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MINIMAX ESTIMATION OF THE PARAMETERS OF THE MULTIVARIATE HYPERGEOMETRIC DISTRIBUTION

1. Summary. The paper concerns the problem of minimax estimation for the parameter $M = (M_1, \ldots, M_r)$ of the multivariate hypergeometric distribution (1) under the loss function (2). A full solution is given for the three-variate hypergeometric distribution. A minimax estimator is found also in the case when

$$M = (M_{11}, \ldots, M_{1s_1}, \ldots, M_{r1}, \ldots, M_{rs_r})$$

and when the loss function is of the form (17).

2. A theorem for the multivariate hypergeometric distribution. Let $X = (X_1, \ldots, X_r)$ be a random variable with the multivariate hypergeometric distribution

$$(1) \qquad P(X = x) = P(X_1 = x_1, \ldots, X_r = x_r) = \frac{\binom{M_1}{x_1}\cdots\binom{M_r}{x_r}}{\binom{N}{n}},$$

where

$$\sum_{i=1}^{r} M_i = N, \qquad \sum_{i=1}^{r} x_i = n.$$

Denote $M = (M_1, \ldots, M_r)$. It is well known that

$$m_i = \mathrm{E}(X_i | M) = n\frac{M_i}{N},$$

$$\mathrm{E}\,[(X_i-m_i)^2|M] = \frac{n(N-n)}{N^2(N-1)} M_i(N-M_i) \quad (i = 1, \ldots, r),$$

$$\mathrm{E}\,[(X_i-m_i)(X_j-m_j)|M] = -\frac{n(N-n)}{N^2(N-1)} M_i M_j \quad (i, j = 1, \ldots, r, i \neq j).$$

Suppose that $X = x$ is observed and that we want to estimate the parameter M. Let $a = (a_1, \ldots, a_r)$ be an estimate of $M = (M_1, \ldots, M_r)$ and let the loss associated with estimate a (the loss function) be

$$L(M, a) = \sum_{i,j=1}^{r} c_{ij}(M_i-a_i)(M_j-a_j) \tag{2}$$

where the matrix $C = \|c_{ij}\|$ is positive definite.

An estimator $d^0(x) = (d_1^0(x), \ldots, d_r^0(x))$ of M is called a *minimax* one if

$$\sup_M R(M, d^0) = \inf_d \sup_M R(M, d) \tag{3}$$

where $R(M, d)$ is the risk function for the loss function (2).

Let us consider the estimator $d = (d_1, \ldots, d_r)$ for which

$$d_i(X) = N\frac{X_i+\alpha_i}{n+\alpha} \tag{4}$$

where $\alpha = \sum_{i=1}^{r} \alpha_i$. In this case

$$R(M, d) = \frac{1}{(n+\alpha)^2}\left\{\sum_{i,j=1}^{r} c_{ij}\left[\left(\alpha^2-n\frac{N-n}{N-1}\right)M_i M_j + N(N\alpha_i-2\alpha M_i)\alpha_j\right]+ + \sum_{i=1}^{r} c_{ii}\, nN\frac{N-n}{N-1}M_i\right\}. \tag{5}$$

Let us put

$$\alpha = \sqrt{n\frac{N-n}{N-1}}, \quad \alpha_i = \beta_i\sqrt{n\frac{N-n}{N-1}} \quad (i = 1, \ldots, r). \tag{6}$$

Then

$$R(M, d) = \frac{N^2 n\dfrac{N-n}{N-1}}{\left(n+\sqrt{n\dfrac{N-n}{N-1}}\right)^2}\left\{\sum_{i,j=1}^{r} c_{ij}\beta_i\beta_j + \sum_{i,j=1}^{r} (c_{ii}-2c_{ij})\beta_j\frac{M_i}{N}\right\}. \tag{7}$$

THEOREM 1. *If there exist constants v, $\beta_1, \ldots, \beta_r$ and a set $A \subset R = \{1, \ldots, r\}$, $|A| \geqslant 2$, such that*

(a) $$\sum_{j\in A} (c_{ii}-2c_{ij})\beta_j = v \quad \text{for} \quad i \in A,$$

(b) $$\sum_{j\in A} (c_{ii}-2c_{ij})\beta_j \leqslant v \quad \text{for} \quad i \in R-A,$$

$\beta_j > 0$ *for* $j \in A$, $\beta_j = 0$ *for* $j \in R-A$, $\sum_{j\in A} \beta_j = 1$, *then the estimator d defined by* (4) *and* (6), *with β_i fulfilling the above conditions, is a minimax estimator.*

Proof. Let the conditions of Theorem 1 hold. Then

$$R(M, d) = \frac{N^2 n \frac{N-n}{N-1}}{\left(n+\sqrt{n\frac{N-n}{N-1}}\right)^2} \{\sum_{i,j=1}^{r} c_{ij}\beta_i\beta_j + v\} \stackrel{\text{df}}{=} c$$

for $M_i = 0$ where $i \in R-A$ and

$$R(M, d) \leqslant c$$

for any M. Denote $A = \{i_1, \ldots, i_s\}$, $R-A = \{i_{s+1}, \ldots, i_r\}$. Then the theorem follows from the fact that estimator defined by (4), where $\alpha_i > 0$ if $i \in A$, $\alpha_i = 0$ if $i \in R-A$, is a Bayes estimator with respect to the a priori distribution of parameter M such that

$$P(M_{i_1} = m_{i_1}, \ldots, M_{i_s} = m_{i_s}) = K\frac{\Gamma(m_{i_1}+a_{i_1})\ldots\Gamma(m_{i_s}+a_{i_s})}{m_{i_1}!\ldots m_{i_s}!},$$

$$P(M_{i_{s+1}} = \ldots = M_{i_r} = 0) = 1$$

if

$$a_{i_j} = \beta_{i_j}\frac{N\sqrt{n\frac{N-n}{N-1}}}{N-n-\sqrt{n\frac{N-n}{N-1}}} \qquad (i = 1, \ldots, s),$$

$$N > n+1$$

and

$$P(M_{i_1} = m_{i_1}, \ldots, M_{i_s} = m_{i_s}) = \frac{N!}{m_{i_1}!\ldots m_{i_s}!}\beta_{i_1}^{m_{i_1}}\ldots\beta_{i_s}^{m_{i_s}},$$

$$P(M_{i_{s+1}} = \ldots = M_{i_r} = 0) = 1$$

if $N = n+1$ (see [3]).

In the following sections we give some applications of this theorem.

3. Estimation of parameters of the multivariate hypergeometric distribution under a general quadratic loss function. We look for a minimax estimator of parameter M of distribution (1) under the loss function (2). Let the assumptions of Theorem 1 hold. Let $A = \{i_1, \ldots, i_s\}$. Then

$$\begin{gathered}\sum_{j\in A} (c_{ii}-2c_{ij})\beta_j - v = 0 \quad \text{for} \quad i \in A,\\ \sum_{j\in A} \beta_j = 1.\end{gathered} \tag{8}$$

Solving this system of equations we obtain (after simple computations)

$$\beta_j = M_A^{(j)}/M_A$$

where

$$M_A^{(j)} = \frac{(-1)^{s+j+l+1}}{2} \begin{vmatrix} 0 & d_{i_1 i_2 i_l} & \ldots & d_{i_1 i_{j-1} i_l} & d_{i_1 i_{j+1} i_l} & \ldots & d_{i_1 i_s i_l} \\ d_{i_2 i_1 i_l} & 0 & \ldots & d_{i_2 i_{j-1} i_l} & d_{i_2 i_{j+1} i_l} & \ldots & d_{i_2 i_s i_l} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ d_{i_{l-1} i_1 i_l} & d_{i_{l-1} i_2 i_l} & & d_{i_{l-1} i_{j-1} i_l} & d_{i_{l-1} i_{j+1} i_l} & \ldots & d_{i_{l-1} i_s i_l} \\ d_{i_{l+1} i_1 i_l} & d_{i_{l+1} i_2 i_l} & & d_{i_{l+1} i_{j-1} i_l} & d_{i_{l+1} i_{j+1} i_l} & \ldots & d_{i_{l+1} i_s i_l} \\ \cdots & \cdots & \cdots & \cdots & \cdots & & \cdots \\ d_{i_s i_1 i_l} & d_{i_s i_2 i_l} & & d_{i_s i_{j-1} i_l} & d_{i_s i_{j+1} i_l} & \ldots & 0 \end{vmatrix} \tag{9}$$

where the terms on the diagonal are equal to zero with the exception of the column with the terms $d_{i_k i_l i_l}$ for l fixed

$$d_{ijk} = c_{ii} + c_{jk} - c_{ij} - c_{ik} \qquad (i \neq j,\ i \neq k) \tag{10}$$

($M_A^{(j)}$ does not depend on l for $j \neq l$),

$$M_A = \begin{vmatrix} d_{i_l i_1} & d_{i_l i_1 i_2} & & d_{i_l i_1 i_{l-1}} & d_{i_l i_1 i_{l+1}} & \ldots & d_{i_l i_1 i_s} \\ d_{i_l i_2 i_1} & d_{i_l i_2} & & d_{i_l i_2 i_{l-1}} & d_{i_l i_2 i_{l+1}} & \ldots & d_{i_l i_2 i_s} \\ & & & & & & \ldots \\ d_{i_l i_{l-1} i_1} & d_{i_l i_{l-1} i_2} & & d_{i_l i_{l-1}} & d_{i_l i_{l-1} i_{l+1}} & \ldots & d_{i_l i_{l-1} i_s} \\ d_{i_l i_{l+1} i_1} & d_{i_l i_{l+1} i_2} & & d_{i_l i_{l+1} i_{l-1}} & d_{i_l i_{l+1}} & \ldots & d_{i_l i_{l+1} i_s} \\ \cdots & \cdots & \cdots & \cdots & \cdots & & \ldots \\ d_{i_l i_s i_1} & d_{i_l i_s i_2} & & d_{i_l i_s i_{l-1}} & d_{i_l i_s i_{l+1}} & \ldots & d_{i_l i_s} \end{vmatrix} \tag{11}$$

(M_A does not depend on l),

(12) $$d_{ij} \stackrel{\text{df}}{=} d_{ijj} = c_{ii}+c_{jj}-2c_{ij} \quad (i \neq j).$$

Let us consider the positive definite quadratic form

$$W = \sum_{i,j=1}^{r} c_{ij} x_i x_j$$

and consider x_i such that $\sum_{i\in A} x_i = 1$ and $x_i = 0$ for $i \in R-A$. After expressing the x_{i_l} by the other x_i we obtain

(13) $$W = \sum_{\substack{i,j\in A\\ i,j\neq i_l}} d_{i_l i_j} x_i x_j.$$

Since the form (13) is also positive definite we obtain $M_A > 0$.

Now let $A_s = \{i_1, \ldots, i_s\}$ and let $\beta_{i_j}^{(s)}, v_s$ and $\beta_{i_j}^{(s+1)}$, v_{s+1} be the solutions of the system (8) for $A = A_s$ and $A = A_{s+1}$, respectively. Then

$$\beta_{i_j}^{(s)} = M_s^{(j)}/M_s \quad (j = 1, \ldots, s), \quad v_s = N_s/M_s,$$

$$\beta_{i_j}^{(s+1)} = M_{s+1}^{(j)}/M_{s+1} \quad (j = 1, \ldots, s+1), \quad v_{s+1} = N_{s+1}/M_{s+1},$$

where the corresponding determinants are defined according to Cramer's formulae. We obtain

(14) $$M_s = (-1)^{s-1} 2^{s-1} M_{A_s},$$

(15) $$\beta_{i_{s+1}}^{(s+1)} = -\frac{M_s}{M_{s+1}} [(c_{i_{s+1}i_{s+1}} - 2c_{i_{s+1}i_1})\beta_{i_1}^{(s)} + \ldots$$
$$\ldots + (c_{i_{s+1}i_{s+1}} - 2c_{i_{s+1}i_s})\beta_{i_s}^{(s)}] + N_s/M_{s+1}.$$

Now we return to Theorem 1 for $A = A_s$. From its assumption (b) we obtain

$$\sum_{j\in A} (c_{ii} - 2c_{ii_j})\beta_{i_j}^{(s)} - v_s \leqslant 0$$

for each $i \in R - A_s$, what, comparing with (14), (15) and with the fact that $N_s = M_s v_s$, $M_A > 0$ is equivalent to

(16) $$\beta_{i_{s+1}}^{(s+1)} \leqslant 0$$

for each $i_{s+1} \in R - A_s$.

Suppose that conditions (a), (b) of Theorem 1 hold for $A = A_1$. Then there exists an index $i \in R$ such that

$$c_{ii} + c_{jj} - 2c_{ij} \leqslant 0 \quad \text{for each } j \in R - \{i\},$$

or taking into account the notation (12)

$$d_{ij} \leqslant 0 \quad \text{for each } j \in R - \{i\}.$$

But this is possible for no j since the form

$$\sum_{\substack{j,k=1\\ j,k \neq i}}^{r} d_{ijk} x_j x_k$$

is positive definite.

Let $r = 3$. In this case

$$\beta_{i_1}^{(2)} = \beta_{i_2}^{(2)} = \tfrac{1}{2}$$

and the minimax estimator exists if $\beta_1^{(3)} \leqslant 0$ (in this case $A = \{2, 3\}$) or if $\beta_2^{(3)} \leqslant 0$ ($A = \{1, 3\}$), or if $\beta_3^{(3)} \leqslant 0$ ($A = \{1, 2\}$). A minimax estimator exists also if $\beta_1^{(3)} > 0$, $\beta_2^{(3)} > 0$, $\beta_3^{(3)} > 0$ ($A = \{1, 2, 3\}$). Hence it follows that a minimax estimator always exists. Let us notice that

$$\beta_1^{(3)} = \frac{d_{23} d_{123}}{2M}, \quad \beta_2^{(3)} = \frac{d_{13} d_{213}}{2M}, \quad \beta_3^{(3)} = \frac{d_{12} d_{312}}{2M}$$

where $M = M_{\{1,2,3\}}$ is given by (11).

4. Estimation of the parameter of a population with a hierarchic structure. Let $X = (X_{11}, \ldots, X_{1s_1}, \ldots, X_{r1}, \ldots, X_{rs_r})$ be a random variable distributed according to the multivariate hypergeometric distribution with parameter $M = (M_{11}, \ldots, M_{1s_1}, \ldots, M_{r1}, \ldots, M_{rs_r})$ and let the loss function be

$$L(M, a) = \sum_{i=1}^{r} c_i (M_i - a_i)^2 + \sum_{i=1}^{r} \sum_{j=1}^{s_i} c_{ij} (M_{ij} - a_{ij})^2, \tag{17}$$

where $M_i = \sum_{j=1}^{s_i} M_{ij}$ ($i = 1, \ldots, r$), a_i, a_{ij} are estimates of M_i, M_{ij}, respectively, $c_i \geqslant 0$, $c_{ij} > 0$ for $i = 1, \ldots, r$, $j = 1, \ldots, s_i$. Consider the estimator $d = (d_{11}, \ldots, d_{1s_1}, \ldots, d_{r1}, \ldots, d_{rs_r})$ of M for which

$$d_{ij}(X) = N \frac{X_{ij} + \alpha_{ij}}{n + \alpha}, \quad \text{where} \quad \alpha = \sum_{i=1}^{r} \sum_{j=1}^{s_i} \alpha_{ij}. \tag{18}$$

Let $X_i = \sum_{j=1}^{s_i} X_{ij}$, $\alpha_i = \sum_{j=1}^{s_i} \alpha_{ij}$ and let

$$d_i(X) = \sum_{j=1}^{s_i} d_{ij}(X) = N \frac{X_i + \alpha_i}{n + \alpha} \tag{19}$$

be an estimator of M_i $(i = 1, \ldots, r)$. For

(20) $$\alpha = \sqrt{n\frac{N-n}{N-1}}, \quad \alpha_{ij} = \beta_{ij}\sqrt{n\frac{N-n}{N-1}}, \quad \beta_i = \sum_{j=1}^{s_i} \beta_{ij}$$
$$(i = 1, \ldots, r, j = 1, \ldots, s_i)$$

we obtain

(21) $$R(M, d) = \left(\left(N^2 n\frac{N-n}{N-1}\right)\Big/\left(n+\sqrt{n\frac{N-n}{N-1}}\right)^2\right)\left\{\sum_{i=1}^{r} c_i\left[\beta_i^2+(1-2\beta_i)\frac{M_i}{N}\right]\right.$$
$$\left.+\sum_{i=1}^{r}\sum_{j=1}^{s_i} c_{ij}\left[\beta_{ij}+(1-2\beta_{ij})\frac{M_{ij}}{N}\right]\right\}.$$

We prove that there exists an estimator d of the parameter M of the form (18) for which (19) holds and which is minimax. We also give the method of determining the parameters α_{ij} for this estimator.

Without loss of generality we can assume that

(22) $$c_{i1} \geqslant \ldots \geqslant c_{is_i} \quad \text{for} \quad i = 1, \ldots, r.$$

Let us suppose that there exist integers $L_1, \ldots, L_r$, $0 \leqslant L_i \leqslant s_i$ $(i = 1, \ldots, r)$ and constants β_{ij} $(i = 1, \ldots, r, j = 1, \ldots, s_i)$ such that

(23) $$c_i(1-2\beta_i)+c_{ij}(1-2\beta_{ij}) = v \quad \text{for} \quad j \leqslant L_i$$

and

(24) $$c_i(1-2\beta_i)+c_{ij} \leqslant v \quad \text{for} \quad j > L_i$$

and that $\beta_{ij} > 0$ for $j \leqslant L_i$, $\beta_{ij} = 0$ for $j > L_i$, $\sum_{i=1}^{r}\sum_{j=1}^{L_i} \beta_{ij} = 1$.

Solving the system of equation (23) with respect to β_{ij} we obtain

(25) $$\beta_{ij} = \{c_i[c_{ij}\sum_{k=1}^{L_i}(1/c_{ik})-(L_i-1)]+c_{ij}-v\}/2c_{ij}(c_i\sum_{k=1}^{L_i}(1/c_{ik})+1)$$

for $j \leqslant L_i$ and

(26) $$\beta_i = \frac{c_i+L_i/\sum_{k=1}^{L_i}(1/c_{ik})-v}{2(c_i+1/\sum_{k=1}^{L_i}(1/c_{ik}))}$$

for each i such that $L_i > 0$. Let $\bar{L} = (L_1, \ldots, L_r)$ and let $c_{\bar{L}}$ be the set of all indices j such that $L_j > 0$.

From the condition $\sum_{j \in c_L} \beta_j = 1$ we obtain

(27) $$v = v(L_1, \ldots, L_r) = \frac{1}{\sum_{j \in c_L} \left(1/(c_j + 1/\sum_{k=1}^{L_j} (1/c_{jk}))\right)} \left(\sum_{j \in c_L} \frac{c_j + L_j / \sum_{k=1}^{L_j} (1/c_{jk})}{c_j + 1/\sum_{k=1}^{L_j} (1/c_{jk})} - 2 \right).$$

Let

(28) $$A_i(l_i) = c_i \left[c_{il_i} \sum_{k=1}^{l_i} \frac{1}{c_{ik}} - (l_i - 1) \right] + c_{il_i}$$

and

(29) $$B_i(l_1, \ldots, l_r) = A_i(l_i) \sum_{j \in c_l} \frac{1}{c_j + 1/\sum_{k=1}^{l_i} (1/c_{jk})} - \sum_{j \in c_l} \frac{c_j + l_j / \sum_{k=1}^{l_j} (1/c_{jk})}{c_j + 1/\sum_{k=1}^{l_j} (1/c_{jk})} + 2$$

for $i = 1, \ldots, r$, $0 \leqslant l_i \leqslant s_i$, c_l is the set of all indices j such that $l_j \geqslant 0$. From (25), (27), (28) and (29) it follows that

(30) $$\beta_{iL_i} = q_i(L_1, \ldots, L_r) B_i(L_1, \ldots, L_r) \quad \text{where} \quad q_i > 0.$$

Method I. We define the method of determining $L_1, \ldots, L_r$ as follows:

(a) In the first step choose i such that

$$A_i(1) = c_i + c_{i1} = \max(c_1 + c_{11}, \ldots, c_r + c_{r1}) = \max(A_1(1), \ldots, A_r(1))$$

and put $l_i = 1$ and $l_k = 0$ for $k \neq i$.

(b) Suppose that $l_1, \ldots, l_r$, where $\sum_{i=1}^{r} l_i = l$, have been determined. In the $(l+1)$-th step choose i such that

$A_i(l_i + 1) = \max(A_1(l_1 + 1), \ldots, A_r(l_r + 1))$, where $A_j(l_j + 1) = -\infty$ if $l_j = s_j$.

(c) Let in the l-th step i_1 be chosen, where $l = \sum_{i=1}^{r} l_i$, and let in the $(l+1)$-th step be chosen i_2. If

$$B_{i_1}(l_1, \ldots, l_{i_1}, \ldots, l_r) > 0, \qquad B_{i_2}(l_1, \ldots, l_{i_2} + 1, \ldots, l_r) \leqslant 0$$

then stop the choosing and put $L_i = l_i$ $(i = 1, \ldots, r)$. If you reach $l = \sum_{i=1}^{r} s_i$ put $L_i = s_i$ $(i = 1, \ldots, r)$.

Lemma 1. *Let $L_1, \ldots, L_r$ be the numbers determined by Method* I *and let $L = \sum_{i=1}^{r} L_i$. Then $L \geqslant 2$.*

LEMMA 2. *Let $l_1, \ldots, l_r$ be determined according to Method* I *and* $\sum_{i=1}^{r} l_i = l$. *If i_1 is chosen in step l and i_2 in step $l+1$, then*

$$A_{i_2}(l_{i_2}+1) \leqslant A_{i_1}(l_{i_1}), \tag{31}$$

$$B_{i_2}(l_1, \ldots, l_{i_2}+1, \ldots, l_r) \leqslant B_{i_1}(l_1, \ldots, l_r). \tag{32}$$

Proof. Let $i_2 \neq i_1$. Then

$$A_{i_2}(l_{i_2}+1) = \max\big(A_1(l_1+1), \ldots, A_{i_1}(l_{i_2}+1), \ldots, A_r(l_r+1)\big) \leqslant A_{i_1}(l_{i_1}).$$

When $i_2 = i_1$ the proof results from (22) and the definition (28).

Now we prove (32). Let $l_{i_2} > 0$. Then

$$B_{i_2}(l_1, \ldots, l_{i_2}+1, \ldots, l_r) - B_{i_1}(l_1, \ldots, l_r)$$

$$= A_{i_2}(l_{i_2}+1)\left[\sum_{\substack{j \in T \\ j \neq i_2}} \frac{1}{c_j + 1/\sum_{k=1}^{l_j} (1/c_{jk})} + \frac{1}{c_{i_2} + 1/\sum_{k=1}^{l_{i_2}+1} (1/c_{i_2 k})}\right] - \frac{c_{i_2} + (l_{i_2}+1)/\sum_{k=1}^{l_{i_2}+1} (1/c_{i_2 k})}{c_{i_2} + \sum_{k=1}^{l_{i_2}+1} (1/c_{i_2 k})} -$$

$$-A_{i_1}(l_{i_1})\left[\sum_{\substack{j \in T \\ j \neq i_2}} \frac{1}{c_j + 1/\sum_{k=1}^{l_j} (1/c_{jk})} + \frac{1}{c_{i_2} + 1/\sum_{k=1}^{l_{i_2}} (1/c_{i_2 k})}\right] + \frac{c_{i_2} + l_{i_2}/\sum_{k=1}^{l_{i_2}} (1/c_{i_2 k})}{c_{i_2} + 1/\sum_{k=1}^{l_{i_2}} (1/c_{i_2 k})}$$

$$\leqslant A_{i_2}(l_{i_2}+1)\frac{1}{c_{i_2} + 1/\sum_{k=1}^{l_{i_2}+1} (1/c_{i_2 k})} - \frac{c_{i_2} + (l_{i_2}+1)/\sum_{k=1}^{l_{i_2}+1} (1/c_{i_2 k})}{c_{i_2} + 1/\sum_{k=1}^{l_{i_2}+1} (1/c_{i_2 k})} -$$

$$-A_{i_2}(l_{i_2}+1)\frac{1}{c_{i_2} + 1/\sum_{k=1}^{l_{i_2}} (1/c_{i_2 k})} + \frac{c_{i_2} + l_{i_2}/\sum_{k=1}^{l_{i_2}} (1/c_{i_2 k})}{c_{i_2} + 1/\sum_{k=1}^{l_{i_2}} (1/c_{i_2 k})} = 0.$$

In the case of $l_{i_2} = 0$ we obtain

$$B_{i_2}(l_1, \ldots, l_{i_2}+1, \ldots, l_r) - B_{i_1}(l_1, \ldots, l_r)$$

$$= A_{i_2}(1)\left[\sum_{j \in T} \frac{1}{c_j + 1/\sum_{k=1}^{l_j} (1/c_{jk})} + \frac{1}{c_{i_2}+c_{i_2 1}}\right] - 1 - A_{i_1}(l_{i_1}) \sum_{j \in T} \frac{1}{c_j + 1/\sum_{k=1}^{l_j} (1/c_{jk})}$$

$$\leqslant A_{i_2}(1)\frac{1}{c_{i_2}+c_{i_2 1}} - 1 = 0.$$

THEOREM 2. *The estimator defined by* (19) *and* (20) *where* β_{ij} *are given by* (25) *and* (27) *and* $L_1, \ldots, L_r$ *are determined according to Method* I, *is a minimax estimator.*

Proof. The proof is based on Theorem 1. At first we prove that for $L_1, \ldots, L_r$ determined by Method I, $\beta_{ij} > 0$ for $j \leqslant L_i$. Let $L = \sum_{i=1}^{r} L_i$, and assume that in the Lth step the index i_0 was chosen. According to (29) $B_{i_0}(L_1, \ldots, L_r)$ can be presented in the form

$$B_{i_0}(L_1, \ldots, L_r) = A_{i_0}(L_{i_0}) D(L_1, \ldots, L_r) - E(L_1, \ldots, L_r),$$

where $D > 0$. Let i be an index such that $L_i > 0$. Then the index i was chosen before the Lth step. But from (31) it follows that $A_{i_0}(L_{i_0}) \leqslant A_i(L_i)$ and

$$\begin{aligned} (33) \qquad B_i(L_1, \ldots, L_r) &= A_i(L_i) D(L_1, \ldots, L_r) - E(L_1, \ldots, L_r) \\ &\geqslant A_{i_0}(L_{i_0}) D(L_1, \ldots, L_r) - E(L_1, \ldots, L_r) = B_{i_0}(L_1, \ldots, L_r) > 0 \end{aligned}$$

by (c) of Method I. Then from (30) we have $\beta_{iL_i} > 0$ if $L_i > 0$, and $\beta_{ij} > 0$ for $j \leqslant L_i$, because $c_{ij+1} \leqslant c_{ij}$ for $j = 1, \ldots, s_i - 1$.

Now, let $l > L_i$. Since $L \geqslant 2$, to apply Theorem 1 it is necessary to show that

$$c_i(1-2\beta_i) + c_{ij} \leqslant v \quad \text{for} \quad j > L_i.$$

Since $c_{ij+1} \leqslant c_{ij}$ $(j = 1, \ldots, s_i - 1)$ it is sufficient to prove that

$$(34) \qquad c_i(1-2\beta_i) + c_{iL_i+1} \leqslant v.$$

Let $L_i > 0$. By (26) we infer that the above inequality is equivalent to

$$(35) \qquad c_i\left(c_{iL_i+1} \sum_{k=1}^{L_i+1} \frac{1}{c_{ik}} - L_i\right) + c_{iL_i+1} \leqslant v.$$

Let $L_i = 0$. Then $\beta_i = 0$ and (35) is the same as (34).

Using the notation of the functions A, D and E introduced earlier we can rewrite (35) in the form

$$A_i(L_i+1)D(L_1, \ldots, L_r)-E(L_1, \ldots, L_r) \leqslant 0.$$

Let i_2 be the index chosen according to Method I in the $(L+1)$-th step. We obtain

$$\begin{aligned}
&A_i(L_i+1)D(L_1, \ldots, L_r)-E(L_1, \ldots, L_r)\\
&\quad \leqslant A_{i_2}(L_{i_2}+1)D(L_1, \ldots, L_r)-E(L_1, \ldots, L_r)\\
&\quad = A_{i_2}(L_{i_2}+1)D(L_1, \ldots, L_{i_2}+1, \ldots, L_r)-E(L_1, \ldots, L_{i_2}+1, \ldots, L_r)\\
&\quad = B_{i_2}(L_1, \ldots, L_{i_2}+1, \ldots, L_r) \leqslant 0.
\end{aligned}$$

The first equation follows from the proof of Lemma 2. Thus the theorem is proved.

5. Final remarks. It is well known (see Hodges and Lehmann [1]) that

$$d(X) = N\frac{X+\frac{1}{2}\sqrt{n\frac{N-n}{N-1}}}{n+\sqrt{n\frac{N-n}{N-1}}}$$

is the minimax estimator of the parameter of the hypergeometric distribution under the quadratic loss function. A minimax estimator of $(M_1, \ldots, M_r)$ for the multivariate hypergeometric distribution and the loss function (2) was given by the author in [3] in the case when C is an arbitrary nonnegative definite diagonal matrix. This result was generalized in [2] to the case when there is an additional loss dependent on observation.

Some results for the multinomial distribution similar to these presented in the paper were obtained by the author in [4].

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