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CYCLIC SYSTEM WITH PREEMPTIVE PRIORITY

1. Introduction. Let us assume that $N+1$ machines working continuously and independently are served by a single repairman. Suppose that the working times of machines are independent random variables each of them having an exponential distribution with parameter λ and that the repair times are independent random variables having a common probability distribution with distribution function $F(x)$. An additional Poissonian input stream with arrival rate ν is introduced into this system. We assume that the system of $N+1$ machines has preemptive priority, i.e. the additional items are served only in periods when all machines are working, and if at the moment of failure of any machine an additional item is being served, then the machine service begins immediately and the additional item rejoins the queue. This item continues its interrupted service after the next end of the repairman's busy period. The service times of additional items are independent with common distribution function $H(x)$. We assume that the necessary expected values exist, $H(0+) = 0$ and we write

$$\frac{1}{\mu} = \int_0^\infty x dH(x).$$

The system of $N+1$ machines without the additional input stream has been investigated by Takács in [8], where, among others, the Laplace-Stieltjes transform $g_N^*(s)$ of the distribution function $G_N(x)$ of the repairman's busy period was found. It is given by the formula

$$(1)\qquad g_N^*(s) = \int_0^\infty e^{-sx} dG_N(x) = \frac{\sum_{j=0}^{N} \binom{N}{j} \prod_{k=0}^{j-1} \frac{1-f^*(s+k\lambda)}{f^*(s+k\lambda)}}{\sum_{j=0}^{N+1} \binom{N+1}{j} \prod_{k=0}^{j-1} \frac{1-f^*(s+k\lambda)}{f^*(s+k\lambda)}},$$

where the empty product is equal to one and

$$f^*(s) = \int_0^\infty e^{-sx} dF(x).$$

2. Problem. The system of additional items can be interpreted as a system in which the service channel can break down both during the service of additional item as well as in the period when there are no additional items in the system. The breakdown probability of the service channel in the time interval $[t, t+h)$ is equal to $(N+1)\lambda h + o(h)$ and the repair time of the service channel has the distribution function $G_N(x)$. Klimov (see [2], p. 85-102) has considered a system with Poissonian input stream, arbitrary distribution of service time and with a channel subject to breakdowns under the assumption that both the working time and the breakdown time of the service channel have arbitrary distribution functions. The generating function of the number of items being in the system immediately after moments of finishing services of items is also given there.

Here we are interested in the probability distribution of the states of the process $n(t)$ defined as the number of additional items being in the system at the moment t. Let $\{t_r\}$ denote successive moments of the beginning of the repairman's busy period in the system of machines and let $\{s_r\}$ denote successive moments of ending the additional items service. Assuming that the process $n(t)$ is stationary, we write

$$P_n^- = \Pr(n(t_r) = n), \quad P_n = \Pr(n(t) = n), \quad n = 0, 1, \ldots,$$
$$P_n^* = \Pr(n(s_r - 0) = n), \quad n = 1, 2, \ldots$$

In order to find the probability distributions $\{P_n\}$ and $\{P_n^-\}$ we introduce the new time variable t' contracting to points the repairman's busy periods in the system of machines. Let $n'(t')$ denote the number of additional items being in the system at the moment t' and let $\{t_r'\}$ denote the consecutive moments of batch arrivals of additional items to the system, items which have arrived at the first time variable t during the repairman's busy period in the system of machines. Let $\{s_r'\}$ be the consecutive moments of ends of additional items services at the time t'. If the process $n(t)$ is stationary, then the process $n'(t')$ is also stationary, and we write

$$Q_n^- = \Pr(n'(t_r' - 0) = n), \quad Q_n = \Pr(n'(t') = n), \quad n = 0, 1, \ldots,$$
$$Q_n^* = \Pr(n'(s_r' - 0) = n), \quad n = 1, 2, \ldots$$

It is easy to see that the following equalities are fulfilled:

$$Q_n^- = P_n^-, \; n = 0, 1, \ldots, \quad Q_n^* = P_n^*, \; n = 1, 2, \ldots \tag{2}$$

At a time t' the input stream is a mixture of a simple Poissonian input stream with arrival rate ν, a batched Poissonian input stream with arrival rate $(N+1)\lambda$ and the probability distribution $\{u_k\}$ of the number of items in a batch, where

$$u_k = \int_0^\infty \frac{(\nu x)^k}{k!} e^{-\nu x} dG_N(x), \quad k = 0, 1, \ldots$$

The generating function of the probability distribution $\{u_k\}$ is given by

$$u(s) = \sum_{k=0}^{\infty} u_k s^k = g_n^*(\nu - \nu s).$$

Omitting empty batches in the mixture we obtain a batched Poissonian input stream with arrival rate $\nu_1 = \nu + (N+1)\lambda(1-u_0)$ and the following probability distribution of the number of items in a batch:

$$a_k = \begin{cases} \dfrac{1}{\nu_1}\big((N+1)\lambda u_1 + \nu\big), & k = 1, \\ \dfrac{1}{\nu_1}(N+1)\lambda u_k, & k = 2, 3, \ldots \end{cases}$$

The generating function of this probability distribution is given by

$$a(s) = \sum_{k=1}^{\infty} a_k s^k = \frac{1}{\nu_1}\left|(N+1)\lambda\big(g_N^*(\nu - \nu s) - u_0\big) + \nu s\right|.$$

Denote by a the expected value of the number of items in a batch. It is easy to verify that

$$a = \frac{\nu}{\nu_1}\left(1 + (N+1)\lambda\frac{1}{\gamma}\right), \quad \text{where } \frac{1}{\gamma} = \int_0^\infty x dG_N(x).$$

In the considered system, the generating function of the stationary probability distribution $\{Q_n\}$ is given (see Gaver [2]) by

$$Q(s) = \sum_{n=0}^{\infty} Q_n s^n = \frac{(1-\varrho)(1-s)h^*\big(\nu_1(1-a(s))\big)}{h^*\big(\nu_1(1-a(s))\big) - s}, \tag{3}$$

where

$$\varrho = \frac{\nu_1 a}{\mu} \quad \text{and} \quad h^*(s) = \int_0^\infty e^{-sx} dH(x).$$

3. Probability distribution $\{P_n^-\}$.

THEOREM 1. *The stationary probability distributions $\{Q_n^-\}$ and $\{Q_n\}$, provided they exist, satisfy the equality*

$$Q_n^- = Q_n, \quad n = 0, 1, \ldots \tag{4}$$

This equality is intuitive (see also [6] and [7]), since the distances between the moments have an exponential distribution function. The exact proof can be obtained using the relations between $\{Q_n^-\}$, $\{Q_n\}$ and $\{Q_n^*\}$, given in the following two theorems:

THEOREM 2. *The stationary probability distributions* $\{Q_n^*\}$ *and* $\{Q_n\}$, *provided they exist, satisfy the relation*

$$(5)\qquad Q_n^* = \frac{\nu_1}{\mu(1-Q_0)}\sum_{i=0}^{n-1} A_{n-1-i}Q_i, \qquad n = 1, 2, \ldots,$$

where

$$1-Q_0 = \frac{\nu_1 a}{\mu}, \qquad A_k = \sum_{j=k+1}^{\infty} a_j, \; k = 0, 1, \ldots$$

This theorem is a generalization of Corollary 4 in [5], where a simple input stream is considered.

THEOREM 3. *The stationary probability distributions* $\{Q_n^-\}$, $\{Q_n\}$ *and* $\{Q_n^*\}$, *provided they exist, satisfy the relation*

$$(6)\qquad (N+1)\lambda\Big(Q_n^- - \sum_{i=0}^{n} u_{n-i}Q_i^-\Big) + \nu(Q_n - Q_{n-1}) + \mu(1-Q_0)(Q_n^* - Q_{n+1}^*) = 0,$$
$$n = 0, 1, \ldots,$$

where $Q_{-1} = Q_0^* = 0$.

This result can be found also by using the Kuczura method presented in [6] and [7].

Proof of Theorem 2. Here we use the method employed in [4] (or [5]). If, for $n'(t') > 0$, we denote by $Y'(t')$ the time necessary to finish an additional item service, which is going on at the moment t', then the stochastic process $[n'(t'), Y'(t')]$ is Markovian. We write

$$P(n, y) = \Pr\big(n'(t') = n, 0 < Y'(t') < y\big), \qquad n = 1, 2, \ldots$$

Considering the state of the process $[n'(t'), Y'(t')]$ at the moments $t'+h$ and t', using the known calculation technique and assuming $h \to 0$, we obtain the following system of differential equations:

$$(7)\qquad -\nu_1 Q_0 + \frac{\partial}{\partial y}P(1, 0) = 0,$$

$$(8)\qquad \frac{\partial}{\partial y}P(n, y) - \frac{\partial}{\partial y}P(n, 0) - \nu_1 P(n, y) + \nu_1\sum_{i=1}^{n-1} a_{n-i}P(i, y) +$$
$$+\nu_1 a_n H(y)Q_0 + H(y)\frac{\partial}{\partial y}P(n+1, 0) = 0, \qquad n = 1, 2, \ldots$$

Hence, for $y \to \infty$, we obtain

$$\frac{\partial}{\partial y} P(1, 0) = \nu_1 Q_0,$$

$$-\frac{\partial}{\partial y} P(n, 0) - \nu_1 Q_n + \nu_1 \sum_{i=0}^{n-1} a_{n-i} Q_i + \frac{\partial}{\partial y} P(n+1, 0) = 0, \qquad n = 1, 2, \ldots,$$

or, equivalently,

$$\frac{\partial}{\partial y} P(n, 0) = \nu_1 \sum_{i=0}^{n-1} A_{n-1-i} Q_i, \qquad n = 1, 2, \ldots \tag{9}$$

Let

$$P(y) = \Pr\big(0 < Y'(t') < y\big) = \sum_{n=1}^{\infty} P(n, y).$$

After the sidewise summation of (8) we obtain

$$\frac{\partial}{\partial y} P(y) - \frac{\partial}{\partial y} P(0)\big(1 - H(y)\big) + \left[\nu_1 Q_0 - \frac{\partial}{\partial y} P(1, 0)\right] H(y) = 0.$$

From equation (7) we infer that the expression in the square brackets equals zero, therefore, we have the differential equation

$$\frac{\partial}{\partial y} P(y) = \frac{\partial}{\partial y} P(0)\big(1 - H(y)\big), \qquad \text{where } P(0+) = 0,\ P(\infty) = 1 - Q_0.$$

Thus

$$P(y) = \mu(1 - Q_0) \int_0^y \big(1 - H(u)\big)\, du.$$

It follows

(10)

$$\begin{aligned} \frac{\partial}{\partial y} P(n, 0) &= \lim_{h \to 0} \frac{1}{h} \Pr\big(n'(t') = n, 0 < Y'(t') < h\big) \\ &= \lim_{h \to 0} \frac{1}{h} \Pr\big(0 < Y'(t') < h\big) \Pr\big(n'(t') = n \mid 0 < Y'(t') < h\big) \\ &= \mu(1 - Q_0) Q_n^*, \qquad n = 1, 2, \ldots \end{aligned}$$

From (10) and (9), relation (5) follows. This completes the proof of Theorem 2.

Proof of Theorem 3. The method of the proof is the same as previously. Denote by $X'(t')$ the time from the moment t' to the nearest batch arrival and, for $n'(t') > 0$, denote by $Y'(t')$ the time necessary to end an additional item service which is going on at the moment t'. The process $[n'(t'), X'(t'), Y'(t')]$ is Markovian, since the distances between consecutive batch arrivals have an exponential distribution function and the process $[n'(t'), Y'(t')]$ is Markovian. We write

$$P(0, x) = \Pr\{n'(t') = 0, X'(t') < x\},$$

$$P(n, x, y) = \Pr\{n'(t') = n, X'(t') < x, \ 0 < Y'(t') < y\}, \qquad n = 1, 2, \ldots$$

Considering the state of the process $[n'(t'), X'(t'), Y'(t')]$ at the moments $t'+h$ and t', we obtain, for $h \to 0$, the system of differential equations

$$(11) \qquad \frac{\partial}{\partial x}P(0, x) - \frac{\partial}{\partial x}P(0, 0) - \nu P(0, x) + (1 - e^{-(N+1)\lambda x})u_0\frac{\partial}{\partial x}P(0, 0) + \frac{\partial}{\partial y}P(1, x, 0) = 0,$$

$$(12) \qquad \frac{\partial}{\partial x}P(n, x, y) + \frac{\partial}{\partial y}P(n, x, y) - \frac{\partial}{\partial x}P(n, 0, y) - \frac{\partial}{\partial y}P(n, x, 0) - \nu P(n, x, y) + (1 - e^{-(N+1)\lambda x})\sum_{1=i}^{n} u_{n-i}\frac{\partial}{\partial x}P(i, 0, y) + (1 - e^{-(N+1)\lambda x})u_n H(y)\frac{\partial}{\partial x}P(0, 0) + \nu P(n-1, x, y) + H(y)\frac{\partial}{\partial y}P(n+1, x, 0) = 0, \qquad n = 1, 2, \ldots,$$

where $P(0, x, y) = P(0, x)H(y)$.

From this, for $x \to \infty$, we obtain

$$(13) \qquad -\frac{\partial}{\partial x}P(0, 0) - \nu Q_0 + u_0\frac{\partial}{\partial x}P(0, 0) + \frac{\partial}{\partial y}P(1, \infty, 0) = 0,$$

$$(14) \qquad \frac{\partial}{\partial y}P(n, \infty, y) - \frac{\partial}{\partial x}P(n, 0, y) - \frac{\partial}{\partial y}P(n, \infty, 0) - \nu P(n, \infty, y) + \sum_{i=1}^{n} u_{n-i}\frac{\partial}{\partial x}P(i, 0, y) + u_n H(y)\frac{\partial}{\partial x}P(0, 0) + \nu P(n-1, \infty, y) + H(y)\frac{\partial}{\partial y}P(n+1, \infty, 0) = 0, \qquad n = 1, 2, \ldots,$$

where $P(0, \infty, y) = H(y)Q_0$. Hence, for $y \to \infty$, we obtain

$$(15) \qquad -\frac{\partial}{\partial x}P(n, 0, \infty) - \frac{\partial}{\partial y}P(n, \infty, 0) - \nu Q_n + \sum_{i=0}^{n} u_{n-i}\frac{\partial}{\partial x}P(i, 0, \infty) + \nu Q_{n-1} + \frac{\partial}{\partial y}P(n+1, \infty, 0) = 0, \qquad n = 0, 1, \ldots,$$

where

$$Q_{-1} = 0 = \frac{\partial}{\partial y} P(0, \infty, 0), \qquad \frac{\partial}{\partial x} P(0, 0, \infty) = \frac{\partial}{\partial x} P(0, 0).$$

Let

$$P(y) = \Pr\{0 < Y'(t') < y\} = \sum_{n=1}^{\infty} P(n, \infty, y).$$

After the sidewise summation of (14) we obtain

$$\frac{\partial}{\partial y} P(y) - \frac{\partial}{\partial y} P(0)\{1 - H(y)\} + \\ + \left[(1-u_0)\frac{\partial}{\partial x} P(0, 0) + \nu Q_0 - \frac{\partial}{\partial y} P(1, \infty, 0)\right] H(y) = 0.$$

From (13) it follows that the expression in the square brackets equals zero. Obviously, we have $P(0+) = 0$ and $P(\infty) = 1 - Q_0$. Hence

$$P(y) = \mu(1-Q_0)\int_0^y \{1 - H(u)\} du. \tag{16}$$

It is known from the renewal theory that, for the stationary process $X'(t')$, we have

$$\Pr\{X'(t') < x\} = (N+1)\lambda \int_0^x e^{-(N+1)\lambda u} du.$$

From this and from (16) it follows

$$\begin{aligned} \frac{\partial}{\partial x} P(n, 0, \infty) &= \lim_{h\to 0} \frac{1}{h} \Pr\{n'(t') = n,\, X'(t') < h,\, 0 < Y'(t') < \infty\} \\ &= \lim_{h\to 0} \frac{1}{h} \Pr\{X'(t') < h,\, 0 < Y'(t') < \infty\} \times \\ &\quad \times \Pr\{n'(t') = n \mid X'(t') < h,\, 0 < Y(t') < \infty\} \\ &= (N+1)\lambda Q_n^-, \qquad n = 0, 1, \ldots, \\ \frac{\partial}{\partial y} P(n, \infty, 0) &= \lim_{h\to 0} \frac{1}{h} \Pr\{n'(t') = n,\, X'(t') < \infty,\, 0 < Y'(t') < h\} \\ &= \lim_{h\to 0} \frac{1}{h} \Pr\{X'(t') < \infty,\, 0 < Y'(t') < h\} \times \\ &\quad \times \Pr\{n'(t') = n \mid X'(t') < \infty,\, 0 < Y'(t') < h\} \\ &= \mu(1-Q_0)Q_n^*, \qquad n = 1, 2, \ldots \end{aligned} \tag{17}$$

Substituting (17) into (15), we obtain (6). This completes the proof of Theorem 3.

Proof of Theorem 1. From equalities (5) and (6), equality (4) follows. This completes the proof of Theorem 1.

Equalities (2) and (4) allow us to find the probability distribution $\{P_n^-\}$ of the number of additional items being in the system at the moments of the beginning of the repairman's busy period in the system of machines.

4. Probability distribution $\{P_n\}$. The following theorem gives the relation between the probability distributions $\{P_n\}$ and $\{P_n^*\}$:

THEOREM 4. *The stationary probability distributions $\{P_n\}$ and $\{P_n^*\}$, provided they exist, satisfy the relation*

$$P_{n+1}^* = P_n, \quad n = 0, 1, \ldots \tag{18}$$

If the service channel can break down only during the item service, then this equality is the corollary to the same equality for $M/G/1$ system in which the service channel is reliable (see [1]). In our system the service channel can break down also in the period when there are no additional items in the system.

Proof of Theorem 4. Define the stochastic process $m(t)$ by $m(t) = 1$ when at a moment t all machines are working and the additional item, if it is, can be served, and by $m(t) = 0$ when at a moment t a machine service is going. For $n(t) > 0$, let $Y(t)$ denote for $m(t) = 1$ the remaining time to end an additional item service and for $m(t) = 0$ let $Y(t)$ denote the time necessary to end an additional item service from the moment of its renewal after the moment t. If $m(t) = 0$, let $Z(t)$ denote the remaining time from the moment t to the end of the repairman's busy period in the system of machines. Then the stochastic process

$$X(t) = \begin{cases} [n(t), m(t), Y(t), Z(t)] & \text{if } n(t) > 0 \text{ and } m(t) = 0, \\ [n(t), m(t), Z(t)] & \text{if } n(t) = 0 \text{ and } m(t) = 0, \\ [n(t), m(t), Y(t)] & \text{if } n(t) > 0 \text{ and } m(t) = 1, \\ [n(t), m(t)] & \text{if } n(t) = 0 \text{ and } m(t) = 1 \end{cases}$$

is Markovian. We introduce the following notation:

$$P(n, 0, y, z) = \Pr\big(n(t) = n, m(t) = 0, 0 < Y(t) < y, Z(t) < z\big), \quad n = 1, 2, \ldots,$$

$$P(0, 0, z) = \Pr\big(n(t) = 0, m(t) = 0, Z(t) < z\big),$$

$$P(n, 1, y) = \Pr\big(n(t) = n, m(t) = 1, 0 < Y(t) < y\big), \quad n = 1, 2, \ldots,$$

$$P(0, 1) = \Pr\big(n(t) = 0, m(t) = 1\big).$$

Considering the state of the process $X(t)$ at the moments $t+h$ and t, we obtain, for $h\to 0$, the following system of differential equations:

$$\frac{\partial}{\partial z}P(0,0,z)-\frac{\partial}{\partial z}P(0,0,0)-\nu P(0,0,z)+(N+1)\lambda P(0,1)G_N(z)=0, \tag{19}$$

$$-\big(\nu+(N+1)\lambda\big)P(0,1)+\frac{\partial}{\partial y}P(1,1,0)+\frac{\partial}{\partial z}P(0,0,0)=0, \tag{19'}$$

$$\begin{aligned}\frac{\partial}{\partial z}P(1,0,y,z)-\frac{\partial}{\partial z}P(1,0,y,0)-\nu P(1,0,y,z)+\\ +\nu H(y)P(0,0,z)+(N+1)\lambda G_N(z)P(1,1,y)=0,\end{aligned} \tag{20}$$

$$\begin{aligned}\frac{\partial}{\partial z}P(n,0,y,z)-\frac{\partial}{\partial z}P(n,0,y,0)-\nu P(n,0,y,z)+\\ +\nu P(n-1,0,y,z)+(N+1)\lambda G_N(z)P(n,1,y)=0, \quad n=2,3,\dots,\end{aligned} \tag{20'}$$

$$\begin{aligned}\frac{\partial}{\partial y}P(1,1,y)-\frac{\partial}{\partial y}P(1,1,0)-\nu P(1,1,y)+\nu H(y)P(0,1)-\\ -(N+1)\lambda P(1,1,y)+H(y)\frac{\partial}{\partial y}P(2,1,0)+\frac{\partial}{\partial z}P(1,0,y,0)=0,\end{aligned} \tag{20''}$$

$$\begin{aligned}\frac{\partial}{\partial y}P(n,1,y)-\frac{\partial}{\partial y}P(n,1,0)-\nu P(n,1,y)+\nu P(n-1,1,y)-\\ -(N+1)\lambda P(n,1,y)+H(y)\frac{\partial}{\partial y}P(n+1,1,0)+\frac{\partial}{\partial z}P(n,0,y,0)=0,\\ n=2,3,\dots\end{aligned} \tag{20'''}$$

Taking the limits for $y\to\infty$ and $z\to\infty$, we obtain

$$-\frac{\partial}{\partial z}P(0,0,0)-\nu P(0,0,\infty)+(N+1)\lambda P(0,1)=0, \tag{21}$$

$$-\big(\nu+(N+1)\lambda\big)P(0,1)+\frac{\partial}{\partial y}P(1,1,0)+\frac{\partial}{\partial z}P(0,0,0)=0, \tag{21'}$$

$$\begin{aligned}-\frac{\partial}{\partial z}P(n,0,\infty,0)-\nu P(n,0,\infty,\infty)+\nu P(n-1,0,\infty,\infty)+\\ +(N+1)\lambda P(n,1,\infty)=0, \quad n=1,2,\dots,\end{aligned} \tag{22}$$

$$\begin{aligned}-\frac{\partial}{\partial y}P(n,1,0)-\nu P(n,1,\infty)+\nu P(n-1,1,\infty)-\\ -(N+1)\lambda P(n,1,\infty)+\frac{\partial}{\partial y}P(n+1,1,0)+\frac{\partial}{\partial z}P(n,0,\infty,0)=0,\\ n=1,2,\dots,\end{aligned} \tag{22'}$$

where $P(0, 0, \infty, \infty) = P(0, 0, \infty)$, and $P(0. 1, \infty) = P(0, 1)$. Obviously, the following equalities hold:

$$P_0 = P(0, 0, \infty) + P(0, 1),$$
$$P_n = P(n, 0, \infty, \infty) + P(n, 1, \infty), \quad n = 1, 2, \ldots$$

Hence and from (21), (21′), (22) and (22′) we obtain

$$\frac{\partial}{\partial y} P(n+1, 1, 0) = \nu P_n, \quad n = 0, 1, \ldots \tag{23}$$

Let

$$p(1, y) = \Pr\big(m(t) = 1, 0 < Y(t) < y\big) = \sum_{n=1}^{\infty} P(n, 1, y).$$

Taking the limits in (20′) for $z \to \infty$ and adding sidewise the equations of the obtained system for $n = 1, 2, \ldots$, we have

$$\frac{\partial}{\partial y} p(1, y) - \frac{\partial}{\partial y} p(1, 0)\big(1 - H(y)\big) + \left[\nu P_0 - \frac{\partial}{\partial y} P(1, 1, 0)\right] H(y) = 0. \tag{24}$$

It follows from (23), for $n = 0$, that the expression in the square brackets equals zero. The solution of equation (24) with initial conditions

$$p(1, 0+) = 0$$

and

$$p(1, \infty) = \Pr\big(m(t) = 1, 0 < Y(t) < \infty\big)$$
$$= \Pr\big(m(t) = 1\big) - \Pr\big(n(t) = 0, m(t) = 1\big) = \frac{\gamma}{(N+1)\lambda + \gamma} - P(0, 1)$$

is given by the function

$$p(1, y) = \mu\left(\frac{\gamma}{(N+1)\lambda + \gamma} - P(0, 1)\right) \int_0^y \big(1 - H(u)\big) du.$$

Hence we have

$$\frac{\partial}{\partial y} P(n, 1, 0) = \lim_{h \to 0} \frac{1}{h} \Pr\big(n(t) = n, m(t) = 1, 0 < Y(t) < h\big) \tag{25}$$
$$= \lim_{h \to 0} \frac{1}{h} \Pr\big(m(t) = 1, 0 < Y(t) < h\big) \Pr\big(n(t) = n \mid m(t) = 1, 0 < Y(t) < h\big)$$
$$= \mu\left(\frac{\gamma}{(N+1)\lambda + \gamma} - P(0, 1)\right) P_n^*, \quad n = 1, 2, \ldots$$

Substituting (25) into (23) and dividing sidewise by the expression

$$\mu\left(\frac{\gamma}{(N+1)\lambda + \gamma} - P(0, 1)\right),$$

we obtain

$$(26)\qquad P^*_{n+1} = \frac{\nu\big((N+1)\lambda+\gamma\big)}{\mu\big(\gamma\big(1-P(0,1)\big)-(N+1)\lambda P(0,1)\big)}P_n, \qquad n = 0, 1, \ldots$$

Obviously,

$$\sum_{n=0}^{\infty} P^*_{n+1} = \sum_{n=0}^{\infty} P_n = 1.$$

Hence, from this and from (26) we obtain (18). This completes the proof of Theorem 4.

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SYSTEM CYKLICZNY Z PRIORYTETEM BEZWZGLĘDNYM

STRESZCZENIE

W pracy rozpatruje się system konserwacji maszyn, do którego wprowadza się dodatkowo poissonowski strumień zgłoszeń. Jednostki dodatkowe są obsługiwane tylko w okresach, gdy wszystkie maszyny pracują, a więc, jeżeli podczas obsługi jednostki zepsuje się dowolna maszyna, to natychmiast rozpoczyna się konserwacja

tej maszyny, a przerwana obsługa jednostki dodatkowej będzie kontynuowana po zakończeniu okresu zajętości konserwatora obsługą maszyn. Czasy obsługi jednostek są niezależnymi zmiennymi losowymi o jednakowym rozkładzie prawdopodobieństwa.

Dla tego systemu definiuje się proces $n(t)$ jako liczbę jednostek w systemie w chwili t oraz proces $n'(t')$ jako liczbę jednostek w systemie w chwili t', gdzie t' jest nową rachubą czasu — po ściągnięciu do punktów wszystkich okresów zajętości konserwatora obsługą maszyn. Przez $\{t_r\}$ oznacza się kolejne momenty rozpoczęcia okresów zajętości konserwatora obsługą maszyn, przez $\{s_r\}$ — kolejne momenty zakończenia obsługi jednostek w czasie t, przez $\{t_r'\}$ — kolejne momenty wejścia do systemu grup utworzonych przez te jednostki, które przybyły w czasie t w okresie zajętości konserwatora obsługą maszyn (z możliwością pustych grup), przez $\{s_r'\}$ zaś kolejne momenty zakończenia obsługi jednostek w czasie t'.

Główny rezultat pracy polega na znalezieniu związków między rozkładami prawdopodobieństwa stanu rozpatrywanych procesów i pewnych włożonych łańcuchów Markowa w warunkach stacjonarności, a mianowicie między rozkładami

$$P_n^- = \Pr(n(t_r) = n), \quad P_n = \Pr(n(t) = n), \quad P_{n+1}^* = \Pr(n(s_r - 0) = n+1),$$
$$Q_n^- = \Pr(n'(t_r' - 0) = n), \quad Q_n = \Pr(n'(t') = n), \quad Q_{n+1}^* = \Pr(n'(s_r' - 0) = n+1)$$

dla $n = 0, 1, \ldots$