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PURSUIT IN PHYSICAL SPACES

0. Introductory remarks. This paper is a continuation of the author's previous work ([4], [5], [6], [7]) on the theory of pursuit in the open Euclid's E_3 and Riemann's R_4 spaces. One assumes here that in the E_3 space there exists a potential field determined by the laws of classical physics. We assume also the existence of a gravitational field (satisfying Einstein's equations) and of an electromagnetic field (satisfying Maxwell's equations generalized to non-inertial systems) in the space-time R_4. Through some physical conditions (such as e.g. the principle of conservation of energy) these fields bound the strategy classes of both the pursuing and the evading objects. These fields influence also essentially the changes of the game value and of the optimum pursuit trajectories; Thus they influence those quantities which describe the physical condition of the given system and which simultaneously determine the course of the game. This description fully realizes the conflict aspect between the pursuing and evading objects since it is in conformity with the minimax principle.

The main purpose of the theory of pursuit is to provide a mathematical rule which would inform explicite both players about their optimum strategies and about the game value in the physical spaces E_3 and R_4. Such a rule may be defined mathematically by means of a canonical Hamilton-Jacobi formalism, since it turns out that the optimum pursuit trajectories may be determined from Hamilton's canonical equations, and the game value from a Hamilton-Jacobi equation with partial derivatives. Thus, the main problem is reduced to the construction of a suitable Hamiltonian, and the way of solution is that of applying canonical transformations. It turns out that the form of the Hamiltonian is more involved and different in the theory of pursuit than in theoretical mechanics.

The above sketched purpose of the paper has been completely achieved for two-person games. Of course, it may be enlarged in a natural manner to many-person games.

The reader is asked to observe that in the whole paper an index (tensor) notation is used and, in addition, that 1° a summation convention is used on repeated indices, 2° the indices i, k sum in the E_3 space from 1 to 3 and in the R_4 space from 1 to 4.

1. Notion and principles of a two-person pursuit game in a potential field. In Newton's classical mechanics the notion of potential force F is defined by the formula

$$F = -\operatorname{grad} V,$$

which in tensor notation has the form

$$F_i = -V_{\cdot i}. \tag{1}$$

The scalar field $V = V(x_k)$ which appears here is named potential of the force F_i or potential energy of a material point in the position x_k. The known principle of conservation of energy is satisfied in this field:

If a material point with mass m underlies the potential force (1), *then the sum of the kinetic energy $T = mv^2/2$ and the potential energy V is constant during the motion, thus for any moment t the following equality holds*:

$$\frac{m}{2} v_i v_i + V(x_k) = E = \text{const}.$$

This theorem may be formulated in an analogous manner for a system of N material points. We shall, however, consider in the sequel only systems of two pursuing objects which will be identified with two material points having the masses m and n. Therefore we shall restrict ourselves to the formulation of formulae for the principle of conservation of energy in two cases only, namely

a) If the two material points m and n do not influence each other and if in the exterior potential field they are situated in positions x_k and y_k, respectively, then for every material point holds

$$\begin{aligned} \frac{m}{2} u_i u_i + U(x_k) &= A = \text{const}, \\ \frac{n}{2} v_i v_i + V(y_k) &= B = \text{const}. \end{aligned} \tag{2}$$

b) If the two material points m and n are situated in an own potential field with energy of mutual influence $V(x_k, y_k)$ then during the motion holds

$$\frac{m}{2} u_i u_i + \frac{n}{2} v_i v_i + V(x_k, y_k) = E = \text{const} \tag{3}$$

for the mass system m and n.

Thus, case a) expresses the principle of conservation of energy for the system of two pursuing objects in an exterior potential field, and case b) expresses the same principle in an interior potential field.

Denote by E_3 the three-dimensional Euclidean space with either potential fields $U(x_k)$ and $V(y_k)$ or potential field $V(x_k, y_k)$. Now let us define the admissible strategy classes X and Y; these are classes of differentiable functions of two variables (the coordinates of the pursuer and of the evader) $x_i, y_i \epsilon E_3$, the values of which are restricted in E_3 either by the conditions following from (2)

$$\begin{aligned} mu_i u_i &= 2(A-U), \quad u_i \epsilon X, \\ nv_i v_i &= 2(B-V), \quad v_i \epsilon Y, \end{aligned} \tag{4}$$

or by the condition following from (3)

$$mu_i u_i + nv_i v_i = 2(E-V), \quad u_i \epsilon X, v_i \epsilon Y. \tag{5}$$

The equilibrium conditions of the conflict situation between the two players 1 and 2 in E_3 are included in the principle of optimality ([1] and [7]):

If $W(x_i, y_i)$ is a non-negative and differentiable function for $x_i, y_i \epsilon E_3$, and if $u_i^0 \epsilon X$ and $v_i^0 \epsilon Y$ are functions such that for every $u_i \epsilon X$ and $x_i, y_i \epsilon E_3$ holds

$$p_i u_i + q_i v_i^0 \geqslant -1, \tag{6a}$$

and for every $v_i \epsilon Y$ and $x_i, y_i \epsilon E_3$ holds

$$p_i u_i^0 + q_i v_i \leqslant -1, \tag{6b}$$

and for every $x_i \epsilon E_3$ holds $W(x_i, x_i) = 0$, then u_i^0 and v_i^0 are optimum strategies and $W(x_i, y_i)$ is the game value for the position x_i, y_i.

Monge's functions p_i, q_i are defined by $p_i = \partial W/\partial x_i$ and $q_i = \partial W/\partial y_i$, and the potential fields U and V are given scalar functions which on the ground of (4) and (5) bound the strategies in the classes X and Y. It is easy to see that from (6a, b) it follows immediately the important

CONCLUSION 1. *If the equality*

$$p_i u_i^0 + q_i v_i^0 = -1 \tag{6c}$$

holds, then the pursuit is a uniformly closed game in E_3.

Formula (6c) may be written in simplified form as $dW/dt = -1$, where the game value $W(x_i^0, y_i^0) = \tau$ determines in E_3 the pursuit time τ on the optimum trajectories $x_i^0(t)$ for the pursuer and $y_i^0(t)$ for the evader.

2. Canonical form of the pursuit equations. On the basis of (4), (5) and (6a)-(6c) the following fundamental theorem will be proved.

THEOREM 1. *If a pursuit game in a potential field is played according to both the principle of conservation of energy and the principle of optimality then the game value $W(x_i, y_i)$ satisfies the Hamilton-Jacobi equation*

$$H\left(x_i, y_i; \frac{\partial W}{\partial x_i}, \frac{\partial W}{\partial y_i}\right) = 1 \tag{7}$$

and the optimum pursuit trajectories satisfy the canonical Hamilton equations

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x_i}, \quad \frac{dy_i}{dt} = \frac{\partial H}{\partial q_i}, \quad \frac{dq_i}{dt} = -\frac{\partial H}{\partial y_i}, \quad (i = 1, 2, 3), \tag{8}$$

where the Hamiltonian of the system of two pursuing each other objects takes the form

a) *in an exterior potential field*

$$H = H(x_i, y_i; p_i, q_i) = \pm\sqrt{\frac{2}{m}(A-U)\, p_i p_i} \pm \sqrt{\frac{2}{n}(B-V) q_i q_i}, \tag{9a}$$

b) *in the potential field of mutual influence*

$$H = H(x_i, y_i; p_i, q_i) = (E-V)\left(\frac{2}{m} p_i p_i + \frac{2}{n} q_i q_i\right). \tag{9b}$$

Proof. From the principle of optimality follows that the linear from $p_i u_i + q_i v_i$ attains its extremum with respect to u_i and v_i for the optimum strategies u_i^0 for the pursuer and v_i^0 for the evader, respectively. Applying the method of Lagrange multipliers, we obtain

$$f(u_k, v_k; \lambda_1, \lambda_2) = p_i u_i + q_i v_i + \lambda_1 [m u_i u_i + 2(U-A)] + \\ + \lambda_2 [n v_i v_i + 2(V-B)],$$

$$\begin{aligned} \frac{\partial f}{\partial u_k} &= p_k + 2m\lambda_1 u_k^0 = 0, \\ \frac{\partial f}{\partial v_k} &= q_k + 2n\lambda_2 v_k^0 = 0 \quad (k = 1, 2, 3), \quad (df = 0), \\ \frac{\partial f}{\partial \lambda_1} &= m u_k^0 u_k^0 + 2(U-A) = 0, \\ \frac{\partial f}{\partial \lambda_2} &= n v_k^0 v_k^0 + 2(V-B) = 0. \end{aligned} \tag{10}$$

This system of 8 algebraic equations of 8 unknowns allows a reduction of the Lagrange multipliers λ_1 and λ_2, and permits an explicit determination of the components of the vectors u_k^0 and v_k^0 which in turn determine the optimum directions of pursuit and evasion in an exterior potential field. After a simple transformation we obtain from (10)

$$u_k^0 = -p_k/2m\lambda_1, \quad v_k^0 = -q_k/2n\lambda_2. \tag{11}$$

This and the two last scalar equations of (10) give

$$\lambda_1 = \pm\sqrt{\frac{p_k p_k}{8m(A-U)}}, \quad \lambda_2 = \pm\sqrt{\frac{q_k q_k}{8n(B-V)}}. \tag{12}$$

In this manner, using (11) and (12), we are able to express the optimum strategies of the game through the partial derivatives of $W(x_i, y_i)$, as follows:

$$u_k^0 = \mp\sqrt{\frac{2}{m}(A-U)}\cdot\frac{p_k}{\sqrt{p_i p_i}}, \quad v_k^0 = \mp\sqrt{\frac{2}{n}(B-V)}\cdot\frac{q_k}{\sqrt{q_i q_i}}. \tag{13}$$

From the conditions (4) bounding the admissible strategies u_i and v_i we obtain easily suitable expressions for the absolute values of the pursuer's velocity u and the evader's velocity v, as follows

$$u = \sqrt{\frac{2}{m}(A-U)}, \quad v = \sqrt{\frac{2}{n}(B-V)}.$$

Substitution into (13) gives

$$u_k^0 = \mp\frac{up_k}{\sqrt{p_i p_i}}, \quad v_k^0 = \mp\frac{vq_k}{\sqrt{q_i q_i}}.$$

Conclusion 1 and (13) lead, after some reductions, to the equation

$$\pm\sqrt{\frac{2}{m}(A-U)p_i p_i} \pm \sqrt{\frac{2}{n}(B-V)q_i q_i} = 1,$$

or to the equivalent one

$$\pm u\sqrt{p_i p_i} \pm v\sqrt{q_i q_i} = 1.$$

This is a differential equation with partial derivatives for the game value $W(x_k, y_k)$, thus a Hamilton-Jacobi equation of type (7); the left hand side of this equation determines the Hamiltonian (9a) in the exterior potential field.

An analogous way leads to the wanted result in the case of an interior potential field. In the Lagrange multiplier method condition (5) is taken instead of the conditions (4). We obtain then

$$
(14)\quad
\begin{aligned}
f(u_k, v_k; \lambda) &= p_i u_i + q_i v_i + \lambda[m u_i u_i + n v_i v_i + 2(V-E)],\\
\frac{\partial f}{\partial u_k} &= p_k + 2m\lambda u_k^0 = 0 \qquad (k = 1, 2, 3), \qquad (df = 0),\\
\frac{\partial f}{\partial v_k} &= q_k + 2n\lambda v_k^0 = 0,\\
\frac{\partial f}{\partial \lambda} &= m u_k^0 u_k^0 + n v_k^0 v_k^0 + 2(V-E) = 0.
\end{aligned}
$$

Solving this system of 7 algebraic equations we come to explicit expressions for the Lagrange multiplier λ and for the optimum strategies

$$
(15)\quad
\begin{aligned}
u_k^0 &= \mp \frac{\sqrt{2(E-V)}}{m} \cdot \frac{p_k}{\sqrt{p_i p_i/m + q_i q_i/n}},\\
v_k^0 &= \mp \frac{\sqrt{2(E-V)}}{n} \cdot \frac{q_k}{\sqrt{p_i p_i/m + q_i q_i/n}}
\end{aligned}
$$

in an potential field with energy of mutual influence $V(x_i, y_i)$.

Conclusion 1 and (15) give the following equation of first order partial derivatives

$$
2(E-V)(p_i p_i/m + q_i q_i/n) = 1
$$

determining the game value $W(x_k, y_k)$. This is a Hamilton-Jacobi equation of type (7), the left hand side of which is the Hamiltonian (9b) in an interior potential field.

From Cauchy's theory of integrating equations with partial derivatives it is known that the following system of characteristic equations

$$
(16)\quad \dot{x}_i = H_{p_i}, \quad \dot{p}_i = -H_{x_i} - p_i H_W, \quad \dot{y}_i = H_{q_i}, \quad \dot{q}_i = -H_{y_i} - q_i H_W,
$$

$$
(17)\quad \dot{W} = p_i H_{p_i} + q_i H_{q_i}
$$

corresponds to the differential equation (7). Since $H(x_k, y_k; p_k, q_k)$ does not explicite contain the wanted function $W(x_k, y_k)$, the equations (16) take a simpler form, as follows

$$
\dot{x}_i = H_{p_i}, \quad \dot{p}_i = -H_{x_i}, \quad \dot{y}_i = H_{q_i}, \quad \dot{q}_i = -H_{y_i}.
$$

Now, equation (17) expresses, according to Conclusion 1, the necessary and sufficient condition for the pursuit in the space E_3 to be a uniformly closed game. Thus, the characteristic equations are, in the case of (7), equivalent to the system of canonical Hamilton equations (8). This ends the proof.

From the proof of theorem 1 follows immediately

THEOREM 2. *If the pursuit in a potential field takes place according to the principle of conservation of energy and the principle of optimality then the optimum pursuing and evading strategies are giben by*

a) *formula* (13) *in the case of an exterior field,*

b) *formula* (15) *in the case of an interior field.*

The final, explicit form of the strategies u_i^0, v_i^0 and of the optimum trajectories $x_i^0(t)$, $y_i^0(t)$ depends upon the knowledge of a general solution of the Hamilton-Jacobi equation (7) for the game value $W = W(x_k, y_k)$. This equation is the key to a solution of the pursuit problem in the potential fields U and V determined in E_3. That justifies us to call equations (7) and (8) the fundamental equations of classical pursuit theory. „Classical theory" is to be understood in the same sense as Newton's „classical mechanics" (i.e. nonrelativistic mechanics) is understood in theoretical physics. We would like to mention that the fundamental pursuit equations, similarly as the motion equations in analytical mechanics, are in accordance with the relativity principle of Galileo. This principle requires that the laws of Newton's classical mechanics have the same mathematical form in all inertial systems which are linked by linear Galilean transformations

$$\bar{r}' = \bar{r} - \bar{c}t, \qquad t' = t,$$

where $\bar{c}$ denotes the constant velocity between the coordinate systems $\bar{r}$ and $\bar{r}'$. The above relativity principle may be formulated in a different manner, as follows: The motion equations in classical mechanics are invariant with respect to Galilean transformations. A similar property have the pursuit equations which in E_3 are bounded by the strategies (velocities) being much snaller than the velocity of light in vacuum.

3. Integration of pursuit equations by the method of canonical transformations. The most widely used and most effective method of integrating the system of Hamilton equations (8) is the so-called Hamilton-Jacobi method which is based on canonical transformations. These transformations give a strict connection between the integration surface of equation (7) and the first integrals of the canonical system (8). The idea of this method is based on a known theorem of Jacobi.

THEOREM 3. *If $W(x_k, y_k; a_k, c_k)$ is any complete integral of the Hamilton-Jacobi equation* (7) *then the first integrals of the system of Hamilton equations* (8) *may be written as*

$$\frac{\partial W}{\partial a_i} = b_i, \qquad \frac{\partial W}{\partial c_i} = d_i \qquad (i = 1, 2, 3), \tag{18}$$

where a_i, b_i, c_i and d_i are arbitrary constants.

A proof of theorem 3, based on the theory of canonical transformations, may be found in [2]. In this proof it suffices to identify the game value $W(x_i, y_i; a_i, c_i)$ and Monge's functions p_i, q_i with the generating function of the canonical transformation and with the „generalized momentum" canonically conjugated with x_i, y_i, respectively. So the independent variables x_i, y_i, p_i and q_i form a phase space in contradiction to the variables x_i and y_i which belong to the configuration space.

The first integrals of the motion (18) determine in full the optimum pursuit trajectories in the configuration space (x_i, y_i), and in connection with

$$\frac{\partial W}{\partial x_i} = p_i, \quad \frac{\partial W}{\partial y_i} = q_i \quad (i = 1, 2, 3)$$

determine completely the trajectories in the phase space $(x_i, y_i; p_i, q_i)$.

4. An example. We shall give now an example which will illustrate the Hamilton-Jacobi method for a pursuit game in a given (homogeneous, Newtonian) gravitational field.

Problem. Find the optimum pursuit trajectories and the game value of a two-person game in a homogeneous gravitational field.

Solution. The homogeneity condition for a gravitational field is as follows

$$g = \text{const in every point of } E_3,$$

where g is the strength of the gravitational field, or the gravitational acceleration. On the virtue of (1) we have

$$mg_i = -U_{.i} = \text{const}, \quad ng_i = -V_{.i} = \text{const} \quad (i = 1, 2, 3).$$

If the coordinate system is chosen in such a manner that one of the coordinate axes is directed along the field g then

$$g_1 = g_2 = 0, \quad g_3 = -g = \text{const}$$

and

$$mg = U_{,3} = dU/dx_3, \quad ng = V_{,3} = dV/dy_3.$$

Integrating these equations and taking into account the appropriate boundary conditions leads us to

$$U(x_3) = mgx_3, \quad V(y_3) = ngy_3. \tag{19}$$

A separation of the variables

$$W(x_i, y_i) = W_1(x_i) + W_2(y_i) \tag{20}$$

in the fundamental pursuit equation (7) with the Hamiltonian (9a) gives two independent of each other equations with partial derivatives

$$\pm\sqrt{\frac{2}{m}(A-U)\frac{\partial W_1}{\partial x_i}\frac{\partial W_1}{\partial x_i}} = K,$$

(21)

$$\pm\sqrt{\frac{2}{n}(B-V)\frac{\partial W_2}{\partial y_i}\frac{\partial W_2}{\partial y_i}} = L,$$

where the separation constants K and L satisfy the condition $K+L = 1$. By simple transformation of (19) and (21) we obtain

$$(22)\qquad (A-mgx_3)\frac{\partial W_1}{\partial x_i}\frac{\partial W_1}{\partial x_i} = M, \quad (B-ngy_3)\frac{\partial W_2}{\partial y_i}\frac{\partial W_2}{\partial y_i} = N,$$

where $M = mK^2/2$, $N = nL^2/2$.

Observe that $x_1, x_2; y_1, y_2$ are cyclic variables, hence the complete integrals of the equations (22) may be sought in the form

$$(23)\qquad \begin{aligned} W_1(a_k; x_k) &= a_1x_1+a_2x_2+w_1(a_1, a_2; x_3)+a_3, \\ W_2(c_k; y_k) &= c_1y_1+c_2y_2+w_2(c_1, c_2; y_3)+c_3. \end{aligned}$$

Due to (22), the functions w_1 and w_2 satisfy the equations

$$(24)\qquad \begin{aligned} \frac{\partial w_1}{\partial x_3} &= \pm\sqrt{\frac{M}{A-mgx_3}-a_1^2-a_2^2}, \\ \frac{\partial w_2}{\partial y_3} &= \pm\sqrt{\frac{N}{B-ngy_3}-c_1^2-c_2^2}. \end{aligned}$$

Now, introducing the notation

$$(25)\qquad \begin{aligned} a &= a_1^2+a_2^2, & x &= A-mgx_3, \\ c &= c_1^2+c_2^2, & y &= B-ngy_3, \end{aligned}$$

and integrating equations (24) we obtain

$$(26)\qquad \begin{aligned} w_1(a; x) &= \mp\frac{1}{mg}\int\sqrt{\frac{M}{x}-a}\,dx, \\ w_2(c; y) &= \mp\frac{1}{ng}\int\sqrt{\frac{N}{y}-c}\,dy. \end{aligned}$$

Hence

$$(27)\qquad \begin{aligned} w_1 &= \mp\frac{1}{mg}\left[\sqrt{(M-ax)x}+\frac{M}{2\sqrt{a}}\arcsin\frac{M-2ax}{M}\right] \quad (a>0), \\ w_2 &= \mp\frac{1}{ng}\left[\sqrt{(N-cy)y}+\frac{N}{2\sqrt{c}}\arcsin\frac{N-2cy}{N}\right] \quad (c>0). \end{aligned}$$

The expressions (27) together with the formulae (23) and (20) give the explicit form of the game value $W(a_k, c_k; x_k, y_k)$. The optimum pursuit trajectories may be determined — on the basis of Jacobi's theorem 3 — from equations (18) which in our example take the form

$$
\begin{aligned}
&x_1+\frac{\partial w_1}{\partial a_1} = b_1, \qquad y_1+\frac{\partial w_2}{\partial c_1} = d_1,\\
(28)\qquad &\\
&x_2+\frac{\partial w_1}{\partial a_2} = b_2, \qquad y_2+\frac{\partial w_2}{\partial c_2} = d_2.
\end{aligned}
$$

While differentiating the functions w_1 and w_2 with respect to a_j and $c_j (j = 1, 2)$, respectively, formulae (25) have to be placed into (27). Our way of silution will, however, be different. We will calculate the differentials not directly from (27) but indirectly from (26). To do so it is necessary first to differentiate (26) with respect to a_j and c_j $(j = 1, 2)$ and then to integrate them with respect to x and y, respectively. We obtain, with the notation (27) in mind,

$$
\begin{aligned}
&\frac{\partial w_1}{\partial a_1} = \frac{a_1}{a} w_1, \qquad \frac{\partial w_2}{\partial c_1} = \frac{c_1}{c} w_2,\\
(29)\qquad &\\
&\frac{\partial w_1}{\partial a_2} = \frac{a_2}{a} w_1, \qquad \frac{\partial w_2}{\partial c_2} = \frac{c_2}{c} w_2.
\end{aligned}
$$

A substitution of (29) into (28) gives the final form of the optimum pursuit trajectory equations, as follows

$$
\begin{aligned}
x_1 &= b_1 - a_1 w_1/a, \qquad y_1 = d_1 - c_1 w_2/c,\\
x_2 &= b_2 - a_2 w_1/a, \qquad y_2 = d_2 - c_2 w_2/c,
\end{aligned}
$$

where the constants a, c and the arguments x, y are determined by (25) and the functions w_1, w_2 are given by (27). The arbitrary constants a_j, b_j, c_j and d_j $(j = 1, 2)$ as well as M and N satisfying

$$\pm\sqrt{2M/m} \pm \sqrt{2N/n} = 1$$

are to be calculated from the initial conditions and one boundary condition, namely $W(x_k, x_k) = 0$. These conditions lead in effect to 10 algebraic equations with 10 unknowns a_j, b_j, c_j, d_j, M and N.

5. Notion and principles of a two-person pursuit game in general relativity theory. Let R_4 denote the four-dimensional Riemannian space-time with gravitational field g^{ik} and electromagnetic field A_i. We consider the classes of admissible strategies X and Y; they consist of all differentiable functions of two variables (the positions in space-time of the pursuer and evader) $x^i, y^i \in R_4$ with values in R_4 which are bounded by the metric conditions

$$(30)\qquad \begin{aligned} g^{ik}(u_i - aA_i)(u_k - aA_k) &= -1, \quad u_i \in X, \\ g^{ik}(v_i - bA_i)(v_k - bA_k) &= -1, \quad v_i \in Y, \end{aligned}$$

where a and b denote the ratios of charge and mass of the pursuer and the evader, respectively.

The equilibrium conditions of the conflict situation between the two players 1 and 2 in R_4 are included in the principle of optimality [7].

If $W(x^i, y^i)$ *is a nonnegative and differentiable function for* $x^i, y^i \in R_4$, *and if* $u_i^0 \in X$ *and* $v_i^0 \in Y$ *are such functions that for every* $u_i \in X$ *and* $x^i, y^i \in R_4$ *holds*

$$(31a)\qquad ug^{ik}p_i u_k + vg^{ik}q_i v_k^0 \geqslant -1$$

and for every $v_i \in Y$ *and* $x^i, y^i \in R_4$ *holds*

$$(31b)\qquad ug^{ik}p_i u_k^0 + vg^{ik}q_i v_k \leqslant -1$$

and for every $x^i \in R_4$ *holds* $W(x^i, x^i) = 0$, *then* u_i^0 *and* v_i^0 *are optimum strategies and* $W(x^i, y^i)$ *is the game value in the position* x^i, y^i.

The Monge functions p_i and q_i are given by the formulae $p_i = \partial W/\partial x^i$ and $q_i = \partial W/\partial y^i$; the scalar functions u and v with values in R_4 are given numbers bounding the strategies in the classes X and Y. It is easy to observe that from (31a, b) follows immediately the important

CONCLUSION 2. *If the equality*

$$(31c)\qquad ug^{ik}p_i u_k^0 + vg^{ik}q_i v_k^0 = -1$$

holds then the pursuit is a uniformly closed game in space-time R_4.

The equation (31c) may be written in a simpler form as $dW/ds = -1$, where the game value $W(x_0^i, y_0^i) = \sigma$ determines the pursuit time σ on the optimum trajectories $x_0^i(s)$ of the pursuing and $y_0^i(s)$ of the evading objects in R_4.

In virtue of (30) and (31a, b, c) we are able to formulate the following

THEOREM 4. *If a pursuit game in the fields* A_i *and* g^{ik} *is played according to* (30) *and* (31) *then the optimum pursuit trajectories satisfy the canonical Hamilton equations*

$$(32)\qquad \begin{aligned} \frac{dx^i}{ds} &= \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{ds} = -\frac{\partial H}{\partial x^i}, \\ \frac{dy^i}{ds} &= \frac{\partial H}{\partial q_i}, \quad \frac{dq_i}{ds} = -\frac{\partial H}{\partial y^i}, \end{aligned} \qquad (i = 1, 2, 3, 4)$$

and the game value $W(x^i, y^i)$ *satisfies the Hamilton-Jacobi equation*

$$H\left(x^i, y^i; \frac{\partial W}{\partial x^i}, \frac{\partial W}{\partial y^i}\right) = 1, \tag{33}$$

where the Hamiltonian of the system of two pursuing each other objects has the form

$$\begin{aligned} H &= H(x^i, y^i; p_i, q_i) \\ &= u(\pm\sqrt{-g^{ik}p_ip_k} - ag^{ik}A_ip_k) + v(\pm\sqrt{-g^{ik}q_iq_k} - bg^{ik}A_iq_k). \end{aligned}$$

The proof is analogous to the proof of theorem 1. Worth mentioning is, however, the fact that the four-potential A_i and the metric tensor g^{ik} both are known solutions of the equations of the Maxwellian electromagnetic field in the general theory of relativity and of the fundamental gravitational equations of Einstein.

If we put Kronecker's symbol as the metric tensoe g^{ik} then we automatically change from Riemann's space-time R_4 to Minkovski's space-time M_4. Physically, this corresponds to a changeover from the general theory of relativity to the special one where only exists the elctromagnetic field A_i. In that case the pursuit theory in the electromagnetic field A_i is based on the Hamiltonian

$$H = u(\pm\sqrt{-p_ip_i} - aA_ip_i) + v(\pm\sqrt{-q_iq_i} - bA_iq_i).$$

THEOREM 5. *If a pursuit in the fields* A_j *and* g^{ik} *is played according to* (30) *and* (31) *then the optimum pursuing and evading strategies are given by*

$$\left.\begin{aligned} u_j^0 &= aA_j \mp p_j/\sqrt{-g^{ik}p_ip_k}, \\ v_j^0 &= bA_j \mp q_j/\sqrt{-g^{ik}q_iq_k}. \end{aligned}\right. \qquad (j = 1, 2, 3, 4),$$

The proof is much the same as the proof of theorem 2.

The explicit form of the strategies u_i^0, v_i^0 and the optimum trajectories $x_0^i(s), y_0^i(s)$ are directly connected with the knowledge of the complete integral of the Hamilton-Jacobi equation (33) for the game value $W(x^k, y^k)$. This equation provides a key to the solution of the pursuit problem in the fields A_i and g^{ik} in R_4. That justifies calling the canonical equations (32) and (33) from theorem 4 fundamental equations of relativistic pursuit theory. Of course, the so constructed canonical formalism in relativistic pursuit theory is in conformity with Einstein's pronciples of general covariance and of local equivalence. These principles which form the basis of the general theory are formulated as follows.

The principle of general covariance. *The mathematical form of the laws of nature is identical in all non-inertial systems, or, speaking differently,* *the equations of physics are covariant with respect to any transformations of coordinates.*

The principle of local equivalence. *The mathematical form of the laws of nature is identical in any non-inertial systems and in certain gravitational fields.*

The fundamental pursuit equations (32) and (33) satisfying these principles appeared to be solvable by the method of canonical transformations in the following two cases: 1° in Schwarzschild's gravitational field, and 2° in a homogeneous electric field. A detailed discussion of these solutions has been presented by the author in [5] and [7].

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POŚCIG W PRZESTRZENIACH FIZYCZNYCH

STRESZCZENIE

W pracy sformułowano zasady gry pościgowej na wzór formalizmu Hamiltona-Jacobiego. Część pierwsza pracy poświęcona jest wyprowadzeniu podstawowych równań pościgu w obecności zewnętrznego i wewnętrznego pola potencjalnego, a więc równań spełniających klasyczną zasadę względności Galileusza. W części drugiej otrzymano podobne równania na optymalne tory pościgu i wartość gry w zadanych polach grawitacyjnym i elektromagnetycznym. Postać tych równań jest tak dobrana, aby uczynić zadość relatywistycznym zasadom ogólnej współzmienniczości i lokalnej równoważności Einsteina, leżących u podstaw ogólnej teorii względności. Rozwiązano także przykład dwuosobowej gry pościgowej w jednorodnym polu grawitacyjnym.

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ПОГОНЬ В ФИЗИЧЕСКИХ ПРОСТРАНСТВАХ

РЕЗЮМЕ

В этой статье сформулированы принципы гоночных игр на образец канонического формализма Гамильтона-Якоби. Первая часть статьи относится к получению основных уравнений погони во внешнем и внутреннем потенциальных полях, т. е. уравнений согласных с классическим принципом Галилея. Во второй части выведены аналогические уравнения для оптимальных траекторий погони и цены игры в гравитационном и электромагнитном полях. Эти уравнения сформулированы так, чтобы согласовать их с принципами общей инвариантности и локальной эквивалентности Эйнштейна.

Достигнуто также решение проблемы парной игры в однородном гравитационном поле.