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PARTIAL DERIVATIVES OF THE VANDERMONDE DETERMINANT AND THEIR APPLICATION TO THE SYNTHESIS OF LINEAR CONTROL SYSTEMS

1. Introduction. In the theory of synthesis of control systems the design of linear stationary systems for specified poles and zeros by the use of the method of state variables can be reduced to the solution of the set of linear equations (see [2] and [3]). The determinants of the coefficient matrices of these sets of linear equations have the form of Vandermonde determinants in the case of distinct poles and zeros, and the form of partial derivatives of Vandermonde determinants in the case of multiple poles and zeros of the transfer function of the system.

The purpose of this paper is to investigate properties of partial derivatives of Vandermonde determinants and their application to the synthesis of linear stationary control systems for specified poles and zeros of the closed-loop transfer functions.

2. Partial derivatives of the Vandermonde determinant. For the sake of simplicity we assume the following notation:

$$\prod_{j=k}^{n} a_j = \begin{cases} a_k a_{k+1} \ldots a_n & \text{for } k \leqslant n, \\ 1 & \text{for } k > n, \end{cases}$$

$$\hat{n!} = 1!\,2!\ldots n!,$$

$$V_n = \begin{vmatrix} 1 & s_1 & s_1^2 & \ldots & s_1^{n-1} \\ 1 & s_2 & s_2^2 & \ldots & s_2^{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & s_n & s_n^2 & \ldots & s_n^{n-1} \end{vmatrix},$$

$$(1)\qquad \frac{\partial^{\hat{r}} V_n}{\partial s_j^{\hat{r}}} = \frac{\partial^{r(r-1)/2} V_n}{\partial s_{j+1}\partial s_{j+2}^2 \ldots \partial s_{j+r-1}^{r-1}}\Bigg|_{s_{j+1}, s_{j+2}, \ldots, s_{j+r-1} = s_j}.$$

It can be easily verified that

$$\frac{\partial^{\hat r} V_n}{\partial s_j^{\hat r}} = \begin{vmatrix} 1 & s_1 & s_1^2 & & \cdots & & s_1^{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & s_j & s_j^2 & & \cdots & & s_j^{n-1} \\ 0 & 1 & 2s_j & & \cdots & & (n-1)s_j^{n-2} \\ 0 & 0 & 2 & 3!s_j & \cdots & & (n-1)(n-2)s_j^{n-3} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0\ldots(r-1)! & & \cdots & & (n-1)\ldots(n+1-r)s_j^{n-r} \\ 1 & s_{j+r} & s_{j+r}^2 & & \cdots & & s_{j+r}^{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & s_n & s_n^2 & & \cdots & & s_n^{n-1} \end{vmatrix}.$$

We use the following form of formula (1):

$$\text{(1')}\qquad \frac{\partial^{\hat r} V_n}{\partial s_j^{\hat r}} = \frac{\partial^{r-1}}{\partial s_{j+r-1}^{r-1}}\left(\cdots \frac{\partial^2}{\partial s_{j+2}^2}\left(\frac{\partial V_n}{\partial s_{j+1}}\Big|_{s_{j+2}=s_j}\right)\Big|_{s_{j+2}=s_j}\cdots\right)\Big|_{s_{j+r-1}=s_j}.$$

Taking into account the well-known formula

$$\text{(2)}\qquad V_n = \prod_{\substack{k,l=1\\k>l}}^{n} (s_k - s_l),$$

we obtain

$$\frac{\partial^{\hat 2} V_n}{\partial s_1^{\hat 2}} = \prod_{\substack{k,l=3,\\k>l}}^{n} (s_k - s_l) \prod_{i=3}^{n} (s_i - s_1)^2.$$

Using the principle of mathematical induction and formula (1'), we get a general formula of the form

$$\text{(3)}\qquad \frac{\partial^{\hat r_1} V_n}{\partial s_1^{\hat r_1}} = \prod_{\substack{k,l=r_1+1\\k>l}}^{n} (s_k - s_l) \prod_{i=r_1+1}^{n} (s_i - s_1)^{r_1} (r_1 - 1)\hat{!}.$$

In the case $n = r_1$ formula (3) reduces to the form

$$\frac{\partial^{\hat n} V_n}{\partial s_1^{\hat n}} = (n-1)\hat{!}.$$

It can be easily verified that

$$\frac{\partial^{\hat{2}}}{\partial s_{r_1+1}^{\hat{2}}}\left(\frac{\partial^{\hat{r}_1} V_n}{\partial s_1^{\hat{r}_1}}\right)$$

$$= \prod_{\substack{k,l=r_1+3\\ k>l}}^{n} (s_k - s_l) \prod_{i=r_1+3}^{n} (s_i - s_{r_1+1})^2 \prod_{i=r_1+1}^{n} (s_i - s_1)^{r_1} (s_{r_1+1} - s_1)^{2r_1} (r_1 - 1)\hat{!}.$$

From the principle of mathematical induction, we obtain

$$(4) \quad \frac{\partial^{\hat{r}_2}}{\partial s_{r_1+1}^{\hat{r}_2}}\left(\frac{\partial^{\hat{r}_1} V_n}{\partial s_1^{\hat{r}_1}}\right)$$

$$= \prod_{\substack{k,l=r_1+r_2+1,\\ k>l}}^{n} (s_k - s_l) \prod_{i=r_1+r_2+1}^{n} [(s_i - s_1)^{r_1} (s_i - s_{r_1+1})^{r_2}] (s_{r_1+1} - s_1)^{r_1 r_2} \cdot$$

$$\cdot (r_1 - 1)\hat{!}(r_2 - 1)\hat{!} \quad \text{for } r_1 \geqslant r_2$$

and

$$(5) \quad \frac{\partial^{\hat{r}_j}}{\partial s_{r_{j-1}+1}^{\hat{r}_j}}\left(\dots \frac{\partial^{\hat{r}_2}}{\partial s_{r_1+1}^{\hat{r}_2}}\left(\frac{\partial^{\hat{r}_1} V_n}{\partial s_1^{\hat{r}_1}}\right)\dots\right)$$

$$= \prod_{\substack{k,l=r_1+\dots+r_j+1,\\ k>l}}^{n} (s_k - s_l) \prod_{i=r_1+\dots+r_j+1}^{n} [(s_i - s_1)^{r_1} \dots (s_i - s_{r_{j-1}+1})^{r_j}] \cdot$$

$$\cdot [(r_1 - 1)\hat{!}(r_2 - 1)\hat{!} \dots (r_j - 1)\hat{!}] [(s_{r_1+1} - s_1)^{r_1 r_2} \dots$$

$$\dots (s_{r_1+\dots+r_{j-1}+1} - s_1)^{r_1 r_j}] \dots (s_{r_1+\dots+r_{j-1}+1} - s_{r_1+\dots+r_{j-2}+1})^{r_{j-1} r_j}$$

$$\text{for } r_1 \geqslant r_2 \geqslant \dots \geqslant r_j \geqslant 1, \; r_1 + r_2 + \dots + r_j \leqslant n.$$

Examples. From formula (4) it results that

$$\frac{\partial^{\hat{3}}}{\partial s_4^{\hat{3}}}\left(\frac{\partial^{\hat{3}} V_6}{\partial s_1^{\hat{3}}}\right) = \begin{vmatrix} 1 & s_1 & s_1^2 & s_1^3 & s_1^4 & s_1^5 \\ 0 & 1 & 2s_1 & 3s_1^2 & 4s_1^3 & 5s_1^4 \\ 0 & 0 & 2 & 6s_1 & 12s_1^2 & 20s_1^3 \\ 1 & s_4 & s_4^2 & s_4^3 & s_4^4 & s_4^5 \\ 0 & 1 & 2s_4 & 3s_4^2 & 4s_4^3 & 5s_4^4 \\ 0 & 0 & 2 & 6s_4 & 12s_4^2 & 20s_4^3 \end{vmatrix} = 4(s_4 - s_1)^9.$$

In a similar way we obtain

$$\frac{\partial^2}{\partial s_4^2}\left(\frac{\partial^3 V_6}{\partial s_1^3}\right) = \begin{vmatrix} 1 & s_1 & s_1^2 & s_1^3 & s_1^4 & s_1^5 \\ 0 & 1 & 2s_1 & 3s_1^2 & 4s_1^3 & 5s_1^4 \\ 0 & 0 & 2 & 6s_1 & 12s_1^2 & 20s_1^3 \\ 1 & s_4 & s_4^2 & s_4^3 & s_4^4 & s_4^5 \\ 0 & 1 & 2s_4 & 3s_4^2 & 4s_4^3 & 5s_4^4 \\ 1 & s_6 & s_6^2 & s_6^3 & s_6^4 & s_6^5 \end{vmatrix} = 2(s_6-s_1)^3(s_6-s_4)^2(s_4-s_1)^6.$$

In the case $r_1 = r_2 = \ldots = r_j = 1$, $j = n$, formula (5) reduces to formula (2).

3. Some properties of polynomials. Taking into account formulas (1)-(5) from section 2 we prove now the following

THEOREM. *If $M(s)$ and $M_0(s)$ are polynomial functions of degree n over the field of complex numbers and*

$$M(s) = M_0(s) + \sum_{i=1}^{p} k_i s^{i-1} \qquad (p \leqslant n) \tag{6}$$

for some coefficients $k_1, \ldots, k_p$, then the constants $k_1, \ldots, k_p$ are uniquely determined by any p roots of the polynomial $M(s)$.

Proof. Let $s_1, \ldots, s_p$ be some roots of polynomial $M(s)$, not necessarily of order 1. Suppose the roots $s_1, \ldots, s_p$ have been ordered in such a way that

$$\begin{gathered} s_1 = \ldots = s_{r_1}, \\ s_{r_1+1} = \ldots = s_{r_1+r_2}, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ s_{r_1+\ldots+r_{j-1}+1} = \ldots = s_{r_1+\ldots+r_j}, \end{gathered}$$

where $r_1 \geqslant r_2 \geqslant \ldots \geqslant r_j \geqslant 1$ and $r_1 + \ldots + r_j = p$.

Let us now observe that if s_l is an r-tuple root of the polynomial $M(s)$, then (6) implies the following sequence of equalities:

$$0 = \frac{d^m}{ds^m} M_0(s)\Big|_{s=s_l} + \frac{d^m}{ds^m} \sum_{i=1}^{p} k_i s^{i-1}\Big|_{s=s_l}, \qquad m = 0, 1, \ldots, r-1. \tag{7}$$

Since the roots $s_1, s_{r_1+1}, \ldots, s_{r_1+\ldots+r_{j-1}+1}$ are r_1-tuple, r_2-tuple, ..., r_j-tuple, respectively, from (7) arises a set of linear equations with p unknowns $k_1, \ldots, k_p$. Let $Qk = M$ be a matrix form of this set of equations. It remains to show that the matrix Q is invertible. In the simplest case $r_1 = \ldots = r_j = 1$, $j = p$, we have

$$Q = \begin{bmatrix} 1 & s_1 & s_1^2 & \ldots & s_1^{p-1} \\ 1 & s_2 & s_2^2 & \ldots & s_2^{p-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & s_p & s_p^2 & \ldots & s_p^{p-1} \end{bmatrix},$$

that is $\det Q = V_p \neq 0$, by (2). It is easy to see that, in the general case,

$$\det Q = \frac{\partial^{\hat{r}_j}}{\partial s_{r_{j-1}+1}^{\hat{r}_j}} \left(\ldots \left(\frac{\partial^{\hat{r}_2}}{\partial s_{r_1+1}^{\hat{r}_2}} \left(\frac{\partial^{\hat{r}_1} V_p}{\partial s_1^{\hat{r}_1}} \right) \right) \ldots \right). \tag{8}$$

Therefore, it follows by (5) that the inequality $\det Q \neq 0$ is also true if some of the roots are of order higher than 1. This completes the proof of the theorem.

4. Synthesis of linear stationary systems for specified poles and zeros of the transfer function of control systems. Let be given a linear stationary control object with input $u = u(t)$, output $y = y(t)$ and transfer function

$$G_0(s) = \frac{L_0(s)}{M_0(s)},$$

where

$$\begin{aligned} L_0(s) &= b_m s^m + \ldots + b_1 s + b_0 \\ M_0(s) &= s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0 . \end{aligned} \qquad (n \geqslant m)$$

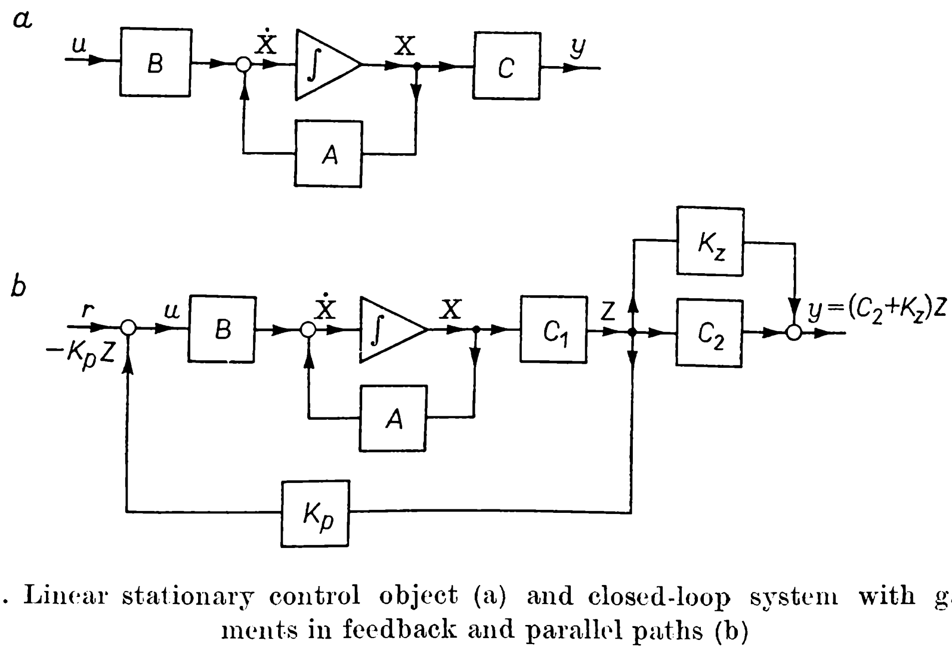

Fig. 1. Linear stationary control object (a) and closed-loop system with gain elements in feedback and parallel paths (b)

The control object in the state space (Fig. 1a) is described by

$$\dot{X} = AX + Bu, \qquad y = CX,$$

where X is an n-vector representing the state of the control object and A, B and C are $n \times n$, $n \times 1$ and $1 \times n$ constant real matrices, respectively.

Let us consider a closed-loop system (Fig. 1b) which contains in the feedback path a gain element described by the constant gain matrix

$$K_p = [k_{p1}, k_{p2}, \ldots, k_{pp}], \tag{9}$$

and in the parallel path also a gain element described by the constant gain matrix

$$K_z = [k_{z1}, k_{z2}, \ldots, k_{zp}], \tag{10}$$

where p is a number of directly measurable state variables (components of the state vector X) [2].

Let the transfer function of the system have the form

$$G(s) = \frac{L(s)}{M(s)}, \tag{11}$$

where

$$L(s) = d_m s^m + \ldots + d_1 s + d_0,$$

$$M(s) = s^n + c_{n-1} s^{n-1} + \ldots + c_1 s + c_0.$$

Determine the elements of the gain matrices (9) and (10) so that p poles and p zeros of the transfer function (11) assume the desired values $s_1, s_2, \ldots, s_p$ and $s_1^0, s_2^0, \ldots, s_p^0$, respectively.

In the paper [2] it has been shown that the polynomials $M(s)$, $M_0(s)$ and $L(s)$, $L_0(s)$ are related by

$$M(s) = M_0(s) + \sum_{i=1}^{p} k_{pi} s^{i-1} \tag{12}$$

and

$$L(s) = L_0(s) + \sum_{i=1}^{p} k_{zi} s^{i-1}, \tag{13}$$

respectively. In the case where the poles and zeros are distinct we choose the elements of gain matrices (9) and (10) so that

$$M(s_i) = 0 \quad \text{for } i = 1, 2, \ldots, p \tag{14}$$

and

$$L(s_i) = 0 \quad \text{for } i = 1, 2, \ldots, p. \tag{15}$$

In this case from (12) and (14) we obtain the set of p linear equations

$$Q_p K_p^T = M_p, \tag{16}$$

where

$$Q_p = \begin{bmatrix} 1 & s_1 & \dots & s_1^{p-1} \\ 1 & s_2 & \dots & s_2^{p-1} \\ \dots & \dots & \dots & \dots \\ 1 & s_p & \dots & s_p^{p-1} \end{bmatrix}, \quad M_p = -\begin{bmatrix} M_0(s_1) \\ M_0(s_2) \\ \dots \\ M_0(s_p) \end{bmatrix},$$

and, from relations (13) and (15),

$$Q_z K_z^T = L_z, \tag{17}$$

where

$$Q_z = \begin{bmatrix} 1 & s_1^0 & \dots & (s_1^0)^{p-1} \\ 1 & s_2^0 & \dots & (s_2^0)^{p-1} \\ \dots & \dots & \dots & \dots \\ 1 & s_p^0 & \dots & (s_p^0)^{p-1} \end{bmatrix}, \quad L_z = -\begin{bmatrix} L_0(s_1^0) \\ L_0(s_2^0) \\ \dots \\ L_0(s_p^0) \end{bmatrix}.$$

Q_p and Q_z are Vandermonde matrices and they are non-singular for distinct poles and zeros. Therefore, the sets of linear equations (16) and (17) have a unique solution. Let, in the general case,

$$s_1 = s_2 = \dots = s_{r_1}, \quad s_{r_1+1} = s_{r_1+2} = \dots = s_{r_1+r_2}, \dots,$$
$$s_{r_1+\dots+r_{l-1}} = s_{r_1+\dots+r_{l-1}+1} = \dots = s_p$$
$$(r_1+r_2+\dots+r_l = p)$$

and, respectively,

$$s_1^0 = s_2^0 = \dots = s_{r_1^0}^0, \quad s_{r_1^0+1}^0 = s_{r_1^0+2}^0 = \dots = s_{r_1^0+r_2^0}^0, \dots,$$
$$s_{r_1^0+\dots+r_{l-1}^0}^0 = s_{r_1^0+\dots+r_{l-1}^0+1}^0 = \dots = s_p^0$$
$$(r_1^0+r_2^0+\dots+r_l^0 = p).$$

In this case the elements of gain matrices (9) and (10) are chosen so that

$$\left.\frac{d^i M(s)}{ds^i}\right|_{s=s_{r_1+\dots+r_{j-1}+1}} = 0 \quad \text{for } i = 0, 1, \dots, r_j-1,\ j = 1, 2, \dots, l \tag{18}$$

and

$$\left.\frac{d^i L(s)}{ds^i}\right|_{s=s_{r_1+\dots+r_{j-1}+1}} = 0 \quad \text{for } i = 0, 1, \dots, r_j^0-1,\ j = 1, 2, \dots, l. \tag{19}$$

From relations (12), (18) and (13), (19), respectively, we obtain similar sets of linear equations as (16) and (17). In this case the matrices Q_p, M_p and Q_z, L_z have the form

$$Q_p = \begin{bmatrix} 1 & s_1 & s_1^2 & \dots & s_1^{p-1} \\ 0 & 1 & 2s_1 & \dots & (p-1)s_1^{p-2} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & (p-1)\dots(p-r_1+1)s_1^{p-r_1} \\ 1 & s_{r_1+1} & s_{r_1+1}^2 & \dots & s_{r_1+1}^{p-1} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & (p-1)\dots(p-r_p+1)s_p^{p-r_p} \end{bmatrix},$$

$$M_p = -\begin{bmatrix} M_0(s_1) \\ M_0(s_1) \\ \dots \\ M_0^{r_1-1}(s_1) \\ M_0(s_{r_1+1}) \\ \dots \\ M_0^{r_p-1}(s_p) \end{bmatrix},$$

$$Q_z = \begin{bmatrix} 1 & s_1^0 & (s_1^0)^2 & \dots & (s_1^0)^{p-1} \\ 0 & 1 & 2s_1^0 & \dots & (p-1)(s_1^0)^{p-2} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & (p-1)\dots(p-r_1^0+1)(s_1^0)^{p-r_1^0} \\ 1 & s^0_{r_1^0+1} & (s^0_{r_1^0+1})^2 & \dots & (s^0_{r_1^0+1})^{p-1} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & (p-1)\dots(p-r_p^0+1)(s_p^0)^{p-r_p^0} \end{bmatrix},$$

$$L_z = -\begin{bmatrix} L_0(s_1^0) \\ L_0(s_1^0) \\ \dots \\ L_0^{r_1^0-1}(s_1^0) \\ L_0(s^0_{r_1^0+1}) \\ \dots \\ L_p^{r_p^0-1}(s_p^0) \end{bmatrix}.$$

From formulae (15) and (18) it results that the matrices Q_p and Q_z are non-singular.

References

[1] T. Kaczorek, *Design of a single-input system for specified poles and zeros using gain elements when not all state variables are measurable,* Bull. Acad. Polon. Sci., Sér. Sci. Techn. 20 (1972), p. 17-23.

[2] — *Relation between the coefficients of the open-loop and closed-loop transfer functions of the linear stationary system,* ibidem 21 (1972), p. 45-54.

[3] — *Design of linear stationary discrete — time systems for specified poles and zeros using gain elements,* ibidem 21 (1972), p. 489-497.

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POCHODNE CZĄSTKOWE WYZNACZNIKA VANDERMONDE'A ORAZ ICH ZASTOSOWANIE DO SYNTEZY LINIOWYCH UKŁADÓW STEROWANIA

STRESZCZENIE

W pracy wykazano, że pochodne cząstkowe wyznaczników Vandermonde'a są wyznacznikami niezerowymi. Ponadto udowodniono, że jeżeli dla wielomianów n-tego stopnia $M(s)$ i $M_0(s)$ nad ciałem liczb zespolonych istnieją stałe $k_1, k_2, \ldots, k_p$, takie że

$$M(s) = M_0(s) + \sum_{i=1}^{p} k_i s^{i-1} \qquad (p \leqslant n),$$

to stałe te są jednoznacznie generowane przez pierwiastki $s_1, s_2, \ldots, s_p$ wielomianu $M(s)$.

Następnie pokazano zastosowanie pochodnych cząstkowych wyznaczników Vandermonde'a i podanego twierdzenia do syntezy liniowych układów sterowania o zadanych z góry wartościach biegunów i zer transmitancji operatorowej układu.
