

(9) $$\begin{aligned} &P_0(t) = 1, \quad P_1(t) = t - \frac{1}{3}, \\ &P_{n+2}(t) = \left[t - \frac{(n+2)^2}{2n+5} + \frac{(n+1)^2}{2n+3}\right] P_{n+1}(t) - \frac{(n+1)(n+2)}{4(2n+3)^2} P_n(t), \\ &\qquad n = 0, 1, \ldots, \end{aligned}$$

holds. Hence, proceeding by induction, we have

$$P_n(1) = \frac{(n+1)!}{(2n+1)^{(n)}}, \qquad n = 0, 1, \ldots$$

Applying formula (9) and the identity

$$\int_0^1 P_n(t)\,dt = \int_0^1 tP_n(t)\,dt, \qquad n \geqslant 1,$$

resulting from the orthogonality of the functions $P_n(t)$ and $P_0(t)$, we obtain by induction

$$\int_0^1 P_n(t)\,dt = \frac{n!}{(2n+1)^{(n)}(n+1)}, \qquad n = 0, 1, \ldots \tag{10}$$

Let us denote by A_k ($k = 1, 2, \ldots, n-1$) the coefficients of the Gauss-Jacobi quadrature formula in the interval $\langle 0, 1\rangle$ with the weight function $\varrho(t) = 1-t$ such that

$$\int_0^1 (1-t)f(t)\,dt = \sum_{k=1}^{n-1} A_k f(u_k) + R_{n-1}(f).$$

It is known [3] that

$$A_k = \int_0^1 \frac{(1-t)P_{n-1}(t)\,dt}{P'_{n-1}(u_k)(t-u_k)} = \left[\frac{(n-1)!}{(2n-1)^{(n-1)}P'_n(u_k)}\right]^2 \frac{1}{u_k(1-u_k)}, \tag{11}$$

$$k = 1, 2, \ldots, n-1,$$

and that there exists a value η, $0 < \eta < 1$, such that

$$R_{n-1}(f) = K_n f^{(2n-2)}(\eta), \qquad \text{where } K_n = \frac{n[(n-1)!]^4}{(4n-2)[(2n-2)!]^2}.$$

Analogously as the Bouzitat quadrature formulae are obtained in [2] (p. 102), we can calculate from (2), (6) and (11)

$$a_k = \frac{A_k}{1-u_k}, \qquad k = 1, 2, \ldots, n-1,$$

and from (10)

$$a_n = \frac{n!}{(2n-1)^{(n)}(n+1)}.$$

Obviously, a_k are the coefficients of the following Bouzitat quadrature formula:

$$\int_0^1 f(t)\,dt = \sum_{i=1}^{n-1} a_i f(u_i) + a_n f(1) + B_{n-1}(f). \tag{12}$$

From formulae (7) and (8), integrating n times by parts, we obtain

$$\int_0^1 (1-t)[P_n(t)]^2 dt = (-1)^n \frac{1}{(2n+1)^{(n)}} \int_0^1 P_n(t) \frac{d^n}{dt^n} [t^n(1-t)^{n+1}] dt$$
$$= \frac{n!}{(2n+1)^{(n)}} \int_0^1 t^n(1-t)^{n+1} dt = \frac{1}{2}\left[\frac{n!}{(2n+1)^{(n)}}\right]^2.$$

Hence and from [2], p. 105, we conclude that

(13)
$$B_{n-1}(f) = \frac{1}{2(2n-1)!}\left[\frac{(n-1)!}{(2n-1)^{(n-1)}}\right]^2 f^{(2n-1)}(\xi), \quad \text{where } 0<\xi<1.$$

It follows from [5], p. 122, that if we replace in each interval (t_l, t_{l+1}), $0 \leqslant l \leqslant i-1$, the first of integrals appearing in (3) by Bouzitat quadrature formula of form (12) with the remainder term given by formula (13) and we add the results, then the error of such a quadrature is given by the formula

$$E_{n-1}(f) = \frac{ih^{2n}}{4^n} B_{n-1}(f) = O(h^{2n}),$$

whereas in [4] the order of error is $O(h^n)$.

It is convenient to express the system of equations (4) in the form

(14) $$A_i y = b$$

and system (5) as

(15) $$B_i y = c,$$

where

$$i = 0, 1, \ldots, I-1, \quad b = [b_1, \ldots, b_n]^T, \quad y = [y_1, \ldots, y_n]^T,$$
$$c = [c_1, \ldots, c_n]^T, \quad A_i = [\tilde{a}_{jk}]_{1\leqslant j,k\leqslant n}, \quad B_i = [b_{jk}]_{1\leqslant j,k\leqslant n},$$
$$b_j = \frac{g(t_{ij})}{h} - \sum_{l=0}^{i-1}\sum_{k=1}^{n} a_k k(t_{ij}, t_{lk}) Y_{lk}, \quad j = 1, 2, \ldots, n,$$
$$\tilde{a}_{jk} = a_{jk} k(t_{ij}, t_{ik}), \quad 1 \leqslant k, j \leqslant n, \quad c_j = b_j/u_j, 1 \leqslant j \leqslant n,$$
$$b_{jk} = \sum_{m=1}^{n} a_m k(t_{ij}, t_i + u_j u_m h) L_k(u_j u_m), \quad 1 \leqslant j, k \leqslant n.$$

The described choice of the points u_i guarantees that the vectors b and c appearing in (14) and (15) are calculated with the maximal order of accuracy when compared with all other methods of classes (4) and (5), i.e. $O(h^{2n})$.

The coefficients a_{jk} for $j = 1, 2, \ldots, n-1$ and $k = 1, 2, \ldots, n$ are proposed to be calculated from the formula (see [4])

$$a_{jk} = u_j \sum_{m=1}^{n} a_m L_k(u_j u_m).$$

We note that the coefficients a_{jk} and a_k for a fixed n should be calculated only once, since they do not depend upon the functions $g(t)$ and $k(t, s)$ appearing in (1). For that reason the method (4) is practically more useful because it shortens much the time of calculation, when compared with (5).

3. Numerical examples. For solving the systems of n linear equations (14) and (15) there was used a method of Gauss-Jordan ([5], p. 409) which was developed as an ALGOL-60 procedure *sleGJ* published in [1]. The values u_k and A_k needed to calculate both a_{jk} and the coefficients of system (15) were taken from [3], where they are given for $n = 2, 3, \ldots$ $\ldots, 15$ with fifteen exact decimal places after the point. The program was written for methods (4) and (5) and calculations were done for two equations of type (1) satisfying assumptions (i)-(iv), namely

$$-1+t+\exp(-t) = \int_0^t (1+t-s)y(s)ds, \quad 0 \leqslant t \leqslant 20, \tag{16}$$

and

$$\sin t = \int_0^t \exp(t-s)y(s)ds, \quad 0 \leqslant t \leqslant 10, \tag{17}$$

which have, respectively, exact solutions

$$y(t) = t\exp(-t) \quad \text{and} \quad y(t) = \cos t - \sin t.$$

TABLE 1 ($y(t) = t\exp(-t)$)

t	Method (4)			Method (5)		
	$h = 0.5$, $n = 5$	$h = 1.0$, $n = 7$	$h = 4.0$, $n = 11$	$h = 0.5$, $n = 5$	$h = 1.0$, $n = 7$	$h = 4.0$, $n = 11$
4.0	$2.4_{10}-8$	$-2.8_{10}-8$	$4.0_{10}-8$	$-1.9_{10}-8$	$-1.9_{10}-9$	$-5.6_{10}-9$
8.0	$-2.1_{10}-8$	$-1.8_{10}-8$	$-1.0_{10}-8$	$5.6_{10}-9$	$-1.3_{10}-8$	$9.0_{10}-9$
12.0	$-1.6_{10}-8$	$5.8_{10}-8$	$-4.9_{10}-8$	$-1.8_{10}-8$	$-9.1_{10}-9$	$-2.9_{10}-9$
16.0	$-6.1_{10}-8$	$-4.1_{10}-8$	$3.9_{10}-9$	$1.3_{10}-8$	$-2.4_{10}-8$	$2.9_{10}-8$
20.0	$1.4_{10}-8$	$-2.0_{10}-7$	$-1.6_{10}-8$	$-6.5_{10}-8$	$4.1_{10}-8$	$3.5_{10}-9$
Time of calculation	228	134	52	317	295	304

The absolute errors of obtained approximate solutions of these equations and the time of calculation in seconds (the program was executed on the ODRA 1204 computer with 37 bits floating-point mantissa) are confronted in tables 1 and 2.

TABLE 2 ($y(t) = \cos t - \sin t$)

t	Method (4)			Method (5)		
	$h = 1.0$, $n = 4$	$h = 1.0$, $n = 5$	$h = 1.0$, $n = 6$	$h = 1,0$, $n = 4$	$h = 1.0$, $n = 5$	$h = 1.0$, $n = 6$
2.0	$-8.3_{10}-3$	$5.7_{10}-4$	$-7.4_{10}-6$	$1.4_{10}-3$	$-1.5_{10}-5$	$-3.8_{10}-6$
4.0	$-4.8_{10}-3$	$1.3_{10}-4$	$-3.8_{10}-5$	$-8.4_{10}-4$	$-6.8_{10}-5$	$2.2_{10}-6$
6.0	$4.4_{10}-3$	$-6.8_{10}-4$	$3.9_{10}-5$	$-7.4_{10}-4$	$7.2_{10}-5$	$1.9_{10}-6$
8.0	$-8.4_{10}-3$	$4.3_{10}-4$	$4.8_{10}-6$	$1.5_{10}-3$	$1.0_{10}-5$	$-6.9_{10}-6$
10.0	$2.6_{10}-3$	$3.3_{10}-4$	$-4.8_{10}-5$	$-4.8_{10}-4$	$-8.0_{10}-5$	$-1.0_{10}-8$
Time of calculation	30	43	55	46	72	111

Table 3 gives a comparison of the results obtained by using the modified method (4) and basing on the nodes (see [4])

$$u_1 = \frac{1}{3}, \quad u_2 = \frac{2}{3}, \quad u_3 = 1 \tag{18}$$

and

$$u_1 = \frac{1}{4}, \quad u_2 = \frac{1}{2}, \quad u_3 = \frac{3}{4}, \quad u_4 = 1. \tag{19}$$

In both cases the calculations were performed with coefficients a_{jk} the errors of which were not greater than 2^{-37}. Comparing the results from tables 1 and 2 with those given in table 3, one sees that a decrease of the

TABLE 3. Method (4)

Equation (16)			Equation (17)		
t	method from this paper	method (18)	t	method from this paper	method (19)
4.0	$1.1_{10}-6$	$1.2_{10}-6$	2.0	$-4.2_{10}-7$	$-4.4_{10}-7$
8.0	$3.0_{10}-8$	$-4.9_{10}-8$	4.0	$-1.7_{10}-7$	$-2.2_{10}-8$
12.0	$-1.4_{10}-8$	$1.4_{10}-7$	6.0	$-6.6_{10}-7$	$-1.1_{10}-6$
16.0	$4.7_{10}-8$	$-3.1_{10}-7$	8.0	$-1.5_{10}-6$	$-1.1_{10}-5$
20.0	$1.8_{10}-7$	$-2.8_{10}-7$	10.0	$1.5_{10}-6$	$4.6_{10}-7$
Time of calculation	1727	1726		1224	1222

length of step h far prolongs the calculation time. In addition, the solution depends essentially upon the rounding errors. Therefore, we propose to perform the calculations with fixed h and increasing n.

References

[1] *Algol procedures of the Odra 1204 computer*, Elwro, Wrocław 1970.
[2] A. Ghizzetti and A. Ossicini, *Quadrature formulae*, Akademie Verlag, Berlin 1970.
[3] E. N. Glonti (Э. Н. Глонти), *Таблицы корней и квадратурных коэффициентов полиномов Якоби*, Вычисл. Центр АН СССР, Москва 1971.
[4] F. de Hoog and R. Weiss, *On the solution of Volterra integral equations of the first kind*, Numer. Math. 1 (1973), p. 22-32.
[5] A. Ralston, *A first course in numerical analysis*, Polish edition, Warszawa 1971.

DEPT. OF NUMERICAL METHODS
M. CURIE-SKŁODOWSKA UNIVERSITY
20-031 LUBLIN

Received on 15. 7. 1974

R. SMARZEWSKI (Lublin)

PEWNA METODA ROZWIĄZYWANIA CAŁKOWEGO RÓWNANIA VOLTERRY PIERWSZEGO RODZAJU

STRESZCZENIE

W pracy omówiono dwie metody rozwiązywania liniowego całkowego równania Volterry pierwszego rodzaju, otrzymane z dwu klas metod (przedstawionych w [4]) przez przyjęcie dodatkowego założenia, że węzły u_i są zerami pewnego wielomianu Jacobiego. Metody te są łatwiejsze do zaprogramowania na maszynie cyfrowej oraz dokładniejsze od metod rozważanych w [4].