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AN ADAPTIVE SEQUENTIAL TEST

In many practical problems such as signal detection, automatic control or identification of the properties of systems arises the need to synthesize decision making systems which would operate effectively even though some statistical properties of the observed sequences are unknown. There are also two additional requirements; the proposed system is expected to make correct decisions after a minimum number of observations and it should be put into the form of a finite automaton convenient for use with digital techniques.

1. Formulation of the problem. Let us consider a decision problem for which we shall propose a sequential test adaptable to the parameters of the transmitted signal. Let this signal be a vector $\boldsymbol{s}$ with components s_l $(l = 1, 2, \ldots, L)$ dependent on elementary messages x_l and on passive parameters which influence the shape of the transmitted signal although they do not carry messages themselves. Let x_l be the elements of a binary set of messages $X = \{0, 1\}$, and parameters ϑ the elements of a certain set Θ [1]. Moreover, we assume that if $x_l = 0$, then the transmitted signal $s_l(x; \vartheta) = 0$. A sequence of elementary messages $x_1, x_2, \ldots, x_L$ represents a transmitted message.

Let $\boldsymbol{x}^l$ be the vector with l identical binary components $x_1, \ldots, x_l$, $l \leqslant N$. Thus

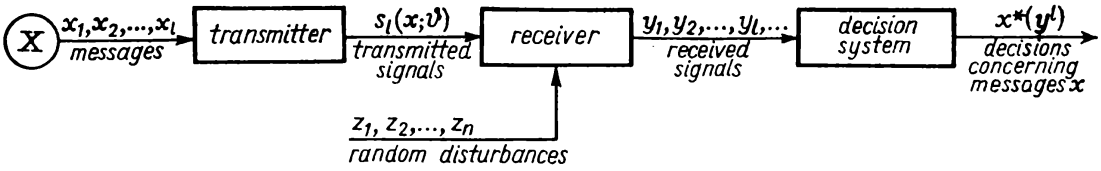
$$\boldsymbol{x}^l = \underbrace{(x, x, \ldots, x)}_{l \text{ times}}, \qquad \text{where } x \in X,$$

and $\boldsymbol{x}^l$ can be identified with the message $x \in X$. Let the received signals y_n, $n = 1, 2, \ldots, N$, for a given value of ϑ, be realizations of independent random variables Y_n with the same probability distribution. We write

$$\boldsymbol{y} = (y_1, y_2, \ldots, y_N) \quad \text{and} \quad Y = (Y_1, Y_2, \ldots, Y_N).$$

[1] The set of passive parameters Θ contains $2N$ 2-dimentional elements ϑ which represent the phase and the parasitic fluctuation of the amplitude of elementary signals transmitted in the sequence of length N known as a transmitted signal packet.

The point decisions, which are denoted by $x^*(\boldsymbol{y})$ and which belong to the set of messages X, are taken on the basis of observation of $\boldsymbol{y}$. The vector $\boldsymbol{y}^l$ $(l \leqslant N)$ represents the sequence of the first l elementary signals received by the decision system from some environment. In particular, it can be a sequence of l binary pulses at the output of the receiving system. Hence the following arrangement can serve as a model for our considerations:



Randomness of the received signals Y is the result of the randomness of a parameter ϑ as well as of the stochastic interferences Z_n in the communication channel. Moreover, let us assume that consecutive observations will be analysed sequentially and with full memorization.

Let us introduce the notation

$$\mathrm{P}(y_n \mid i, \vartheta) = \Pr\{Y_n = y_n \mid x = i, \vartheta\}, \qquad i = 0, 1, \ \vartheta = (\vartheta_1, \vartheta_2) \in \Theta.$$

We assume that $\mathrm{P}(y_n \mid 0, \vartheta)$ does not depend on ϑ:

$$\mathrm{P}(y_n \mid 0, \vartheta) = \mathrm{P}(y_n \mid 0) = \begin{cases} 1 & \text{for } y_n = 0, \\ 0 & \text{for } y_n \neq 0. \end{cases}$$

The identity

$$\mathrm{P}(y_n \mid i', \vartheta') \equiv \mathrm{P}(y_n \mid i'', \vartheta'')$$

holds only if $i' = i'' = 0$ or $i' = i'' = 1$ and $\vartheta' = \vartheta''$.

Moreover, let the probability $\mathrm{P}(y_1; x = 0)$ be known while the probability $\mathrm{P}(y_1; x = 1; \vartheta)$ is unknown due to the fact that we have not made any assumption as regards the *a priori* distribution of the parameter ϑ and the messages x. Let us denote by $\mathrm{P}(\boldsymbol{y}^l; i; \vartheta)$, $i = 0$, the probability of the result $\boldsymbol{y}^l$ for an l-element observation vector Y^l. Also, let us introduce the following shortened notation:

$$p_1(\boldsymbol{y}^l) \stackrel{\mathrm{df}}{=} \mathrm{P}(\boldsymbol{y}^l; 1; \vartheta), \tag{1.1}$$

$$p_0(\boldsymbol{y}^l) \stackrel{\mathrm{df}}{=} \mathrm{P}(\boldsymbol{y}^l; 0). \tag{1.2}$$

In this paper we shall restrict ourselves to the case where the received sygnals are s-dimensional random variables assuming only the values in the binary set $Y = \{0, 1\}$.

The physical meaning of the above-mentioned limitation could, for example, be the assumption that a sequence of elementary signals appears at the output of the receiving system following two-level quantization of the amplitude.

The statistical properties of the parameter $\vartheta \in \Theta$ are not known beforehand.

Note that, owing to the fact that the random variables Y_n are independent and of binary nature, the functions (1.1) and (1.2) become

$$p_0(\boldsymbol{y}^l) = \beta^{y_1+y_2+\ldots+y_l}(1-\beta)^{l-(y_1+y_2+\ldots+y_l)}, \tag{1.3}$$

$$p_1(\boldsymbol{y}^l) = (1-\alpha)^{y_1+y_2+\ldots+y_l}\alpha^{l-(y_1+y_2+\ldots+y_l)}, \tag{1.4}$$

where $\beta = \mathrm{P}(1 \mid 0)$ is the probability of the received elementary signal y_l taking the value of 1 for $x = 0$, and $\alpha = \mathrm{P}(0 \mid 1; \vartheta)$ is the probability of the received elementary signal y_l being equal to 0 for $x = 1$. From the previous assumption the quantity β is known. On the other hand, we assume that α is unknown and that it can take values from $0 < \alpha < \beta$.

The problem is to find a sequential decision rule which we denote by x^* and which would be optimal in relation to the established criterion, that is which would assign the best decisions $x^*(\boldsymbol{y}^l)$ to the sequences $(y_1, y_2, \ldots, y_l)$ of the received signals.

2. Optimization criteria. The criterion of optimization will be specified on the basis of the following argument:

if the decision $x^* = 1$ is made after the message $x = 0$ has been transmitted, then it will result in a false alarm.

Let us denote by $\mathrm{P_{FA}}$ the probability of this happening. Similarly, let us denote by $\mathrm{P_{FS}}$ the probability of a lack of alarm. This situation occurs when the decision $x^* = 0$ is made after the message $x = 1$ has been transmitted. We agree to make the decisions sequentially after investigating observation sequences $\boldsymbol{y}^l = (y_1, y_2, \ldots, y_l)$ for each $l = 1, 2, \ldots, L \leqslant N$. At any step $l < L$ we may either accept the decision $x(\boldsymbol{y}^l) = 0, 1$ or decide to proceed to the examination of $\boldsymbol{y}^{l+1}$, which will be denoted by $x^*(\boldsymbol{y}^l) = -1$. The final decision must be reached at last for $l = L$ and the value of l for which the final decision is taken is called the *duration* of the investigation. Now, let us work out, for a moment, the procedure which would minimize the losses resulting from making an erroneous decision

$$F(L, x^*) \overset{\mathrm{df}}{=} W_0\mathrm{P_{FA}} + W_1\mathrm{P_{FS}}, \tag{2.1}$$

where $W_0 > 0$ and $W_1 > 0$ are constants, and we have a limited number of steps to minimize the expression.

Let Ω be the set of all $2(2^L-1)$ binary sequences of length $l = 1, 2, \ldots, L$, i. e. the set of all possible values of random vectors $\boldsymbol{y}^l$. The symbol $\boldsymbol{y}_L$ will be used to denote an element of Ω.

It can be shown that minimization of function (2.1) for the decision rules $x^*(\boldsymbol{y}^l)$ belonging to the set of all functions $x^*(\cdot)$ with arguments from the space Ω and values from $\{-1, 0, 1\}$, giving the number

$$K = \min_{x^*(\cdot)} \left\{ W_0 \sum_A p_0'(\boldsymbol{y}_L) + W_1 \sum_B p_1'(\boldsymbol{y}_L) \right\},$$

seduces the sequential investigation to inspection of a sample of a fixed rize. A and B are the subsets of Ω discriminated by the decision function $x^*(\cdot)$.

The functions $p_0'(\boldsymbol{y}_L)$ and $p_1'(\boldsymbol{y}_L)$ represent the probabilities specified within Ω for the case $x = 0$ and $x = 1$, respectively. Such a result is not surprising and could have been foreseen intuitively since the accepted rule minimizes (2.1) though, at the same time, it requires the investigation to be continued for as long as necessary, bearing in mind the assumed upper limit of L.

Besides the discussed criterion of minimal losses we are also interested in the possibility of shortening the duration of the investigation. Let us denote by $E(L)$ the expected duration of the investigation in the sequential procedure defined by the decision rule $x^*(\cdot)$. Of course, we should desire to construct such a decision rule which would assure a simultaneous minimum of $E(L)$ and of $F(L, x^*)$. Such intention is, however, indeterminate since it is not clear what is meant by "simultaneous minimum" of the two expressions. Usually we understand this as the demand for the minimum weighted sum of the given functions. The choice of an appropriate weights is, however, somewhat arbitrary (see [4]) and we shall take advantage of this fact when formulating the criterion of quality.

We shall apply a two-stage criterion.

(i) The first stage consists of the superior criterion to shorten the duration of the sequential investigation by making the final decision at the l-th step if the decision would also recur at the steps $l+1$, $l+2, \ldots, l+m$, $m = 1, 2, \ldots, N-1$, although it might perhaps change at some successive step $l = 1, 2, \ldots, L \leqslant N$. At the same time minimization should be assured of the linear combination of expression (2.1) and of the product $C_1 E[l(m)]$, where C_1 denotes the cost of the investigation, and $E[L(m)]$ the expected duration of the investigation as a function of the number of recurrences m. Moreover, we stress that now the probabilities P_{FA} and P_{FS} in (2.1) depend on m. This dependence will become clearer later on in this paper. Also the decision functions $x^*(\cdot)$ now depend on m which will be marked by $x^*(\cdot, m)$ and the set of all possible decision functions will be denoted by χ. Thus the superior criterion K_{I}, with which we judge the quality of the proposed test, becomes

$$(2.2) \qquad K_{\mathrm{I}} \stackrel{\mathrm{df}}{=} \min_{x^*[\cdot, m]\in\chi} \{[W_0 \mathrm{P}_{\mathrm{FA}}(m) + W_1 \mathrm{P}_{\mathrm{FS}}(m)] + C_1 E[L(m)]\}.$$

(ii) The second stage consists in the subordinate criterion to choose such a decision $x^*(\boldsymbol{y}^l)$, which belongs to the set of all decision functions $x^*(\cdot)$ with arguments from the set Ω^l of all 2^l binary sequences $y_1, y_2, \ldots$ $\ldots, y_l$ of length l, and the point in space Ω^l which minimizes the expression

$$(2.3) \qquad F(l, x^*) = W_0 \mathrm{P}'_{\mathrm{FA}} + W_1 \mathrm{P}'_{\mathrm{FS}},$$

where $\mathrm{P}'_{\mathrm{FA}}$ and $\mathrm{P}'_{\mathrm{FS}}$ represent the probabilities of errors in the decision for a given vector $\boldsymbol{y}^l$.

The adoption of the procedure described in (i) can be justified as follows:

1° Although Wald's sequential probability ratio test may be widely used in a majority of problems which require testing hypotheses taking into account costs of the investigation, this test is unacceptable in decision problems because of its arbitrary assumption concerning the probabilities of both kinds of errors in binary decisions. This is because it is incompatible with our assumed concept of optimalization.

2° The optimal properties of the probability ratio test are only preserved for the strictly specified competitive values of the tested parameter which can not be the case in the situation described here.

3° Procurement of the adaptive effect of the test for the unknown value of the parameter of the tested hypothesis on the ground of estimates obtained during the investigation is either extremely difficult or requires repeated recalculations of the whole test for the estimates at each stage.

4° The ratio test has been constructed above all for long samples (with several scores or more elements). In view of the nature of the test, arbitrary cuts can only be introduced after a sufficiently large sample has been obtained.

The characteristics 1°-4° of the ratio test cannot be accepted by us owing to physical limitations and rapidly growing costs of the investigation while l increases. The acceptance of the subordinate criterion specified in (ii), which defines the quality of a current decision, does not require an explanation (2). We only point out that, assuming unitary losses, the

(2) Detailed analysis and the choice of the two-stage criterion of optimization of a sequential investigation may be found in [2], p. 56-58 and p. 128-145.

right-hand side of (2.3) can be treated as a linear combination of the conditional risks $r[x^*(\cdot); x]$ (see Seidler [7], p. 25-26) determined by the expected conditional losses

$$r[x^*(\cdot); x] \overset{\mathrm{df}}{=} \underset{Y|x}{\mathrm{E}}\, R[x, x^*(Y^l)], \tag{2.4}$$

where $\underset{Y|x}{\mathrm{E}}$ stands for the conditional expectation of the random variable Y given $x \in X$ (cf. [1], p. 642), and $R[x, x^*(Y^l)]$ is the loss resulting from making a binary decision $x^*(Y^l)$ if x has been transmitted.

The procedure described in (ii) can be explained by the fact that the subordinate criterion (2.3) of optimization $x^*(\cdot)$ at the l-th step takes a similar form to that of the first element of the superior criterion, except that its probabilities are specified on different sets. It, therefore, minimizes a linear combination of risks specified by (2.4). Hence

$$F(l, x^*) \overset{\mathrm{df}}{=} W_0 r[x^*(\cdot); x = 0] + W_1 r[x^*(\cdot); x = 1],$$

where W_0 and W_1 are present positive weights.

We shall still consider that the losses resulting from erroneous decisions are unitary and equal, while the losses associated with decisions consistent with the actual state are equal to zero.

Owing to this, it is easy to check that the conditional risks in (2.4) reduce to the probabilities of decision errors of the I-st and II-nd kind, respectively,

$$\begin{aligned} \mathrm{P}'_{\mathrm{FS}} &= \mathrm{P}[x^*(\boldsymbol{y}^l) = 0; x = 1], \\ \mathrm{P}'_{\mathrm{FA}} &= \mathrm{P}[x^*(\boldsymbol{y}^l) = 1; x = 0], \end{aligned} \tag{2.5}$$

taken at the l-th step.

We determine probabilities (2.5) by summing (1.3) and (1.4) over appropriate decision areas

$$\mathrm{P}'_{\mathrm{FS}} = \sum_{A^l} p_1(\boldsymbol{y}^l) \quad \text{and} \quad \mathrm{P}'_{\mathrm{FA}} = \sum_{B^l} p_2(\boldsymbol{y}^l),$$

where $A^l, B^l \subset \Omega^l$ are the decision areas for decision 1 and decision 0, respectively, and $\Omega^l = A^l \cup B^l$.

The subordinate criterion K_{II} now becomes

$$K_{\mathrm{II}} \overset{\mathrm{df}}{=} \min_{A^l} \Big\{ W_0 \sum_{A^l} p_0(\boldsymbol{y}^l) + W_1 \sum_{B^l} p_1(\boldsymbol{y}^l) \Big\}. \tag{2.6}$$

We are looking for such an area A^l which would satisfy (2.6) and all previous assumptions.

Since we do note make any assumption concerning the distribution of messages $x_l \in \{0, 1\}$, further considerations will be referred to a model of an appropriately defined game.

3. Interpretation of the problem as a model of a game. We shall discuss our search for the optimal decision rule $x^*(\cdot)$ according to criterion (2.6) on the basis of a two-person recursive game, as defined by McKinsey, with partial information and complete memory. Henceforth, we shall refer to it as the game (see [5], p. 117-135 and p. 158-164). The adoption of such a model approximates sufficiently the reality since in the discussed problem there are no premises for assuming the *a priori* distribution of messages x (hence, only partial information in the game). On the other hand, every result of observation and every decision at the l-th step of the sequential procedure can be memorized. It is technically realizable. The recurrence is justified by recurrence of the message which is transmitted at least l times.

The game comprises a number of stages at which the partners play a play of the rectangular game by picking their choices.

Let the partner I be a source of elementary binary messages $x \in \{0, 1\}$. Partner II is an observer determining the binary point decisions x^*. These decisions belong to the same binary set. Both partners have some information as regards the situation at every stage of the game. The source of messages has empty information while the observer has partial information obtained from the sequence of received independent messages $y_1, y_2, \ldots, y_l$.

Both partners make $l = 1, 2, 3, \ldots, L \leqslant N$ moves (3), but y_l, which belongs to the binary set Y of the received messages, is assigned to the l-th move of the source of messages. The moves of the source of messages take the form of transmission of s_l signals uniquely assigned to the elementary messages 1 or 0. On the other hand, the observer's moves reduce to determining, at the l-th stage of the game, which of the possible messages has been transmitted. The determination is carried out on the basis of the received signals y_l. As we have done previously, we assume that source of messages repeats the moves chosen at the beginning of the game at least $L \leqslant N$ times.

Thus, after each move, the observer does not need to decide which message has been transmitted. He may regard his respective statements

(3) The value of l depends on the criterion for the end of the sequential investigation in the way to be discussed later in this paper.

$x_1(y_1), x_2(y_1, y_2), \ldots, x_l(y_1, y_2, \ldots, y_l)$ as preliminary ones, and he does not have to make the final decision till after the l-th move.

Following the first move ($l = 1$), the observer has a choice of four strategies corresponding to the decision functions (estimators)

$$x_{ij}^*(y) = \begin{cases} i & \text{if } y = 1, \\ j & \text{if } y = 0, \end{cases} \qquad i = 0, 1;\ j = 0, 1,$$

which determine what decision should be made at each result of an observation. At the same time the observer risks a payment which, for a given game, is the expected conditional loss (2.3). After an appropriate simplification (see [5], p. 46-52) and also owing to the lack of *a priori* information, the matrix M_1 of the rectangular game has the form

$$M_1 = \begin{bmatrix} 0 & \beta \\ (1-a) & 0 \end{bmatrix}, \tag{3.1}$$

where a and β are defined as in expressions (1.3) and (1.4). Form (3.1) corresponds to the first two steps in which the partner I chooses a row and the partner II chooses a column, this being equivalent to the number $l = 1$ of moves of each partner, and thus the term placed at the crossing of the chosen row and column represents the payment $K_1(x, x^*)$ for the partner I, for $W_0 = W_1$. In such a case the observer has at his disposal one mixed strategy

$$x^*(y) = \begin{cases} 1 & \text{if } y = 1, \\ 0 & \text{if } y = 0, \end{cases}$$

where the frequencies of making decision 1 or 0 depend on the statistical properties of the random variable Y_n. Note that the expected value of the payment at each play ([4]) of the rectangular game is not equivalent to the expense at the end of a given play of the recursive game.

After $l = 2$ moves of the partners, i. e. after four steps corresponding to them, we deal with an extended game (cf. [5], p. 117-135). We note that the tree of this game among $\nu_4 = 2(2\cdot 2^n - 2) + 2 = 30$ branches contains $\zeta_2 = 4(2^2 - 1) = 12$ branches assigned to the observer. These form a representation of the matrix game (3.1), where n is the number of levels of the tree, equal to the number of steps less one. The tree of an extended game, after two moves, described for a fixed strategy of the sourse of messages (e. g., for $x = 1$) takes the form as in Fig. 3.1.

([4]) We remind that at each stage of the recursive game, i. e. after every pair of moves of the partners I and II, a play of the respective rectangular game is played for a given l.

The branches of the game tree represent the respective values of the binary signals received within a two-element sequence (y_1, y_2). The apexes (y_1, y_2), $y_l = 0, 1$, are the decision points at which the observer determines $x^*(\boldsymbol{y}^2) = i$, $i = 0, 1$.

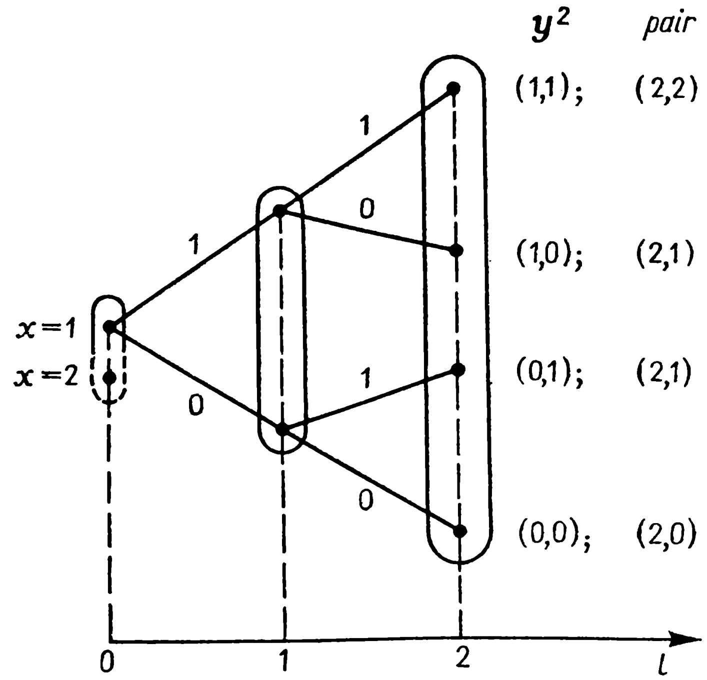


Fig. 3.1. Tree of observation vectors $\boldsymbol{y}^l$ with $l = 2$ components representing the tree of the game (for a fixed strategy of the source)

Let us demonstrate a given play in the form of the following table of results for the observer:

Observation $\boldsymbol{y}^2 = (y_1, y_2)$	Conditional probability of observation		Expected loss $\underset{Y\mid x}{\mathrm{E}}\, R[x, x^*]$			
	$\mathrm{P}_0(\boldsymbol{y}_2^2 \mid 0)$	$\mathrm{P}_1(\boldsymbol{y}^2)$	$R(1, 1)$	$R(1, 0)$	$R(0, 1)$	$R(0, 0)$
(1, 1)	β^2	$(1-\alpha)^2$	0	β^2	$(1-\alpha)^2$	0
(1, 0)	$\beta(1-\beta)$	$(1-\alpha)\alpha$	0	$\beta(1-\beta)$	$(1-\alpha)\alpha$	0
(0, 1)	$(1-\beta)\beta$	$\alpha(1-\alpha)$	0	$(1-\beta)\beta$	$\alpha(1-\alpha)$	0
(0, 0)	$(1-\beta)^2$	α^2	0	$(1-\beta)^2$	α^2	0

As can be seen, at the fourth step of the game, i. e. at the second move of the observer, he is to play one of the four rectangular games. In each of these games, with the matrices $M_2(y_1, y_2)$, the observer risks paying an amount defined by (2.1) and depending upon the payments due to the preceding and actual moves. Thus we have

$$
\begin{aligned}
M_2(1,1) &= \begin{bmatrix} 0 & \beta^2 \\ (1-\alpha)^2 & 0 \end{bmatrix}, \quad M_2(1,0) = \begin{bmatrix} 0 & \beta(1-\beta) \\ (1-\alpha)\alpha & 0 \end{bmatrix}, \\
M_2(0,1) &= \begin{bmatrix} 0 & (1-\beta)\beta \\ \alpha(1-\alpha) & 0 \end{bmatrix}, \quad M_2(0,0) = \begin{bmatrix} 0 & (1-\beta)^2 \\ \alpha^2 & 0 \end{bmatrix}.
\end{aligned} \tag{3.2}
$$

It is not difficult to notice that the rectangular games specified in (3.2) by matrices $M_2(1, 0)$ and $M_2(0, 1)$ are equivalent. Their terms situated at the non-zero diagonal depend only on the number $k = k(\boldsymbol{y}^l) \leqslant l$ of coordinates equal to one of the vectors $\boldsymbol{y}^l$ and do not depend on the places on which they occur. It is a direct result of branch symmetry of the game tree.

We shall make use of this fact and project the vectors of observation $\boldsymbol{y}^2$ onto the plane (l, k),

$$\boldsymbol{y}^2 = (y_1, y_2) \overset{\mathrm{df}}{\to} (2, k), \tag{3.3}$$

where $l(\boldsymbol{y}^2) = 2$, and $k = y_1 + y_2$.

It is obvious that a similar projection (cf. Fig. 3.1) takes place for any l. Thus, the observation space Ω^l may be projected onto the set of pairs (l, k) with the fixed first coordinate and $k = 0, 1, \ldots, l$. The set $\{(l, 0), (l, 1), \ldots, (l, l)\}$ of these images is equivalent to the set $\Omega^l = \{0, 1, \ldots, l\}$ of their second coordinates. Ambiguities resulting from the projection (3.3) are eliminated by introducing appropriate combinatorial coefficients C_l^k.

The matrices $M_l(y_1, \ldots, y_l)$, assigned to the apexes of the game tree, reduce to the matrix $M(l, k)$, $l = 1, 2, \ldots, L \leqslant N$; $k = 0, 1, \ldots, l$,

$$M(l, k) = \begin{bmatrix} 0 & \beta^k(1-\beta)^{l-k} \\ (1-\alpha)^k \alpha^{l-k} & 0 \end{bmatrix}. \tag{3.4}$$

Note that the diagonal terms of (3.4) have the sense of probability function of the observation vectors $\boldsymbol{y}^l$ with l independent elements.

We take the functions

$$p_0(l, k) \overset{\mathrm{df}}{=} \binom{l}{k} \beta^k (1-\beta)^{l-k}, \tag{3.5}$$

$$p_1(l, k) \overset{\mathrm{df}}{=} \binom{l}{k} (1-\alpha)^k \alpha^{l-k} \tag{3.6}$$

defining a projection on the plane (l, k) of the probabilities $p_i(\boldsymbol{y}^l)$, $i = 0, 1$, for vectors $\boldsymbol{y}^l$, with k elements having the value 1. Note that the right-hand side of equation (3.6) is unknown since we do not know the value of a.

In the considered recursive game, an l-term sequence of moves made by both partners, after which the observer decides what message has been transmitted and the payment $K_l(x, x^*)$ takes place, shall be referred to as "play".

We seek such a strategy $x_0^* \in \chi$ of the observer which would fulfil criterion (2.2).

The strategy satisfying this requirement shall be called an *optimal practical strategy* or an *optimal strategy* for a given game([5]).

It is not difficult to notice that the determination of an optimal strategy x_0^* in this case is equivalent to determining the decision rule $x^*(\boldsymbol{y}^l)$ which is optimal according to criterion (2.6).

4. Method of choosing an optimal strategy. Let us assume first that the values of the parameters a and β in (3.5) and (3.6) are fixed and known.

We shall define in $\{0, 1\}$ the characteristic function Ω^l which assigns binary decisions $x_l^*(\boldsymbol{y}^l)$ to the l-th apexes ($l = 1, 2, \ldots, L \leqslant N$) of a particular branch $\boldsymbol{y}^l$ of the game tree.

For this purpose we are going to investigate a given branch $\boldsymbol{y}^l$ with respect to the decisions $x_l^* = 0$ and $x_l^* = 1$. We wish to determine such $\varphi(l, k)$, $k = 0, 1, \ldots, l$, denoted briefly as φ_k^l which would assure that the right-hand side of (2.6) is minimum for any fixed $\langle l, W_0, W_1, a, \beta\rangle$. Note that, owing to (3.3) and to the properties of (3.5) and (3.6), the sum in (2.6) for any fixed l can be written in the form

$$(4.1)\qquad F(l, \boldsymbol{\varphi}^l) = \sum_{k=0}^{l} \binom{l}{k} [\varphi_k^l W_0 \beta^k (1-\beta)^{l-k} + (1-\varphi_k^l) W_1 (1-a)^k a^{l-k}],$$

where

$$\boldsymbol{\varphi}^l \stackrel{\mathrm{df}}{=} (\varphi_0^l, \varphi_1^l, \ldots, \varphi_k^l, \ldots, \varphi_l^l).$$

We conclude that the minima of (2.6) and (4.1) either both exist and are equal to each other or both do not exist. It can be easily checked that the functional $F(l, \boldsymbol{\varphi}^l)$, specified by (4.1), reaches its minimum when

$$(4.2)\qquad \varphi_k^l = \begin{cases} 1 & \text{for } W_0 \beta^k (1-\beta)^{l-k} \leqslant W_1 (1-a)^k a^{l-k}, \\ 0 & \text{for other cases.} \end{cases}$$

Besides, let us agree that we shall make binary decisions $x^*(l, k)$ of such values as those assumed by the component φ_k^l for a given pair (l, k). Thus we have, for each $l = 1, 2, \ldots, L \leqslant N$ and any $k = 0, 1, \ldots, l$, the following strategy:

$$(4.3)\qquad x_l^*(l, k) \stackrel{\mathrm{df}}{=} \varphi(l, k; W_0, W_1, a, \beta).$$

([5]) In distinction to the optimal strategy in the sense of the saddle point.

If, moreover, we agree to include in the set $A_k^l \subset \Omega_k^l$ only those k for which $\varphi_k^l = 1$ and in the set

$$B_k^l \stackrel{\mathrm{df}}{=} \Omega_k^l \setminus A_k^l$$

only those for which $\varphi_k^l = 0$, then A_k^l will be the critical sets of the functional $F(l, \varphi)$. For so defined sets A_k^l which correspond, according to the performed projection, to the formerly used sets A^l we write $F(l, A_k^l)$ instead of $F(l, \varphi)$.

Thus, for a fixed l, the functional $F(l, A_k^l)$ take the form

$$F(l, A_k^l) = \sum_{k \in A^l} W_0 \binom{l}{k} \beta^k (1-\beta)^{l-k} + \sum_{k \in B^l|} W_1 \binom{l}{k} (1-a)^k a^{l-k} \tag{4.4}$$

and has, for each l, the minimum value with respect to the division of the space Ω^l of the received signals $\boldsymbol{y}^l$ into the sets A^l and B^l.

The sets A_k^l and B_k^l can be clearly characterized by

$$\begin{aligned} A_k^l &= \{k\colon\ W_0 \beta^k (1-\beta)^{l-k} \leqslant W_1 (1-a)^k a^{l-k}\}, \\ B_k^l &= \{k\colon\ W_0 \beta^k (1-\beta)^{l-k} > W_1 (1-a)^k a^{l-k}\} \end{aligned} \tag{4.5}$$

and are equivalent to the wanted sets in expression (2.6):

$$\begin{aligned} A^l &= \{\boldsymbol{y}^l\colon\ W_0 \beta^{k(\boldsymbol{y}^l)} (1-\beta)^{l-k(\boldsymbol{y}^l)} \leqslant W_1 (1-a)^{k(\boldsymbol{y}^l)} a^{l-k(\boldsymbol{y}^l)}\}, \\ B^l &= \{\boldsymbol{y}^l\colon\ W_0 \beta^{k(\boldsymbol{y}^l)} (1-\beta)^{l-k(\boldsymbol{y}^l)} > W_1 (1-a)^{k(\boldsymbol{y}^l)} a^{l-k(\boldsymbol{y}^l)}\}. \end{aligned} \tag{4.6}$$

In this way expressions (4.2) and (4.5) define the characteristic function $\varphi(l, k)$. Consequently, the strategy (decision rule) specified by (4.3) and (4.2) is, on account of (4.1) for fixed l's, an optimal strategy. For each l and any k, owing to (4.5), we obtain

$$x_l^*(\cdot) = \begin{cases} 1 & \text{if } k(\boldsymbol{y}^l) \in A^l, \\ 0 & \text{if } k(\boldsymbol{y}^l) \in B^l, \end{cases} \tag{4.7}$$

where A^l and B^l are the sets defined by (4.6).

Since $\boldsymbol{y}^l$ is a realization of the random vector Y^l in which the number K of components having the value "one" is random, so

$$K = \sum_{n=1}^{l} Y_n.$$

The random variable K is specified in the set of arguments (l, k) and, for a fixed l, in the set of arguments k. Hence, after considering (4.7), we classify either in the set $A^l \subset \Omega^l$ or in the set $B^l \subset \Omega^l$ those $k(\boldsymbol{y}^l)$ which, for a fixed l, satisfy the conditions

$$\begin{aligned} \boldsymbol{y}^l \in A^l \quad & \text{if } k(\boldsymbol{y}^l) \geqslant T, \\ \boldsymbol{y}^l \in B^l \quad & \text{if } k(\boldsymbol{y}^l) < T, \end{aligned} \tag{4.8}$$

where T is the threshold dividing the set of the values $k(\boldsymbol{y}^l)$ into two subsets contained in A_k^l and B_k^l, respectively. For a variable l, the threshold T is the function $T(l)$. It is evident from (4.7) that $T(l)$ has the sense of a decision threshold.

By introducing the threshold $T(l)$ into expressions (4.1) and (4.7), we eventually obtain, for each l, an optimal current strategy of the observer,

$$x_l^*(\cdot) = \begin{cases} 1 & \text{if } k(\boldsymbol{y}^l) \geqslant T(l), \\ 0 & \text{if } k(\boldsymbol{y}^l) < T(l), \end{cases} \tag{4.9}$$

and also a modified form of the functional $F(l, A_k^l)$,

$$F(l, A_k^l) = \sum_{k=T}^{l} W_0 \binom{l}{k} \beta^k (1-\beta)^{l-k} + \sum_{k=0}^{T-1} W_1 \binom{l}{k} (1-\alpha)^k \alpha^{l-k}, \tag{4.10}$$

where W_0, W_1, α and β are fixed parameters.

5. Determination of the end of a play. We now have to determine when the observer should make his final decision, i. e. at which l should his current decision (4.9) be declared the final decision of a given play. We proceed to determine the value of l at which we end a given play.

After the l-th move the respective $k(\boldsymbol{y}^l)$ belong to A_k^l or B_k^l. A decision $x_l^*(l, k)$, equal to 1 or 0, is associated with this move. Depending on whether $k(\boldsymbol{y}^{l+1})$ will belong to A_k^{l+1} or to B_k^{l+1}, we shall either keep the decision unchanged or we shall make a reverse decision. One can easily imagine that, if $k(\boldsymbol{y}^l) \in A_k^l$ (or, respectively, to B_k^l), there exist situations such that $k(\boldsymbol{y}^{l+1})$ will still belong to A_k^{l+1} (or, respectively, to B_k^{l+1}). In such situations the next move will not change the decision made at the previous step. We establish that in such a case we end our play at the l-th move [6]. In other words, we end a given play when there is no chance of changing the current decision made at the l-th step at any one of the immediately succeeding steps of the investigation. The l at which we end the investigation is denoted by l_0.

On the basis of (4.9) the observer strategy $x_0^*(\cdot) = x_{l_0}^*(l_0, \cdot)$ which is optimal on account of criterion (2.3) is, for respective weights W_0, W_1 and C_1, described by the expression

$$x_0^*(\cdot) = \begin{cases} 1 & \text{if } x_l^*(l, k) = 1 \text{ and } x_l^*(l, k) = x_{l+1}^*(l+1, k), \\ 0 & \text{if } x_l^*(l, k) = 0 \text{ and } x_l^*(l, k) = x_{l+1}^*(l+1, k) \end{cases} \tag{5.1}$$

[6] One can easily imagine that we end the game after the l-th move when we have the situation that if $k(\boldsymbol{y}^l) \in A_k^l$, then also $k(\boldsymbol{y}^{l+i}) \in A_k^{l+i}$ for $i = 1, 2, \ldots, m$ (and, analogously, for the set B_k^l). However, in this paper we are only interested in the case $m = 1$. This fact does not mean that our considerations are less general for an optimal value of m can be specified numerically from equation (2.3) as explained in [2], p. 71-73.

for the first $l = 1, 2, \ldots, L \leqslant N$ at which it occurs. If none of the cases in (5.1) occurs for $l = L$, then we make the decision $x_L^*(L, k) = 0$. In expression (5.1) the equality

$$x_l^*(l, k) = x_{l+1}^*(l+1, k+1)$$

occurs when

$$x_l^*(l, k) = x_{l+1}^*(l+1, k+0) = x_{l+1}^*(l+1, k+1).$$

The final decision specified by (5.1) ends a given play at the move $l = l_0$. For such an l, expression (4.10) determines the payment $K_{l_0}(x, x^*)$.

6. Estimation of the parameter a. In the previous sections 3 and 4 we have assumed that both parameters a and β are fixed and known. Now, according to the assumptions regarding the model of the game, we find that only β can be regarded as known. On the other hand, the value of the parameter a is unknown to us, and the necessary assessing of it should be included in the observer's strategy. Doing this, we shall retain the assumption that the parameter a, as defined by expression (1.4), has a constant value for all the moves l.

We apply a scheme of estimation of the parameter a based on gradual learning of its value by observing the moves of the source of messages or, otherwise, by sequential observation of the received signal $\boldsymbol{y}^l$. We shall introduce an estimate of the parameter instead of its true value into (4.5) and (4.10).

As is well known, in such a scheme it is necessary to introduce a condition determining the point at which the algorithm of estimation starts to estimate the value of the unknown parameter a of function $p_1(l, k)$. We assume that the occurrence in the vector $\boldsymbol{y}^l$ of a present number $r = 1, 2, \ldots, l$ of successively observed ones constitutes such an arbitrary condition ([7]).

At this point it occurs to us that the choice of r can bear some relation to the value of m determining the end of a given play. Since the study of this relation as well as of its influence on the observer's strategy is a separate problem, we shall not deal with it in this paper.

Due to the assumptions made in this paper, the sufficient and most effective estimate a^* of the parameter a is, as it is well known, the mean of a sample $\boldsymbol{y}^l$ with binary components

$$a^*(\boldsymbol{y}^l) = \frac{l-k}{k}, \tag{6.1}$$

([7]) In fact, we learn the value of a parameter γ which is either a or β, because we cannot be fully certain when a occurs and when does β. Thus there exists the probability $P_{FA}^{(r)}$, $r = 1, 2, \ldots, l$, that we shall wrongly learn the value of β instead of a.

where k is the number of ones in the vector of observation $\boldsymbol{y}^l$, and l the number of components of the vector $\boldsymbol{y}^l$.

The estimate a^* described by (6.1) is stochastically convergent to the true value a. The actual value $a^*(l)$ determines, through (4.5) and (4.9), the division of the space Ω^l into the sets A^{*l} and B^{*l}, and thereby the value of the functional $F(l, A^{*l})$ in the manner shown in Fig 6.1.

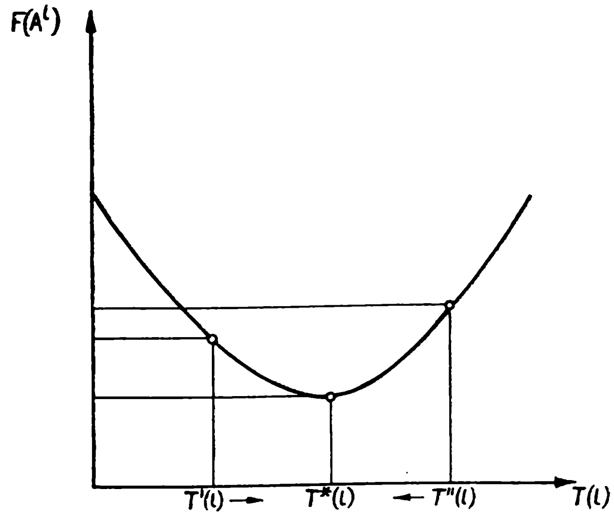


Fig. 6.1. The diagram of $F(A^l) = F(l, A^l)$ as a function of the threshold; W_0, W_1 and l are fixed

Since in practical applications the diagram of $F(A^l)$ is usually asymmetrical, it is relevant from which side the estimate $T(l)$ is convergent to $T^*(l)$. Let us construct a confidence interval

$$\mathrm{P}[\underline{a}^*(l) \leqslant a < \bar{a}^*(l)] \leqslant 1-\varepsilon \tag{6.2}$$

which determines the random relations between the true value boundaries of the confidence interval are one-sidely convergent to the estimated parameter a, where the assurance is given by the confidence level $1-\varepsilon$.

Let us agree to insert the lower limit $\underline{a}^*(l)$ of the estimate of the parameter a, instead of the actual parameter, into expressions (4.5) and (4.10). In this way we obtain the critical sets A_k^{*l} and $B_k^{*l} \subset \Omega_k^l$ while the observer's strategy described by (4.7), in such a case, takes the form

$$x_{l_a}^*(\cdot) = \begin{cases} 1 & \text{if } k(\boldsymbol{y}^l) \in A_k^{*l}, \\ 0 & \text{if } k(\boldsymbol{y}^l) \in B_k^{*l}. \end{cases} \tag{6.3}$$

Of course, the sets A_k^{*l} and B_k^{*l} depend also on the confidence level $1-\varepsilon$. The choice of $1-\varepsilon$ has to be based on operational economic considerations, and we, henceforth, assume that it has been taken together with the choice of appropriate weights W_0 and W_1. Having inserted (for respective l) into (5.1) the values $x^*_{l_a}(\cdot)$ from (6.3) to take the place of $x^*_l(\cdot)$ which are specified by (4.7) and which figured in (5.1) up to now, we obtain the final decision $x^*_a(\cdot)$. Therefore, we have

$$(6.4)\qquad x^*(\cdot) = \begin{cases} 1 & \text{if } x^*_{l_a}(l, k) = 1 \text{ and } x^*_{l_a}(l, k) = x^*_{l_a+1}(l+1, k), \\ 0 & \text{if } x^*_{l_a}(l, k) = 0 \text{ and } x^*_{l_a}(l, k) = x^*_{l_a+1}(l+1, k). \end{cases}$$

The value of the parameter a is estimated with the help of (6.1) for every l-th move. It is easily noticed that the value by which this estimation changes from one move to the next one influences the decision whether or not to end the play. One can, therefore, say that strategy (6.4) of the observer has an adaptive ability — it can adapt itself to the unknown value of a in the probability distribution $p_1(\boldsymbol{y}^l)$ of a given observation.

7. Some properties of the proposed sequential test. Our considerations determine (in some sense) an asymptotically optimal strategy of the observer. This strategy has adaptive properties. It has been found for a model game that represents to a good degree the considered decision problem. Thus, instead of referring to the observer's strategy, we can speak of an adaptive decision rule constructed for a sequential investigation or of an adaptive sequential lest.

The operation of the adaptive sequential test can, having W_0, W_1, a and β, be presented graphically, as in Fig. 7.1. The figure indicates that

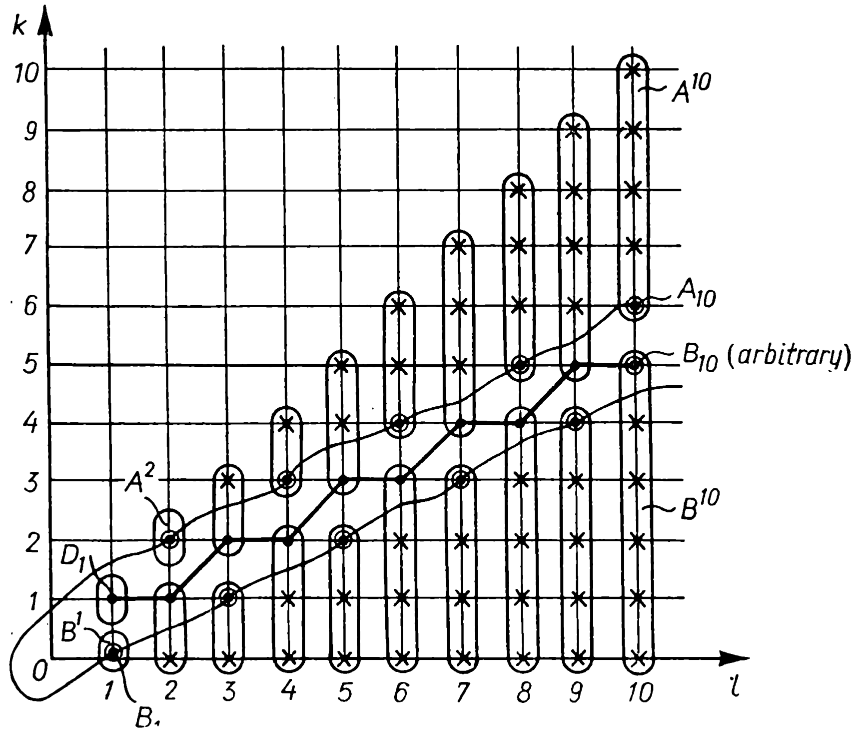


Fig. 7.1. The operation of decision rule for fixed W_0, W_1, a and β ($a = \beta$; $W_0 = W_1 = 1$; $L = 10$)

⊙ — decision points, × × × — regions of elimination, ⊙—⊙—⊙ — decision barrier, ●—●—● — region of no decision the decision points, i. e. the points at which after l steps there first appears a final decision, form the respective barriers of the region of decisions 1 (barrier A) and of the region of decisions 0 (barrier B). The barriers eliminate the points that lie above and below them because observation sequences end when they reach one of the barriers which depend on the weights W_0 and W_1 and on the parameters α and β. The region D of no decision also depends on these quantities. There occurs the relation

$$A \cup B \cup D = \Omega,$$

where Ω is a space of observation vectors, defined as in section 2.

The region D is described by a vector $\boldsymbol{y}^L$ which has L components and at which we cannot decide where we should classify a given observation sequence. Therefore, it ensues that the probability of no decision after the l-th step of the sequential investigation decreases geometrically with increasing l and for a sufficiently large l decreases to zero. Now, we procede with working out a way of determining decision regions such that it would suit the previously indicated applications.

The regions A, B, $D \subset \Omega$ can be described in recursive form. For this purpose we introduce an auxiliary function $\pi(l, k)$, defined as follows:

$$\pi(l, k) = \begin{cases} 1 & \text{if } \varphi_k^l = 1,\ \varphi_k^{l+1} = 1,\ \varphi_{k+1}^{l+1} = 1, \\ 0 & \text{if } \varphi_k^l = 0,\ \varphi_k^{l+1} = 0,\ \varphi_{k+1}^{l+1} = 0, \\ -1 & \text{for other cases.} \end{cases} \tag{7.1}$$

The case $\pi(l, k) = -1$ in expression (7.1) signifies the lack of final decision after the l-th step and, therefore, the necessity to continue the investigation. The region A of final decisions 1 is an alternative of subregions A_l (different from A^l) in which we assign the decision $x^* = 1$ to a given observation vector $\boldsymbol{y}^l$ with l components,

$$A = \bigcup_{l=r}^{L} A_l, \qquad r = 1, 2, \ldots, L \leqslant N,$$

where, as it is shown in Fig. 6.1, for $r = 2$ the subregion (8) $A_2 \overset{\text{df}}{=} \{(2,2)\}$ and, for $l > 2$, $A \ = \emptyset$. For an arbitrary r, there is

$$A_{l+1} = \{(l, k)\colon [\pi(l+1, k+1) = 1] \text{ and } [(l, k) \in D_l \text{ or } (l, k+1) \in D_l]\}.$$

The region D of no decision is an alternative of subregions D_l in which we assign -1 to a given observation vector $\boldsymbol{y}^l$, i. e. we continue the investigation,

$$D = \bigcup_{l=r}^{L} D_l, \qquad r = 1, 2, \ldots, L \leqslant N,$$

(8) Note that the parameter r corresponds to the arbitrary condition of beginning of estimating $\alpha^*(l)$ introduced in section 6 of this paper. For the case of a fixed α, $r \overset{\text{df}}{=} 1$, which is easy to understand.

where, as it is evident from Fig. 7.1, for $l = 1$ the subregion

$$D_1 \overset{\text{df}}{=} \{(1, 1)\}$$

and

$$D_{l+1} = \{(l, k)\colon [\pi(l+1, k+1) = 1 \text{ or } \pi(l+1, k+1) = 0] \text{ and } [(l, k)\in D_l \text{ or } (l, k+1)\in D_l]\}.$$

The region B of decisions 0 is an alternative of subregions B_l, in which we assign $x^* = 0$ to a given vector $\boldsymbol{y}^l$,

$$B = \bigcup_{l=r}^{L} B_l, \quad r = 1, 2, \ldots, L \leqslant N,$$

where, as Fig. 7.1 signifies, for $l = 1$ the subregion

$$B_1 \overset{\text{df}}{=} \{(1, 0)\}$$

and

$$B_{l+1} = \{(l, k)\colon [\pi(l+1, k+1) = 0] \text{ and } [(l, k)\in D_l \text{ or } (l, k+1)\in D_l]\}.$$

Besides that, the region $B_{l+1} = \emptyset$ and is an empty subregion.

For fixed W_0, W_1, a and β, the characteristics of the decision barriers and the region of no decision are fixed. In particular, for each l there exist specified and constant critical sets $A^l, B^l \in \Omega^l$ determined by (4.5).

The above-mentioned properties hold also for the case where the value of the parameter a is unknown. This is so because the algorithm of the final decision (5.1) for each l determines the critical sets $A^{*l}, B^{*l} \subset \Omega^l$ taking into account the upper or lower boundary $\underline{a}^*(l)$ of the estimate of a (cf. section 6). It is easy to notice that the sets A^{*l} and B^{*l} are determined for each a if W_0, W_1, r and β are fixed. The latter condition is easy to satisfy in practical applications. Namely, the sets A_{l+1}, B_{l+1} and D_{l+1} are depending of estimation $\underline{a}^*_{l+1}(k)$.

The operation of decision rule (6.3) at a constant value of the parameter a is illustrated by the example in Fig. (7.2).

Note that the space of no decision narrows with increasing l. After several moves, e.g. $l = 12$, the space is transformed into a single sequence. Thus, the properties of the decision rule constructed with ignorance of the value of the parameter a, after a certain period of adaptation, coincide with the properties of the test constructed for the known value of this parameter, to the accuracy of a set of zero measure.

The period of adaptation is random and depends on that l from which on the number of paths belonging to the space D is not larger than one (see Fig. 7.2).

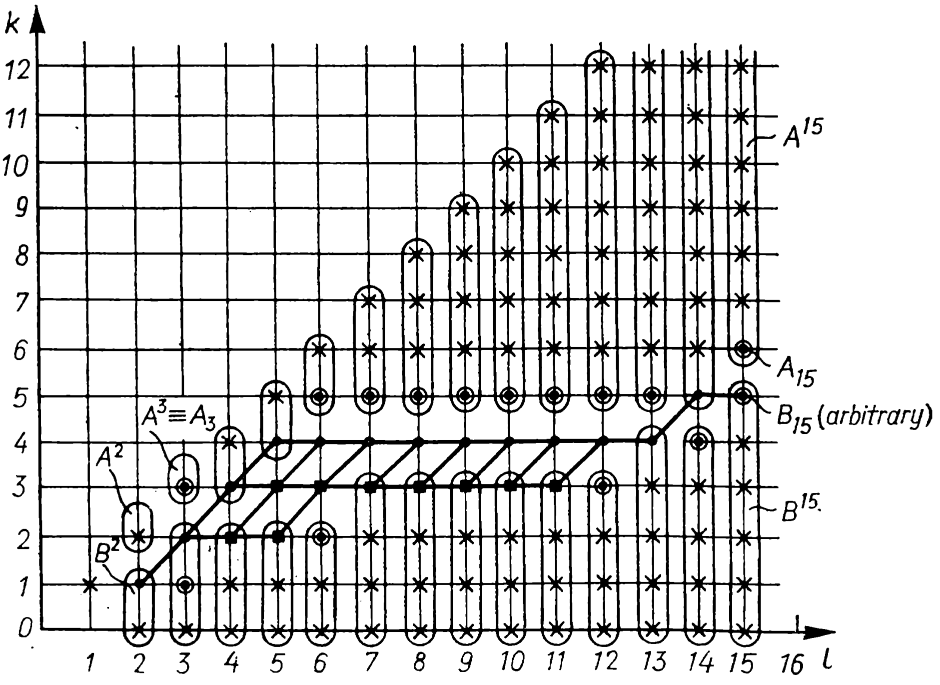


Fig. 7.2. The operation of decision rule (6.3) for unknown a ($W_0 = W_1 = 1$; $\beta = 0,1$; $r = 2$; $L = 15$)

⊙ — decision points, ●—●—● — space of no decision, ■ — singular points

8. Conclusion. During the construction of the proposed test we were minimizing the value of risk $F(l, x^*)$, determined by (2.3), applying a better estimate $a^*(l)$ at every step. In this meaning the adaptive sequentional test is asymptotically optimal.

Referring to features 1°-4° presented in section 2, which characterize the usefullness of the ratio test, we find that:

(i) When applying the adaptive test, we only assume the probability of error of 2-nd kind.

(ii) Asymptotic optimal properties of the adaptive test are also kept for non-parametric hypotheses. This is because there is no need to assume a value of the parameter a which can be of any value within the range $0 < a < 1$.

(iii) Inserting of the lower limit of estimation $\underline{a}^*(l)$ instead of parameter a in expression (5.1) is not difficult.

(iv) The rapid convergence of the effectiveness of the decision procedure for $\underline{a}^*$ to that with a known is of particular interest if the available samples contain only some 10 or 20 elements.

(v) Decision rule (6.4) is determined for any values of $\langle W_0, W_1, \beta, r, c_1\rangle$ and any $0 < a < 1$. It is, therefore, suitable for direct realization in the form of a finite automaton (cf. [6], p. 24-34). This has been verified in practice and registered in the Patent Office(⁹). Statistical properties of the rule (6.4), called *adaptive sequential test* in the sense of Wald [8], as well as a comparison of the adaptive test with classical sequential tests will be presented by the author in [3].

Throughout all this paper we have dealt only with binary random variables Y_n (received sygnals). The whole construction can be easily extended (and really was) for the case of more possible values or even continuous random variables Y_n. In the latter case instead of probabilities one has to deal with probability densities of the distribution of Y_n.

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PEWIEN ADAPTACYJNY TEST SEKWENCYJNY

STRESZCZENIE

W pracy proponuje się pewien adaptacyjny test sekwencyjny i analizuje jego własności.

W kolejnych chwilach $i = 1, 2, \ldots$ odbierane są sygnały Y_i, będące realizacjami niezależnych zmiennych losowych o wartościach 0 i 1. Rozkład prawdopodobieństwa tych zmiennych zależy od tego, czy pewien parametr x przyjmuje wartość 0 czy 1. W pracy definiuje się pewien sekwencyjny estymator parametru x, oparty na obserwacjach Y_i. Estymator ten ma, zdaniem autora, pewne zalety w porównaniu z sekwencyjnym testem ilorazowym i z testem opartym na z góry ustalonej liczbie obserwacji.

(⁹) See patent No. 62589 at the Patent Office, Poland.