ZASTOSOWANIA MATEMATYKI
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A. OLEJNICZAK (Wrocław)

CRANK-NICOLSON-GALERKIN APPROXIMATION OF THE PERIODIC SOLUTIONS OF WEAKLY NONLINEAR PARABOLIC EQUATIONS

1. Introduction. Let Ω be a domain in R^n with the boundary Γ, and let T be a positive constant. Let V be a linear subspace of $H^1(\Omega)$ satisfying $H_0^1(\Omega) \subset V \subset H^1(\Omega)$. The paper deals with the numerical approximation of the solution of the following parabolic equations

(1.1a) $$u_t + A(x, t, u) = f(x, t, u), \quad (x, t) \in \Omega \times (0, T),$$

(1.1b) $$u(\cdot, t) \in V, \quad t \in [0, T],$$

with the periodic condition

(1.1c) $$u(\cdot, 0) = u(\cdot, T).$$

In (1.1a), $A(x, t, u)$ is a weakly nonlinear elliptic operator of second order in the space-variable x, defined in the cylindrical domain $Q = \Omega \times (0, T)$. We assume that the solution u of (1.1) exists and $u \in C^3(\bar{Q})$.

In [2] the problem was studied by the least-squares method for the Laplace operator. Recently, the linear case of (1.1) was studied in [1] by using Fourier expansions.

There are many papers (e.g., [4], [7] and the references given therein), in which a nonlinear initial value problem was studied. Here, instead of (1.1c), one considers the case

$$u(\cdot, 0) = g.$$

Following [4], we use the Crank–Nicolson–Galerkin method for approximation of the periodic solution of problem (1.1). We prove that this scheme is solvable. Also, we give sufficient conditions for the method of successive approximation to be applied. Some estimations of the error are given.

An outline of this paper is as follows. In Section 2 we give the basic definitions. In Section 3, we formulate the Crank–Nicolson–Galerkin scheme and prove that the estimation of the error may be reduced to an approximation problem. Section 4 is devoted to estimating the error for a particular finite-dimensional subspace of V. In Section 5 we consider, in more detail, the linear case and finally, in Section 6 we discuss the existence of the solution of a system of nonlinear equations connected with the presented method.

2. Basic definitions and notations.

A. Let Ω be a bounded domain in R^n with a boundary Γ. We assume that there exists a finite covering $\bar{\Gamma} \subset \bigcup_{j=1}^{p} U_j$ and positive numbers α, β such that the set $\Gamma \cap U_j$ may be, in a suitable given coordinate system in R^{n-1}, described by the equation

$$x_n = \varphi_j(x')$$

where

(i) $x' \in \Delta_j = \{x' \in R^{n-1}\colon |x_s| < \alpha,\ s = 1, 2, \ldots, n-1\}$,

(ii) φ_j is Lipschitz-continuous in $\bar{\Delta}_j$,

(iii) $\{(x', x_n)\colon x' \in \Delta_j,\ \varphi_j(x') < x_n < \varphi_j(x') + \beta\} \subset \Omega$,

(iv) $\{(x', x_n)\colon x' \in \Delta_j,\ \varphi_j(x') - \beta < x_n < \varphi_j(x')\} \subset R^n \setminus \Omega$.

The boundary Γ satisfying these conditions is called *Lipschitz-continuous* [6].

B. All the considered functions are real-valued. All the derivations in the sequel are understood in the distributional sense. Let us denote by $\langle \cdot, \cdot \rangle$ the inner product in $L^2(\Omega)$ and introduce the following notations

$$\begin{aligned} \|v\|_0^2 &= \langle v, v \rangle, && v \in L^2(\Omega), \\ |v|_1^2 &= \sum_{i=1}^{n} \left\langle \frac{\partial v}{\partial x_i}, \frac{\partial v}{\partial x_i} \right\rangle, && \frac{\partial v}{\partial x_i} \in L^2(\Omega), \\ \|v\|_1^2 &= \|v\|_0^2 + |v|_1^2, && v \in H^1(\Omega). \end{aligned}$$

We will be working with the Hilbert space $H^1(\Omega)$. In our assumptions on Ω there exists the trace operator tr: $v \in H^1(\Omega) \to \operatorname{tr} v \in L^2(\Gamma)$ [6] and for $\Gamma_0 \subset \Gamma$ with $\operatorname{meas}(\Gamma_0) > 0$, we may put

$$V = \{v \in H^1(\Omega)\colon \operatorname{tr} v|_{\Gamma_0} = 0\}. \tag{2.1}$$

We have the following

LEMMA 2.1 ([3]). *Let Ω be a bounded domain in R^n with the Lipschitz-continuous boundary Γ. Then the set V defined by* (2.1) *is a closed subspace of* $H^1(\Omega)$. *Additionally, if the measure of Γ_0 is positive then the seminorm $|\cdot|_1$ is equivalent to the norm $\|\cdot\|_1$.* ■

Let the space $L^2(0, T; V)$ denote the set of all functions Φ: $[0, T] \to V$, for which the norm

$$\|\Phi\|^2_{L^2(0,T;V)} = \int_0^T \|\Phi(t)\|^2_V \, dt$$

is finite.

We introduce the following space

$$W(0, T) = \left\{u \in L^2(0, T; V): \frac{\partial u}{\partial t} \in L^2(0, T; H^1(\Omega))\right\}.$$

We have

LEMMA 2.2 ([6]). *For $v \in W(0, T)$ the traces $v(\cdot, 0)$ and $v(\cdot, T)$ are well defined.* ■

Thus, we may introduce the subset of T-periodic functions in $W(0, T)$

$$\tilde{W}(0, T) = \{u \in W(0, T): u(\cdot, 0) = u(\cdot, T)\}.$$

C. We assume that a_{ij}, a_0 and f are in $L^2(Q \times R)$ and for every $w \in V$ is $a_{ij}(\cdot, \cdot, w)$, $a_0(\cdot, \cdot, w)$, $f(\cdot, \cdot, w) \in L^2(Q \times R)$. The nonlinear elliptic operator A is defined by

$$A(x, t, w)u = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(x, t, w)\frac{\partial u}{\partial x_i}\right) + a_0(x, t, w)u, \tag{2.2}$$

where the coefficients are

(i) symmetric, i.e. $a_{ij}(x, t, w) = a_{ji}(x, t, w)$,

(ii) T-periodic, i.e. $a_{ij}(x, t+T, w) = a_{ij}(x, t, w)$, $a_0(x, t+T, w) = a_0(x, t, w)$,

(2.3) (iii) bounded, i.e. $|a_{ij}(x, t, w)| \leqslant N$, $0 < m \leqslant a_0(x, t, w) \leqslant N$,

(iv) and they satisfy

$$\delta_0 \sum_{i=1}^{n} \xi_i^2 \leqslant \sum_{i,j=1}^{n} a_{ij}(x, t, w)\xi_i\xi_j \leqslant c_0 \sum_{i=1}^{n} \xi_i^2 \quad (c_0, \delta_0 > 0).$$

Of the right-hand side of (1.1a) we suppose that it is

(2.4) (i) T-periodic, i.e. $f(x, t+T, w) = f(x, t, w)$,

(ii) bounded, i.e. $|f(x, t, w)| \leqslant N$.

For simplicity of further notations we put

$$\delta = \min\{\delta_0, m\} \quad \text{and} \quad C = \sqrt{2}\max\{Nn, 1\}.$$

Moreover, we assume that a_{ij}, a_0 and f satisfy Lipschitz conditions

$$\begin{aligned} &|a_{ij}(x, t, w)-a_{ij}(x, t, v)| \leqslant L_1 |w-v|, \\ (2.5) \qquad &|a_0(x, t, w)-a_0(x, t, v)| \leqslant L_0 |w-v|, \\ &|f(x, t, w)-f(x, t, v)| \leqslant L_2 |w-v|, \end{aligned}$$

for all $(x, t)\in Q$ and $w, v\in R$.

Later, we make some assumptions on L_0, L_1 and L_2. For $u\in W(0, T)$, we put

$$\|\nabla_x u\|_{L^\infty \times L^\infty} = \max_{1\leqslant i\leqslant n} \sup_{(x,t)\in Q} \operatorname{ess} \left|\frac{\partial u(x, t)}{\partial x_i}\right|,$$

$$\|u\|_{L^\infty \times L^\infty} = \sup_{(x,t)\in Q} \operatorname{ess} |u(x, t)|.$$

In the sequel C, $\tilde{C}$ are generic positive constants not necessarily the same.

3. Crank–Nicolson–Galerkin approximation of the problem. In connection with the problem (1.1) we have the following weak formulation:

Find $u\in \tilde{W}(0, T)$, such that

$$(3.1) \qquad \bigwedge_{v\in V} \langle u_t, v\rangle + a(t, u; u, v) = \langle f(\cdot, t, u), v\rangle$$

holds, where

$$a(t, w; u, v) = \int_\Omega \left[\sum_{i,j=1}^n a_{ij}(x, t, w)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + a_0(x, t, w)uv\right]dx.$$

Under our assumptions, $a(t, w; u, v)$ and the right-hand side of (3.1) have the following properties:

LEMMA 3.1. *Let $w\in V$, then*

(i) $\bigwedge_{v\in V} a(t, w; v, v) \geqslant \delta \|v\|_1^2$,

(ii) $\bigwedge_{u,v\in V} |a(t, w; u, v)| \leqslant C\|u\|_1 \|v\|_1$,

(iii) $\bigwedge_{v\in V} |\langle f(\cdot, t, w), v\rangle| \leqslant \|f(\cdot, t, w)\|_0 \|v\|_0$.

Proof. The proof immediately follows from (2.3)–(2.5) and the Schwarz inequality. ∎

We approximate the solution of (3.1) by requiring that u and v lie in a finite-dimensional subspace of V for each t. Assuming that the set

$\{v_1, v_2, \ldots, v_N\} \in V$ is linearly independent we denote by $\mathscr{V}$ the subspace spanned by $v_1, v_2, \ldots, v_N$. We approximate the solution u of (3.1) by the function

$$U(x, t) = \sum_{i=1}^{N} \alpha_i(t) v_i(x), \tag{3.2}$$

where

$$\alpha_i(0) = \alpha_i(T).$$

The semidiscretization method is defined as follows:

Find $U \in \tilde{W}(0, T) \cap L^2(0, T; \mathscr{V})$ such that

$$\bigwedge_{v \in \mathscr{V}} \left\langle \frac{\partial U}{\partial t}, v \right\rangle + a(t, U; U, v) = \langle f(\cdot, t, U), v \rangle. \tag{3.3}$$

We consider a scheme for approximating the solution of (3.3) in which the variable t is discretized. Let us consider a fixed uniform division of the time interval $[0, T]$. Putting $\Delta t = T/M$ $(M > 0)$, for any $g \in \tilde{W}(0, T)$ we introduce the notations

$$\begin{aligned} g_m &= g_m(x) = g(x, m\Delta t), \\ g_{m+1/2} &= g_{m+1/2}(x) = \tfrac{1}{2}[g_{m+1}(x) + g_m(x)], \\ t_m &= (m + \tfrac{1}{2})\Delta t, \end{aligned} \tag{3.4}$$

for all $0 \leqslant m \leqslant M-1$. From the periodicity follows $g_M = g_0$. Moreover, we denote

$$\begin{aligned} e &= e(x, t) = u(x, t) - U(x, t), \\ \eta &= \eta(x, t) = u(x, t) - \tilde{u}(x, t), \\ \Delta f &= f(x, t, u) - f(x, t, U), \end{aligned} \tag{3.5}$$

where $\tilde{u}$ is any function of $\tilde{W}(0, T)$ and u denotes the exact solution of (3.1). In this notation we formulate the Crank–Nicolson–Galerkin scheme [4]:

Find $\{U_m\}_{m=0}^{M-1} \in \mathscr{V}^M$, such that

$$\bigwedge_{v \in \mathscr{V}} \left\langle \frac{U_{m+1} - U_m}{\Delta t}, v \right\rangle + a(t_m, U_{m+1/2}; U_{m+1/2}, v) = \langle f(\cdot, t_m, U_{m+1/2}), v \rangle \quad \text{and} \quad U_0 = U_M. \tag{3.6}$$

The estimation of the error of this method is given in the following

THEOREM 3.1. *Let u be the exact solution of* (3.1) *and let the following condition be satisfied*

$$K = L_2 + \sqrt{2} \max\{L_1 n^{3/2} \|\nabla_x u\|_{L^\infty \times L^\infty}, L_0 \|u\|_{L^\infty \times L^\infty}\} < \delta. \tag{3.7}$$

Let a_{ij}, a_0 and f satisfy the conditions (2.3)–(2.5). *Then there exists a constant* $L > 0$, *such that*

$$(3.8)\qquad \sum_{m=0}^{M-1} \|e_{m+1/2}\|_1^2 \Delta t$$

$$\leqslant L \inf_{\tilde{u}\in \tilde{\mathscr{W}}(0,T)\cap L^2(0,T;\mathscr{V})} \left[\sum_{m=0}^{M-1} \|\eta_{m+1/2}\|_1^2 \Delta t + \sum_{m=0}^{M-1} \left\|\frac{\eta_{m+1}-\eta_m}{\Delta t}\right\|_0^2 \Delta t\right] + C(\Delta t)^4.$$

Proof. Under the conditions of Theorem 3.1, we can write (3.1) at $t = t_m$ in the following fashion:

$$(3.9)\qquad \bigwedge_{v\in\mathscr{V}} \left\langle \frac{u_{m+1}-u_m}{\Delta t} + \varrho_m, v\right\rangle + a(t_m, u_{m+1/2}+\xi_m; e_{m+1/2}+\xi_m, v)$$
$$= \langle f(\cdot, t_m, u_{m+1/2}+\xi_m), v\rangle,$$
$$u_0 = u_M,$$

where $\|\varrho_m\|_0$, $\|\xi_m\|_0$ and $\|\xi_m\|_1$ are less than $M_1(\Delta t)^2$. The constant M_1 depends on the upper bounds of $\left|\frac{\partial^3 u}{\partial t^3}\right|$ and $\left|\frac{\partial^3 u}{\partial t^2\,\partial x_i}\right|$. Subtracting (3.9) and (3.6) for $v\in\mathscr{V}$, the following identity holds:

$$\bigwedge_{v\in\mathscr{V}} \left\langle \frac{e_{m+1}-e_m}{\Delta t} + \varrho_m, v\right\rangle + a(t_m, u_{m+1/2}+\xi_m; u_{m+1/2}+\xi_m, v) -$$
$$- a(t_m, U_{m+1/2}; U_{m+1/2}, v)$$
$$= \langle \Delta f_m, v\rangle,$$

where

$$\Delta f_m = f(x, t_m, u_{m+1/2}+\xi_m) - f(x, t_m, U_{m+1/2}).$$

This equality may be transformed to the form

$$\left\langle \frac{e_{m+1}-e_m}{\Delta t}, v\right\rangle + a(t_m, U_{m+1/2}; e_{m+1/2}, v) +$$
$$+ [a(t_m, u_{m+1/2}+\xi_m; u_{m+1/2}+\xi_m, v) - a(t_m, U_{m+1/2}; u_{m+1/2}, v)]$$
$$= \langle \Delta f_m, v\rangle - \langle \varrho_m, v\rangle.$$

Next, we consider the above equality with $v = e_{m+1/2} - \eta_{m+1/2} \in \mathscr{V}$. By ordering, we obtain

$$(3.10)\quad \left\langle \frac{e_{m+1}-e_m}{\Delta t}, e_{m+1/2} \right\rangle + a(t_m, U_{m+1/2}; e_{m+1/2}, e_{m+1/2})$$

$$= \langle \Delta f_m, e_{m+1/2} - \eta_{m+1/2} \rangle - \langle \varrho_m, e_{m+1/2} - \eta_{m+1/2} \rangle -$$
$$- [a(t_m, u_{m+1/2} + \xi_m; u_{m+1/2}, e_{m+1/2} - \eta_{m+1/2}) -$$
$$- a(t_m, U_{m+1/2}, u_{m+1/2}, e_{m+1/2} - \eta_{m+1/2})] -$$
$$- a(t_m, u_{m+1/2} + \xi_m; \xi_m, e_{m+1/2} - \eta_{m+1/2}) +$$
$$+ \left\langle \frac{e_{m+1}-e_m}{\Delta t}, \eta_{m+1/2} \right\rangle.$$

In order to estimate the right-hand side of (3.10) we state the following

LEMMA 3.2. *Under the conditions of Theorem* 3.1 *the following estimations hold*:

(i) $|\langle \Delta f_m, e_{m+1/2} - \eta_{m+1/2} \rangle| \leqslant L_2 \|e_{m+1/2} + \xi_m\|_0 \|e_{m+1/2} - \eta_{m+1/2}\|_0$,

(ii) $\bigwedge_{v \in \mathscr{V}} |a(t_m, u_{m+1/2} + \xi_m; u_{m+1/2}, v) - a(t_m, U_{m+1/2}; u_{m+1/2}, v)|$
$\leqslant \sqrt{2} \max\{L_1 n^{3/2} \|\nabla_x u\|_{L^\infty \times L^\infty}, L_0 \|u\|_{L^\infty \times L^\infty}\} \|e_{m+1/2} + \xi_m\|_0 \|v\|_1$,

(iii) $\bigwedge_{v \in \mathscr{V}} |a(t_m, u_{m+1/2} + \xi_m; \xi_m, v)| \leqslant C \|\xi_m\|_1 \|v\|_1$, *where* C *is the same constant as in Lemma* 3.1.

Proof of Lemma 3.2. The estimates given in (i) and (iii) immediately follow by the Schwarz inequality and Lemma 3.1, respectively. We prove only (ii). From the definition of form a, we obtain

$$|a(t_m, u_{m+1/2} + \xi_m; u_{m+1/2}, v) - a(t_m, U_{m+1/2}; u_{m+1/2}, v)|$$

$$\leqslant L_1 \sum_{i,j=1}^{n} \int_\Omega |e_{m+1/2} + \xi_m| \left| \frac{\partial u_{m+1/2}}{\partial x_i} \right| \left| \frac{\partial v}{\partial x_j} \right| dx + L_0 \int_\Omega |e_{m+1/2} + \xi_m| \, |u_{m+1/2}| \, |v| \, dx$$

$$\leqslant L_1 n \|\nabla_x u_{m+1/2}\|_{L^\infty(\Omega)} \sum_{j=1}^{n} \int_\Omega |e_{m+1/2} + \xi_m| \left| \frac{\partial v}{\partial x_j} \right| dx +$$
$$+ L_0 \|u_{m+1/2}\|_{L^\infty(\Omega)} \int_\Omega |e_{m+1/2} + \xi_m| \, |v| \, dx$$

$$\leqslant L_1 n^{3/2} \|\nabla_x u_{m+1/2}\|_{L^\infty(\Omega)} \|e_{m+1/2} + \xi_m\|_0 |v|_1 +$$
$$+ L_0 \|u_{m+1/2}\|_{L^\infty(\Omega)} \|e_{m+1/2} + \xi_m\|_0 \|v\|_0$$

$$\leqslant \sqrt{2} \max\{L_1 n^{3/2} \|\nabla_x u\|_{L^\infty \times L^\infty}, L_0 \|u\|_{L^\infty \times L^\infty}\} \|e_{m+1/2} + \xi_m\|_0 \|v\|_1.$$

Thus the lemma is proved. ■

Using these estimates in (3.10) we come to the inequality

$$(3.11)\quad \frac{1}{2\Delta t}[\|e_{m+1}\|_0^2-\|e_m\|_0^2]+\delta\|e_{m+1/2}\|_1^2$$
$$\leqslant \|\varrho_m\|_0\|e_{m+1/2}-\eta_{m+1/2}\|_0+K\|e_{m+1/2}+\xi_m\|_0\|e_{m+1/2}-\eta_{m+1/2}\|_1+$$
$$+C\|\xi_m\|_1\|e_{m+1/2}-\eta_{m+1/2}\|_1+\left\langle\frac{e_{m+1}-e_m}{\Delta t},\ \eta_{m+1/2}\right\rangle.$$

In the sequel we use the inequality

$$(3.12)\qquad ab\leqslant \varepsilon a^2+\frac{1}{4\varepsilon}b^2,$$

obviously valid for all $\varepsilon>0$ and $a, b\in R$.

From (3.11), we obtain

$$\frac{1}{2\Delta t}[\|e_{m+1}\|_0^2-\|e_m\|_0^2]+\delta\|e_{m+1/2}\|_1^2$$
$$\leqslant\frac{1}{4\varepsilon}\|\varrho_m\|_0^2+\varepsilon\|e_{m+1/2}\|_0^2+\|\varrho_m\|_0^2+\tfrac{1}{4}\|\eta_{m+1/2}\|_0^2+$$
$$+K\|e_{m+1/2}\|_0\|e_{m+1/2}\|_1+\frac{K}{4\varepsilon}\|\xi_m\|_0^2+K\varepsilon\|e_{m+1/2}\|_1^2+K\varepsilon\|e_{m+1/2}\|_1^2+$$
$$+\frac{K}{4\varepsilon}\|\eta_{m+1/2}\|_1^2+K\|\xi_m\|_1^2+\frac{K}{4}\|\eta_{m+1/2}\|_1^2+C\varepsilon\|e_{m+1/2}\|_1^2+$$
$$+\frac{C}{4\varepsilon}\|\xi_m\|_1^2+C\|\xi_m\|_1^2+\frac{C}{4}\|\eta_{m+1/2}\|_1^2+\left\langle\frac{e_{m+1}-e_m}{\Delta t},\ \eta_{m+1/2}\right\rangle.$$

By ordering and estimating the norm $\|\cdot\|_0$ by $\|\cdot\|_1$, we obtain

$$(3.13)\quad \tfrac{1}{2}[\|e_{m+1}\|_0^2-\|e_m\|_0^2]+[\delta-K-\varepsilon(1+2K+C)]\|e_{m+1/2}\|_1^2\Delta t$$
$$\leqslant\left(\frac{1+K+C}{4}+\frac{K}{4\varepsilon}\right)\|\eta_{m+1/2}\|_1^2\Delta t+\left(\frac{1+K+C}{4\varepsilon}+1+K+C\right)M_1^2(\Delta t)^5+$$
$$+\left\langle\frac{e_{m+1}-e_m}{\Delta t},\ \eta_{m+1/2}\right\rangle\Delta t.$$

It is easy to verify that if $e_0=e_M$ and $\eta_0=\eta_M$ then

$$S=\sum_{m=0}^{M-1}\left\langle\frac{e_{m+1}-e_m}{\Delta t},\ \eta_{m+1/2}\right\rangle\Delta t=-\sum_{m=0}^{M-1}\left\langle e_{m+1/2},\ \frac{\eta_{m+1}-\eta_m}{\Delta t}\right\rangle\Delta t.$$

Thus

$$|S| \leqslant \varepsilon \sum_{m=0}^{M-1} \|e_{m+1/2}\|_1^2 \Delta t + \frac{1}{4\varepsilon} \sum_{m=0}^{M-1} \left\| \frac{\eta_{m+1}-\eta_m}{\Delta t} \right\|_0^2 \Delta t .$$

Summing (3.13) for $0 \leqslant m \leqslant M-1$ and using the above inequality, we obtain

$$(3.14) \quad [\delta - K - \varepsilon(2+2K+C)] \sum_{m=0}^{M-1} \|e_{m+1/2}\|_1^2 \Delta t$$
$$\leqslant \left(\frac{1+K+C}{4} + \frac{K}{4\varepsilon} \right) \sum_{m=0}^{M-1} \|\eta_{m+1/2}\|_1^2 \Delta t + \frac{1}{4\varepsilon} \sum_{m=0}^{M-1} \left\| \frac{\eta_{m+1}-\eta_m}{\Delta t} \right\|_0^2 \Delta t +$$
$$+ (1+K+C)\left(\frac{1}{4\varepsilon} + 1 \right) M_1^2 (\Delta t)^4 T$$

and the theorem is proved. ∎

In particular cases, we have

Corollary 3.1. *If the operator A is linear (i.e., $L_0 = L_1 = 0$) and $L_2 < \delta$ then for any solution of (3.1) the estimate (3.8) holds.* ∎

Corollary 3.2. *In the linear case (i.e. $L_0 = L_1 = L_2 = 0$) for any solution of (3.1) we have*

$$\sum_{m=0}^{M-1} \|e_{m+1/2}\|_1^2 \Delta t$$
$$\leqslant \frac{2}{\delta} \max\left(\frac{1+C}{4}, \frac{2+C}{2\delta} \right) \left[\sum_{m=0}^{M-1} \|\eta_{m+1/2}\|_1^2 \Delta t + \sum_{m=0}^{M-1} \left\| \frac{\eta_{m+1}-\eta_m}{\Delta t} \right\|_0^2 \Delta t \right] +$$
$$+ \frac{(1+C)(2+2\delta+C)}{\delta^2} M_1^2 T (\Delta t)^4 .$$

Proof. In this case, we have $K = 0$ and taking in (3.14) $\varepsilon = \delta/2(2+C)$ the estimate follows. ∎

4. The finite-element approximation of V. In this section we introduce a particular subspace of V. For this subspace we estimate the right-hand side of (3.8). We adopt the notations of [3].

Let Ω be a polyhedral domain in R^n. We consider a regular affine-equivalent triangulation T_h of $\bar{\Omega}$, where h denotes the upper bound of the diameters of $K \in T_h$. Let $\mathscr{V}$ be the Lagrange-type finite-element space corresponding to this triangulation [3] and let for each $K \in T_h$ the restriction of $\mathscr{V}$ to K contains all polynomials of order less than r. Moreover, let Π be the interpolation operator Π: $V \to \mathscr{V}$. Then the following approximation theorem holds [3]:

THEOREM 4.1. *Let $r > n/2$. Then for any $v \in H^r(\Omega)$*

$$\|v - \Pi v\|_l \leqslant C_1 h^{r-l} |v|_r \quad (l = 0 \vee 1), \tag{4.1}$$

where the constant C_1 does not depend on v. ∎

Our aim is to estimate the right-hand side of (3.8), thus we are interested in approximation of functions which have continuous derivatives. In this case, we have

LEMMA 4.1. *Let $r > n/2$. If $v \in C^1(\bar{Q})$ then the interpolation operator Π has the property*

$$[\Pi v(\cdot, t)]_t = \Pi v_t(\cdot, t).$$

Proof. The proof immediately follows from the continuity of the derivatives of v. ∎

To obtain the estimation of the error of the Crank–Nicolson–Galerkin scheme we need

LEMMA 4.2. *Let $\eta \in W(0, T)$, then*

$$\sum_{m=0}^{M-1} \left\| \frac{\eta_{m+1} - \eta_m}{\Delta t} \right\|_0^2 \Delta t \leqslant \int_0^T \left\| \frac{\partial \eta}{\partial t} \right\|_0^2 dt.$$

Proof. We have the identity

$$\frac{\eta_{m+1} - \eta_m}{\Delta t} = \frac{1}{\Delta t} \int_{m\Delta t}^{(m+1)\Delta t} \frac{\partial \eta}{\partial t} dt.$$

Using the Schwarz inequality and summing, we obtain

$$\sum_{m=0}^{M-1} \left| \frac{\eta_{m+1} - \eta_m}{\Delta t} \right|^2 \Delta t \leqslant \sum_{m=0}^{M-1} \int_{m\Delta t}^{(m+1)\Delta t} \left| \frac{\partial \eta}{\partial t} \right|^2 dt = \int_0^T \left| \frac{\partial \eta}{\partial t} \right|^2 dt.$$

By integrating on Ω, the lemma follows. ∎

Now, we can prove the main result of this section.

THEOREM 4.2. *Let u be the solution of* (3.1) *satisfying $u(\cdot, t)$, $u_t(\cdot, t) \in H^r(\Omega)$. Let U be the solution of* (3.6). *Then there exists a constant $C_2 > 0$, such that*

$$\sum_{m=0}^{M-1} \|e_{m+1/2}\|_1^2 \Delta t \leqslant C_2 [h^{2r-2} + (\Delta t)^4].$$

Proof. By virtue of Lemma 4.2 the estimate (3.8) may be written in the form

$$(4.2)\qquad \sum_{m=0}^{M-1} \|e_{m+1/2}\|_1^2 \Delta t \leqslant L \inf_{\tilde{u}\in\tilde{W}(0,T)} \left[\sum_{m=0}^{M-1} \|\eta_{m+1/2}\|_1^2 \Delta t + \int_0^T \left\|\frac{\partial\eta}{\partial t}\right\|_0^2 dt\right] + C(\Delta t)^4.$$

We put $\Pi u(\cdot, t)$ instead of $\tilde{u}(\cdot, t)$. Then by Theorem 4.1 and Lemma 4.1, for any fixed $t\in[0, T]$ we have

$$\|\eta\|_1 \leqslant C_1 h^{r-1} |u|_r, \qquad \left\|\frac{\partial\eta}{\partial t}\right\|_0 \leqslant C_1 h^r \left|\frac{\partial\eta}{\partial t}\right|_r$$

and

$$(4.3)\qquad \int_0^T \left\|\frac{\partial\eta}{\partial t}\right\|_0^2 dt \leqslant C_1 h^{2r} \int_0^T \left|\frac{\partial u}{\partial t}\right|_r^2 dt \leqslant C' h^{2r}.$$

Also, we have

$$\|\eta_{m+1/2}\|_1 = \left\|\frac{\eta_{m+1}+\eta_m}{2}\right\|_1 \leqslant C_1 h^{r-1} |u_{m+1/2}|_r,$$

hence

$$(4.4)\qquad \sum_{m=0}^{M-1} \|\eta_{m+1/2}\|_1^2 \Delta t \leqslant \sum_{m=0}^{M-1} C_1 h^{2r-2} |u_{m+1/2}|_r^2 \Delta t \leqslant \tilde{C} h^{2r-2}.$$

Substituting (4.4) and (4.3) into (4.2), the desired estimation holds. ∎

5. The linear case. In this section we discuss, in more detail, the linear case of (3.6). We write $a(t; u, v)$ and $f(x, t)$ instead of $a(t, w; u, v)$ and $f(x, t, w)$. Let the finite-dimensional space $\mathscr{V}$ be given and let the basis $\{v_i\}_{i=1}^N$ be in $H^1(\Omega)$. The solution of (3.6) has the following form

$$(5.1)\qquad U_m = \sum_{i=1}^{N} \alpha_{im} v_i \in \mathscr{V}, \qquad 0 \leqslant m \leqslant M-1.$$

From (3.6) we obtain the system of linear equations

$$(5.2a)\qquad \sum_{i=1}^{N} \alpha_{im+1}\left[\langle v_i, v_j\rangle + \frac{\Delta t}{2} a(t_m; v_i, v_j)\right] + \sum_{i=1}^{N} \alpha_{im}\left[-\langle v_i, v_j\rangle + \frac{\Delta t}{2} a(t_m; v_i, v_j)\right] = \langle f(\cdot, t_m), v_j\rangle$$

for all $1 \leqslant j \leqslant N$ and $0 \leqslant m \leqslant M-1$ and

$$(5.2b)\qquad \alpha_{i0} = \alpha_{iM}, \qquad 1 \leqslant i \leqslant N.$$

If we denote

$$\Lambda = \{\langle v_i, v_j \rangle\}_{i,j=1}^{N}, \qquad A_m = \{a(t_m; v_i, v_j)\}_{i,j=1}^{N},$$
$$F_m = \{\langle f(\cdot, t_m), v_j \rangle\}_{j=1}^{N}, \qquad \alpha_m = \{\alpha_{im}\}_{i=1}^{N}$$

then the linear equation system (5.2) has the block form

$$\text{(5.3)} \quad \begin{vmatrix} -\Lambda+\frac{\Delta t}{2}A_0 & \Lambda+\frac{\Delta t}{2}A_0 & 0 & 0 & 0 & 0 \\ 0 & -\Lambda+\frac{\Delta t}{2}A_1 & \Lambda+\frac{\Delta t}{2}A_1 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & & & -\Lambda+\frac{\Delta t}{2}A_{M-2} & \Lambda+\frac{\Delta t}{2}A_{M-2} \\ \Lambda+\frac{\Delta t}{2}A_{M-1} & 0 & & & 0 & -\Lambda+\frac{\Delta t}{2}A_{M-1} \end{vmatrix} \begin{vmatrix} \alpha_0 \\ \alpha_1 \\ \\ \alpha_{M-2} \\ \alpha_{M-1} \end{vmatrix} = \begin{vmatrix} F_0 \\ F_1 \\ \\ F_{M-2} \\ F_{M-1} \end{vmatrix}$$

We prove the existence of the solution of (5.3). At first we prove three lemmas.

LEMMA 5.1. *Let $v_i \in H^1(\Omega)$. Then there exist the constants d and d_1, such that*

(i) $\inf_{|\xi|=1} \|\sum_{i=1}^{N} \xi_i v_i\|_0 = d > 0$,

(ii) $\sup_{|\xi|=1} \|\sum_{i=1}^{N} \xi_i v_i\|_1 \leqslant d_1 < \infty$,

where for $\xi \in R^N$, we put $|\xi| = (\sum_{i=1}^{N} \xi_i^2)^{1/2}$.

Proof. We have

$$\|\sum_{i=1}^{N} \xi_i v_i\|_0 = \xi^T \Lambda \xi / \xi^T \xi,$$

thus d is equal to the least eigenvalue of Gramm's matrix Λ. Therefore (i) holds. Next, applying the Schwarz inequality to (ii) it is easy to see that

$$d_1 = (\sum_{i=1}^{N} \|v_i\|_1^2)^{1/2}. \quad \blacksquare$$

LEMMA 5.2. *In the linear case let a_{ij}, a_0 and f satisfy* (2.3). *Then*

(i) *the submatrices $\Lambda + \frac{\Delta t}{2} A_m$ and $-\Lambda + \frac{\Delta t}{2} A_m$ are symmetric,*

(ii) *the submatrices* $\Lambda+\frac{\Delta t}{2}A_m$ *are positive definite,*

(iii) *there exists* $\tau>0$ *such that for every* $\Delta t<\tau$ *the submatrices* $-\Lambda+\frac{\Delta t}{2}A_m$ *are negative definite.*

Proof. The property (i) is obvious. In order to prove (ii), we observe that

$$\sum_{i,j=1}^{N}\xi_i\left[\langle v_i, v_j\rangle+\frac{\Delta t}{2}a(t_m; v_i, v_j)\right]\xi_j=\|z\|_0^2+\frac{\Delta t}{2}a(t_m; z, z)$$
$$\geqslant\|z\|_0^2+\frac{\Delta t}{2}\delta\|z\|_1^2\geqslant d^2|\xi|^2,$$

where $z=\sum_{i=1}^{N}\xi_i v_i$. The estimation follows from the condition of ellipticity (2.3iv) and Lemma 5.1. Similarly we have

$$\sum_{i,j=1}^{N}\xi_i\left[-\langle v_i, v_j\rangle+\frac{\Delta t}{2}a(t_m; v_i, v_j)\right]\xi_j=-\|z\|_0^2+\frac{\Delta t}{2}a(t_m; z, z)$$
$$\leqslant-\|z\|_0^2+\frac{\Delta t}{2}C\|z\|_1^2\leqslant\left(-d+\frac{\Delta t}{2}Cd_1\right)|\xi|^2.$$

The last estimation follows from Lemma 5.1 (the constant C is the same as in Lemma 3.1). Therefore, for $\Delta t<\tau=2d/Cd_1$ the right-hand side is negative. ∎

Remark 5.1. We note that τ depends only on the coefficients a_{ij}, a_0 and on the basis of $\mathscr{V}$. This remark is used in the next section. ∎

The matrices $\Lambda+\frac{\Delta t}{2}A_m$ are nonsingular and the system (5.3) may be written as follows

$$(5.4)\qquad \left\|\begin{array}{cccccc} -P_0 & I & 0 & & & 0\\ 0 & -P_1 & I & & & 0\\ & & & & & \\ 0 & 0 & & & -P_{M-2} & I\\ I & 0 & & & 0 & -P_{M-1}\end{array}\right\|\left\|\begin{array}{c}\alpha_0\\ \alpha_1\\ \vdots\\ \alpha_{M-2}\\ \alpha_{M-1}\end{array}\right\|=\left\|\begin{array}{c}\bar F_0\\ \bar F_1\\ \vdots\\ \bar F_{M-2}\\ \bar F_{M-1}\end{array}\right\|,$$

where

$$P_m=\left(\Lambda+\frac{\Delta t}{2}A_m\right)^{-1}\left(\Lambda-\frac{\Delta t}{2}A_m\right)\quad\text{and}\quad\bar F_m=\left(\Lambda+\frac{\Delta t}{2}A_m\right)^{-1}F_m.$$

We have

LEMMA 5.3. *The linear equation system* (5.4) *has a unique solution if and only if*

$$\det(I - P_{M-1} P_{M-2} \cdot \ldots \cdot P_0) \neq 0. \tag{5.5}$$

Proof. Let, for $0 \leqslant m \leqslant M-2$

$$\alpha_{m+1} = Q_m \alpha_0 + R_m. \tag{5.6}$$

From (5.4) it is easy to obtain the forms of Q_m and R_m. Namely,

$$\alpha_1 = P_0 \alpha_0 + \bar{F}_0,$$

thus $Q_0 = P_0$ and $R_0 = \bar{F}_0$. Next

$$\alpha_2 = P_1 \alpha_1 + \bar{F}_1 = P_1 P_0 \alpha_0 + P_1 \bar{F}_0 + \bar{F}_1,$$

whence $Q_1 = P_1 P_0$ and $R_1 = P_1 \bar{F}_0 + \bar{F}_1$. Generally, we obtain the recurrence formulas

$$Q_0 = P_0, \quad Q_k = P_k Q_{k-1},$$

$$R_0 = \bar{F}_0, \quad R_k = P_k R_{k-1} + \bar{F}_k,$$

for $1 \leqslant k \leqslant M-1$. For $k = M-1$, we obtain the linear equation system

$$(I - P_{M-1} P_{M-2} \cdot \ldots \cdot P_0) \alpha_0 = R_{M-1}.$$

If (5.5) holds, then we can compute α_0 and then using (5.6) we can obtain the solution of (5.4). ∎

Now we can prove that for $\Delta t < \tau$ the condition (5.5) holds.

THEOREM 5.1. *Let τ be the same constant as in Lemma* 5.2 *and let $\Delta t < \tau$. Then* (5.5) *is true.*

Proof. We prove that the eigenvalues of all matrices P_m are less than 1. First we note that for $\Delta t < \tau$ these matrices are positive definite. Let λ be an eigenvalue of P_m. Thus $\lambda > 0$ and

$$\left(\Lambda + \frac{\Delta t}{2} A_m\right)^{-1} \left(\Lambda - \frac{\Delta t}{2} A_m\right) x = \lambda x.$$

Hence

$$\frac{1-\lambda}{1+\lambda} \frac{2}{\Delta t} x = \Lambda^{-1} A_m x.$$

This means that $\frac{1-\lambda}{1+\lambda}\frac{2}{\Delta t}$ is an eigenvalue of matrix $\Lambda^{-1}A_m$, which is also positive definite. Therefore

$$\frac{1-\lambda}{1+\lambda}\frac{2}{\Delta t} > 0,$$

and the spectral norm of $P_{M-1}P_{M-2}\cdot\ldots\cdot P_0$ is less than 1. Thus (5.5) holds. ∎

6. The nonlinear case. In this section we discuss the existence, under additional assumptions, of the solution of (3.6). For any $z = (z_0, z_1, \ldots, z_{M-1}) \in \mathscr{V}^M$ and $0 \leqslant m \leqslant M-1$, we denote

$$z_{m+1/2} = \tfrac{1}{2}[z_{m+1}+z_m].$$

In the case $m = M-1$, we put $z_M = z_0$. Let τ be the same constant as in Lemma 5.2 and let the uniform division of the time interval satisfying $\Delta t < \tau$ be done. This division is fixed in the whole section. For any $w \in \mathscr{V}^M$ there exists the unique solution $z = Sw$ of the linear problem

$$\text{(6.1)}\qquad \bigwedge_{v\in\mathscr{V}} \left\langle \frac{z_{m+1}-z_m}{\Delta t}, v\right\rangle + a(t_m, w_{m+1/2}; z_{m+1/2}, v) = \langle f(\cdot, t_m, w_{m+1/2}), v\rangle,$$
$$z_M = z_0.$$

The existence of the solution of (6.1) follows from Theorem 5.1. In this manner we define the operator $S\colon \mathscr{V}^M \to \mathscr{V}^M$. We show that this operator has a fixed point. Exactly, we prove

THEOREM 6.1. *Let the coefficients a_{ij}, a_0 and f satisfy* (2.3)–(2.5) *and let f satisfy the additional condition*

$$\text{(6.2)}\qquad \sup_{w\in\tilde{W}(0,T)} \sum_{m=0}^{M-1} \Delta t \int_{\Omega} f^2(x, t_m, w(x), t_m)\,dx \leqslant c_f < \infty.$$

Moreover, let $\mathscr{V}$ be the space defined in Section 4. *There exists $u \in \mathscr{V}^M$ satisfying $Su = u$.*

Remark 6.1. The condition (6.2) is satisfied, if there exists the constant $K > 0$ and a function $g \in L^2(Q)$, such that

$$\bigwedge_{(x,t)\in Q} |f(x, t, w)| \leqslant Kg(x, t). \ ∎$$

LEMMA 6.1. *Let a_{ij}, a_0 and f satisfy* (2.3)–(2.5) *and* (6.2). *Then the solution z of* (6.1) *satisfies*

$$\sum_{m=0}^{M-1} \|z_{m+1/2}\|_1^2 \Delta t \leqslant c_f/\delta^2.$$

Proof. For any $0 \leqslant m \leqslant M-1$, we consider (6.1) with $v = z_{m+1/2}$. We obtain

$$\tfrac{1}{2}[\|z_{m+1}\|_0^2 - \|z_m\|_0^2] + \delta \|z_{m+1/2}\|_1^2 \Delta t \leqslant \|f(\cdot, t_m, w_{m+1/2})\|_0 \|z_{m+1/2}\|_0.$$

Summing these inequalities and using (3.12), (6.2), we have

$$(\delta - \varepsilon) \sum_{m=0}^{M-1} \|z_{m+1/2}\|_1^2 \Delta t \leqslant \frac{1}{4\varepsilon} \sum_{m=0}^{M-1} \|f(\cdot, t_m, w_{m+1/2})\|_0^2 \Delta t$$

for any $0 < \varepsilon < \delta$.

Taking $\varepsilon = \delta/2$, the estimation follows. ∎

LEMMA 6.2. *For $u, v, w \in \mathscr{V}^M$, we have*

$$|\langle f(\cdot, t_m, w_{m+1/2}) - f(\cdot, t_m, u_{m+1/2}), v\rangle| \leqslant L_2 \|(w-u)_{m+1/2}\|_0 \|v\|_0.$$

Proof. From (2.5) and the Schwarz inequality we obtain

$$\begin{aligned} &|\langle f(\cdot, t_m, w_{m+1/2}) - f(\cdot, t_m, u_{m+1/2}), v\rangle| \\ &\qquad \leqslant \int_\Omega |f(x, t_m, w_{m+1/2}) - f(x, t_m, u_{m+1/2})|\, |v|\, dx \\ &\qquad \leqslant L_2 \int_\Omega |(w-u)_{m+1/2}|\, |v|\, dx \leqslant L_2 \|(w-u)_{m+1/2}\|_0 \|v\|_0. \end{aligned}$$ ∎

LEMMA 6.3. *Let $\mathscr{V}$ be the space defined in Section 4 and let $z = Sw$. Then the quantities*

$$(6.3) \qquad \max_{0 \leqslant m \leqslant M-1} \|\nabla_x z_{m+1/2}\|_{L^\infty(\Omega)} \quad \textit{and} \quad \max_{0 \leqslant m \leqslant M-1} \|z_{m+1/2}\|_{L^\infty(\Omega)}$$

are finite.

Proof. From Lemma 6.1 it follows that the coefficients α_{im} in (5.1) are bounded. Because the functions v_i are piecewise continuous then for any fixed m the quantity $\|\nabla_x z_{m+1/2}\|_{L^\infty(\Omega)}$ is finite. Similarly, the finiteness of $\max_{0 \leqslant m \leqslant M-1} \|z_{m+1/2}\|_{L^\infty(\Omega)}$ follows. ∎

LEMMA 6.4. *The operator S is continuous on $\mathscr{V}^M$.*

Proof. We consider (6.1) for $u, w \in \mathscr{V}^M$. Similarly as in the proof of Theorem 3.1, we obtain the following identity

$$(6.4) \qquad \bigwedge_{v \in \mathscr{V}} \left\langle \frac{\gamma_{m+1} - \gamma_m}{\Delta t}, v \right\rangle + a(t_m, w_{m+1/2}; \gamma_{m+1/2}, v)$$
$$= \langle f(\cdot, t_m, w_{m+1/2}) - f(\cdot, t_m, u_{m+1/2}), v\rangle +$$
$$+ [a(t_m, u_{m+1/2}; (Su)_{m+1/2}, v) - a(t_m, w_{m+1/2}; (Su)_{m+1/2}, v)],$$

where we put $\gamma = Sw - Su$.

We estimate the expression in the square brackets.

$$\mathscr{A} = \left|a\left(t_m, u_{m+1/2}; (Su)_{m+1/2}, v\right) - a\left(t_m, w_{m+1/2}; (Su)_{m+1/2}, v\right)\right|$$
$$\leqslant \int_\Omega \Big[\sum_{i,j=1}^{n} |a_{ij}(x, t_m, u_{m+1/2}) - a_{ij}(x, t_m, w_{m+1/2})| +$$
$$+ |a_0(x, t_m, u_{m+1/2}) - a_0(x, t_m, w_{m+1/2})|\Big] \left|\frac{\partial (Su)_{m+1/2}}{\partial x_i}\right| \left|\frac{\partial v}{\partial x_j}\right| dx.$$

Analogously as in the proof of Lemma 3.2, we obtain

$$\mathscr{A} \leqslant C \|(w-u)_{m+1/2}\|_0 \|v\|_1,$$

where

$$C = \sqrt{2}\max\{L_1 n^{3/2} \max_{0 \leqslant m \leqslant M-1} \|\nabla_x (Su)_{m+1/2}\|_{L^\infty(\Omega)}, L_0 \max_{0 \leqslant m \leqslant M-1} \|(Su)_{m+1/2}\|_{L^\infty(\Omega)}\}.$$

We consider (6.4) for $0 \leqslant m \leqslant M-1$ and $v = \gamma_{m+1/2}$. By summing (6.4) and using Lemma 6.2, we obtain

$$\sum_{m=0}^{M-1} \|(Sw)_{m+1/2} - (Su)_{m+1/2}\|_1^2 \Delta t \leqslant C(u) \sum_{m=0}^{M-1} \|(w-u)_{m+1/2}\|_1^2 \Delta t,$$

where $C(u) = \left((L_2 + C)/\delta\right)^2$. ∎

Proof of Theorem 6.1. We have

$$(Sw)_m = z_m = \sum_{k=1}^{N} \alpha_{km} v_k, \quad 0 \leqslant m \leqslant M-1.$$

Putting $v = v_j$ $(1 \leqslant j \leqslant N)$ in (6.1) for any m, we obtain a linear equation system with $N \times M$ unknowns α_{km}.

Let

$$B_f = \{\alpha_{km} \in R: \sum_{m=0}^{M-1} \|z_{m+1/2}\|_1^2 \Delta t \leqslant C_f/\delta^2\} \subset R^{N \times M}.$$

The set B_f is convex and bounded. By Lemma 6.1, $S(B_f) \subset B_f$. Also, from Lemma 6.4 we know that S is continuous. Then by Schauder's theorem [5] the operator S has a fixed point. ∎

Let $w_0 \in \mathscr{V}^M$. We may generate the sequence

$$w_{k+1} = Sw_k, \quad k \geqslant 0. \tag{6.5}$$

From the practical point of view, it is interesting to know, when this sequence converges to the solution of (3.6). We give a sufficient condition for this in the case when the operator A is linear.

THEOREM 6.2. *Let the operator A in* (1.1) *be linear and let f satisfy* (6.2). *If $L_2 < \delta$ then the sequence $\{w_k\}_{k=0}^{\infty}$ converges to the solution of* (3.6).

Proof. Let $v \in \mathscr{V}^M$. The quantity $(\sum_{m=0}^{M-1} \|v\|_1^2 \Delta t)^{1/2}$ is a norm of v in $\mathscr{V}^M$. From Lemma 6.4 follows that the operator S is a contraction with the contraction constant $\varkappa = L_2/\delta$. The result follows from the Banach theorem [5]. ∎

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INSTITUTE OF COMPUTER SCIENCE
UNIVERSITY OF WROCŁAW
51-151 WROCŁAW

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