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ALGORITHM 2

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CALCULATION OF A RATIONAL FUNCTION DEFINED BY INTERPOLATION CONDITIONS

1. Procedure declaration.

procedure *ratint*$(l, x, m, y, notex)$;
  integer l, m;
  array x, y;
  label *notex*;
  comment The procedure *ratint* finds the rational function $R(x)$ such that

(1) $$R(x_k) = y_k \quad (k = 0, 1, \ldots, l+m).$$

Data:

l, m — the degrees of the nominator $L(x)$ and denominator $M(x)$, respectively, of the function $R(x)$,
$x[0:l+m]$ — array of pairwise different arguments of $R(x)$,
$y[0:l+m]$ — array of values of $R(x)$.

Results:

l, m — numbers not exceeding the input values of l and m, respectively, being the true degrees of the nominator and denominator of the calculated function $R(x)$,
$y[0:l]$ — coefficients of $x^l, x^{l-1}, \ldots, 1$ of the nominator $L(x)$,
$x[0:m]$ — coefficients of $x^m, x^{m-1}, \ldots, 1$ of the denominator $M(x)$
(Remark: during the performance of procedure *ratint* the values of $x[m+1:l+m]$ and $y[l+1:l+m]$ are usually also changed).

Other parameters:

notex — label of the statement (outside the procedure body) where exit is made if the function $R(x)$ does not exist;

begin
  integer $i, j, kmin, kmax, lm, n$;
  real $yi, yj, ymin, ymax$;

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integer array r[0:l+m];
n := l+m;
for i := n step -1 until 1 do
  begin
    kmin := kmax := 0;
    ymin := ymax := abs(y[0]);
    for j := 1 step 1 until i do
      begin
        yj := abs(y[j]);
        if yj < ymin
          then begin
                 ymin := yj;
                 kmin := j
               end yj < ymin
          else if yj > ymax
               then begin
                      ymax := yj;
                      kmax := j
                    end yj > ymax
      end j;
    if ymin = 0
      then j := 1
      else if l ≥ m
          then begin
                 ymax := abs(y[kmax]-y[kmin]);
                 j := 2
               end ymin > 0 ∧ l ≥ m
          else begin
                 j := l;
                 l := m;
                 m := j;
                 for j := 0 step 1 until i do
                   y[j] := 1/y[j];
                 ymax := abs(y[kmin]-y[kmax]);
                 kmin := kmax;
                 j := 3
               end ymin > 0 ∧ l < m;
    r[i] := j;
    yj := x[kmin];
    x[kmin] := x[i];
    x[i] := yj;
    ymin := y[kmin];
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            y[kmin] := y[i];
            y[i] := ymin;
            if ymax = 0
              then go to p2;
            if l = 0
              then go to notex;
            l := l-1;
            for j := i-1 step -1 until 0 do
              y[j] := (y[j]-ymin)/(x[j]-yj)
          end i;
p2:   lm := l+m+1;
      l := m := 0;
      x[0] := 1;
      for i := lm step 1 until n do
        begin
          yi := y[i];
          yj := x[i];
          ymin := y[l];
          y[l+1] := -ymin×yj;
          for j := l step -1 until 1 do
            begin
              ymax := y[j-1];
              y[j] := ymin-ymax×yj;
              ymin := ymax
            end j;
          l := l+1;
          kmax := r[i];
          if kmax > 1
            then begin
                   kmin := l-m;
                   for j := kmin step 1 until l do
                     y[j] := y[j]+yi×x[j-kmin];
                   if kmax = 3
                     then begin
                            for j := 0 step 1 until l do
                              begin
                                ymin := x[j];
                                x[j] := y[j];
                                y[j] := ymin
                              end j;
                            j := l;
                            l := m;
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                m := j
              end r[i] = 3
          end r[i] > 1
    end i
  end ratint
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2. Method used. The procedure *ratint*, the declaration of which is given in § 1, solves the following problem of rational interpolation:

Given nonnegative integers l and m, pairwise different numbers $x_0, x_1, \dots, x_n$ (where $n = l+m$) and numbers $y_0, y_1, \dots, y_n$, it is to find the coefficients of the polynom $L(x)$ of degree at most l and of the polynom $M(x)$ of degree at most m such that the rational function

$$R(x) = \frac{L(x)}{M(x)}$$

satisfies

$$R(x_k) = y_k \quad (k = 0, 1, \dots, n). \tag{1}$$

This problem is solved by the following nearly obvious algorithm (which, however, was published probably only in [1] for the first time):

The algorithm begins with a transformation of the system (1).

1° If there exists any j such that $y_j = 0$, then (1) is replaced by the system

$$\frac{L_1(x_k)}{M(x_k)} = \frac{y_k}{x_k - x_j} \quad (k = 0, 1, \dots, j-1, j+1, \dots, n),$$

where $L_1(x)$ is a polynom of at most degree $l-1$ and such that

$$L(x) = (x - x_j) L_1(x). \tag{2}$$

2° If $y_k \neq 0$ for $k = 0, 1, \dots, n$ and $l \geqslant m$, then for any chosen j $(0 \leqslant j \leqslant n)$ the system (1) is replaced by the system

$$\frac{L_2(x_k)}{M(x_k)} = y_k - y_j \quad (k = 0, 1, \dots, n),$$

which satisfies the assumptions of case 1°. $L_2(x)$ is a polynom of at most degree l and such that

$$L(x) = L_2(x) + y_j M(x). \tag{3}$$

3° If $y_k \neq 0$ for $k = 0, 1, \dots, n$ and $l < m$, then (1) is replaced by the system

$$\frac{M(x_k)}{L(x_k)} = \frac{1}{y_k} \quad (k = 0, 1, \dots, n),$$

satisfying the assumptions of case 2°.

Similarly, one transforms the system which results immediately from (1) and the following systems till one obtains one equation with a rational function with nominator and denominator, both of zero degrees. This part of the calculations is performed by the first for statement in the procedure declaration. In case 2° the number j is chosen such that

$$|y_j| = \min_k |y_k|.$$

This part verifies also if the function $R(x)$ satisfying (1) exists. It does not exist in the case 1° if $\max_k |y_k| > 0$ and $l = 0$, and also if similar inequalities are satisfied for any system of equations obtained from (1) through reduction by the above mentioned method.

After repeated use of formulas (2) and (3) and, eventually (in case 3°), after an exchange of the nominator and denominator, the second part of the calculations (beginning with label *p2* in the procedure declaration) yields the coefficients of the rational functions satisfying the reduced equations systems obtained from (1), finally, gives the coefficients of the sought function $R(x)$.

3. Certification. The procedure *ratint* has been verified on the Elliott 803 computer. Results of the control calculations are given in the table.

TABLE

l, m data	l, m results	x_k	y_k	$R(x)$
1, 2	1, 2	0	-2	$\dfrac{-1.1428571x+2.2857143}{-1.1428571x^2-1.1428572x-1.1428571} \approx \dfrac{x-2}{x^2+x+1}$
		1	$-.333333333 \approx -\frac{1}{3}$	
		4	$.095238095 \approx \frac{2}{21}$	
		6	$.093023256 \approx \frac{4}{43}$	
3, 1	1, 0	0	-6	$2x-6$
		1	-4	
		3	0	
		4	2	
		7	8	
0, 1		2	0	does not exist
		5	16	

Reference

[1] S. Paszkowski, *Zbiór zadań z teorii metod numerycznych*, part 1, Łódź 1969.

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ALGORYTM 2

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OBLICZANIE WSPÓŁCZYNNIKÓW FUNKCJI WYMIERNEJ OKREŚLONEJ WARUNKAMI INTERPOLACYJNYMI

STRESZCZENIE

Procedura *ratint* znajduje funkcję wymierną $R(x)$ określoną warunkami (1).

Dane:

l, m — odpowiednio stopień licznika $L(x)$ i mianownika $M(x)$ funkcji $R(x)$,

$x[0:l+m]$ — tablica argumentów funkcji $R(x)$, liczby parami różne,

$y[0:l+m]$ — tablica wartości funkcji $R(x)$.

Wyniki:

l, m — liczby nie większe odpowiednio od wejściowych wartości l i m, prawdziwe stopnie licznika i mianownika znalezionej funkcji $R(x)$,

$y[0:l]$ — współczynniki licznika $L(x)$ przy $x^l, x^{l-1}, \ldots, 1$,

$x[0:m]$ — współczynniki mianownika $M(x)$ przy $x^m, x^{m-1}, \ldots, 1$

(Uwaga: podczas wykonania procedury *ratint* na ogół zmieniają się również wartości zmiennych $x[m+1:l+m]$ i $y[l+1:l+m]$.)

Inne parametry:

notex — etykieta instrukcji (poza treścią procedury), do której następuje skok, gdy funkcja $R(x)$ nie istnieje.

Metoda użyta w procedurze *ratint* jest zapewne od dawna znana, ale opublikowano ją po raz pierwszy chyba dopiero w [1]. Opisano ją krótko w § 2. Obliczenia kontrolne wykonane na maszynie cyfrowej Elliott 803 (§ 3) wykazały poprawność procedury.

АЛГОРИТМ 2

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ВЫЧИСЛЕНИЕ КОЭФФИЦИЕНТОВ РАЦИОНАЛЬНОЙ ФУНКЦИИ ОПРЕДЕЛЕННОЙ ИНТЕРПОЛЯЦИОННЫМИ УСЛОВИЯМИ

РЕЗЮМЕ

Процедура *ratint* вычисляет рациональную функцию $R(x)$ определенную условиями (1).

Данные:

l, m — соответственно степень числителя $L(x)$ и знаменателя $M(x)$ функции $R(x)$,

$x[0:l+m]$ — массив аргументов функции $R(x)$, числа попарно разные,

$y[0:l+m]$ — массив значений функции $R(x)$.

Результаты:

l, m — числа не превышающие соответственно входных значений l и m, точные степени числителя и знаменателя найденной функции $R(x)$,

$y[0:l]$ — коэффициенты числителя $L(x)$ при $x^l, x^{l-1}, \ldots, 1$,

$x[0:m]$ — коэффициенты знаменателя $M(x)$ при $x^m, x^{m-1}, \ldots, 1$

(Замечание: Во время выполнения процедуры *ratint* обычно изменяются также значения переменных $x[m+1:l+m]$ и $y[l+1:l+m]$).

Остальные параметры:
notex — метка оператора (вне тела процедуры), которому передается управление, если функция $R(x)$ не существует.

В процедуре *ratint* применяется метод, который вероятно уже давно известен, но опубликован в первый раз по видимому только в [1]. Этот метод кратко описан в § 2. Контрольные вычисления (§ 3) выполнены на ЭВМ Эллиотт 803.