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A. KORZENIOWSKI (Wrocław) and A. OPAWSKI (Warszawa)

BOUNDS FOR RELIABILITY IN THE NBU, NWU AND NBUE, NWUE CLASSES

1. Introduction. Marshall and Proschan estimated in [1] certain reliability functions. For reliability functions in New Better than Used (NBU) and New Worse than Used (NWU) classes the bounds were obtained in terms of a percentile. For a fixed mean the authors found the lower bounds of reliability functions for NBU and New Better than Used in Expectation (NBUE) classes, posing the problem of upper bounds. The solution of this problem is presented in this paper. The upper and lower bounds for NWU and New Worse than Used in Expectation (NWUE) classes are also given.

Definitions. A reliability function P defined on $[0, \infty)$ is said to be *NBU* if

$$P(x+y) \leqslant P(x)P(y) \quad \text{for all } x, y \geqslant 0. \tag{1}$$

P is said to be *NBUE* if the mean μ of P is finite and

$$\mu P(t) \geqslant \int_0^\infty P(t+x)\,dx \quad \text{for all } t \geqslant 0. \tag{2}$$

The *NWU* and *NWUE classes* are defined by reversing the inequalities (1) and (2), respectively.

2. Bound of $P(t)$ in the NBU class. The following lemmas 1-3 are basic for this section.

LEMMA 1. *Let b_j, $j = 1, 2, \ldots, 2m$, be a sequence of numbers from the interval $[\beta, 1]$, $0 < \beta < 1$. If $b_{2m+1-j} \geqslant \beta/b_j$, $j = 1, 2, \ldots, m$, then*

$$\sum_{j=1}^{2m}\left(b_j+\frac{\beta^2}{b_j}\right) \geqslant \sum_{j=1}^{m}\left(b_j+\frac{\beta^2}{b_j}\right)+\sum_{j=1}^{m}\left(\frac{\beta}{b_j}+\beta b_j\right) = (1+\beta)\sum_{j=1}^{m}\left(b_j+\frac{\beta}{b_j}\right).$$

Proof. The condition $x \geqslant y \geqslant \beta$ implies

$$x+\frac{\beta^2}{x} \geqslant y+\frac{\beta^2}{y}. \tag{3}$$

LEMMA 2. *Let a_i, $i = 0, 1, \ldots, 2^n$, be a sequence of numbers from the interval* $[0, 1]$ *such that $a_0 = 1$ and*

$$a_{i+j} \leqslant a_i a_j \quad \text{for each } i, j;\ i+j \leqslant 2^n. \tag{4}$$

Then, for each $a \leqslant a_{2^n}$, the sum

$$\left(a_1 + \frac{a}{a_1}\right) + \left(a_3 + \frac{a}{a_3}\right) + \ldots + \left(a_{2^n-1} + \frac{a}{a_{2^n-1}}\right) \tag{5}$$

(*odd items only*) *is not less than*

$$a^{1/2^n} + a^{3/2^n} + \ldots + a^{(2^n-1)/2^n}. \tag{6}$$

Proof. The assumptions of the lemma imply that the sequence $\{a_i\}$ is non-increasing, whence $a_i \geqslant a$, $i = 0, 1, \ldots, 2^n$. Now we use the induction.

For $n = 2$ from (4) we have $a_1^4 \geqslant a_2^2 \geqslant a_4 \geqslant a$. Putting $x = a_1$, $y = a^{1/4}$ and $\beta = a^{1/2}$, we obtain the statement from (3).

For $n \geqslant 3$ we apply lemma 1 to sum (5). Putting $2m = 2^{n-2}$, $b_j = a_{2j-1}$, $j = 1, 2, \ldots, 2m$, and $\beta = a^{1/2}$, from (4) we have

$$b_j^2 = a_{2j-1}^2 \geqslant a_{2^{n-1}}^2 \geqslant a_{2^n} \geqslant a = \beta^2$$

and

$$b_{2m+1-j} \geqslant \beta / b_j, \quad j = 1, 2, \ldots, 2m.$$

Thus

$$\left(a_1 + \frac{a}{a_1}\right) + \ldots + \left(a_{2^{n-2}-1} + \frac{a}{a_{2^{n-2}-1}}\right) + \left(a_{2^{n-2}+1} + \frac{a}{a_{2^{n-2}+1}}\right) + \\ + \ldots + \left(a_{2^{n-1}-1} + \frac{a}{a_{2^{n-1}-1}}\right)$$

is not less than

$$(1 + a^{1/2})\left[\left(a_1 + \frac{a^{1/2}}{a_1}\right) + \ldots + \left(a_{2^{n-2}-1} + \frac{a^{1/2}}{a_{2^{n-2}-1}}\right)\right].$$

Since $a_{2^{n-1}} \geqslant a^{1/2}$, by the inductive statement the sum in the square brackets is not less than

$$(a^{1/2})^{1/2^{n-1}} + \ldots + (a^{1/2})^{(2^{n-1}-1)/2^{n-1}}.$$

Multiplying the latter sum by $1 + a^{1/2}$, we get (6).

LEMMA 3. *Let a and the sequence $\{a_i\}$ be the same as in lemma 2. Then the sum $a_1 + a_3 + \ldots + a_{2^n-1}$ (odd items only) is not less than sum* (6).

Proof. From (4) we have $a_{2^n-j} \geqslant a/a_j$, $j = 1, 3, \ldots, 2^{n-1}-1$, and the result follows from lemma 2.

THEOREM 1. *If P is NBU with mean μ, then*

$$P(t) \leqslant \begin{cases} 1 & \text{for } t \leqslant \mu, \\ e^{-wt} & \text{for } t > \mu, \end{cases}$$

where w satisfies the equation

$$\int_0^t e^{-wx} dx = \mu.$$

The bounds are sharp.

Proof. Let $t > \mu$. Suppose $P(t) > e^{-wt}$ and put $P(t) = a$. Take ξ such that $e^{-\xi t} = a$. The Riemann sum of P on the interval $[0, t]$, i.e.

$$\frac{t}{2^{n-1}}\left[P\left(\frac{t}{2^n}\right) + P\left(\frac{3t}{2^n}\right) + \ldots + P\left(\frac{(2^n-3)t}{2^n}\right) + P\left(\frac{(2^n-1)t}{2^n}\right)\right],$$

by lemma 3 is not less than

$$\frac{t}{2^{n-1}}\left[a^{1/2^n} + a^{3/2^n} + \ldots + a^{(2^n-3)/2^n} + a^{(2^n-1)/2^n}\right].$$

This sum is the Riemann one for $e^{-\xi x}$, $x \in [0, t]$. Therefore,

$$\mu \geqslant \int_0^t e^{-\xi x} dx > \int_0^t e^{-wx} dx = \mu.$$

The upper bound for $t > \mu$ is attained by

$$H(x) = \begin{cases} e^{-wx} & \text{for } x < t, \\ 0 & \text{for } x \geqslant t, \end{cases}$$

which is NBU. The degenerate reliability function concentrating at μ provides the upper bound for $t > \mu$.

3. Bounds of $P(t)$ in the remaining classes.

THEOREM 2. *If P is NBUE with mean μ, then*

$$P(t) \leqslant \begin{cases} 1 & \text{for } t \leqslant \mu, \\ \exp\left(-\dfrac{t-\mu}{\mu}\right) & \text{for } t > \mu. \end{cases}$$

The bounds are sharp.

Proof. In [1] it was shown that if P is NBUE with mean μ, then

(7) $$\int_0^t P(x)\,dx \geqslant \mu\left[1 - \exp\left(-\frac{t}{\mu}\right)\right] \quad \text{for all } t \geqslant 0.$$

Using (7) for $t > \mu$ we have

$$P(t) \leqslant \frac{1}{\mu} \int_{t-\mu}^{t} P(x)\,dx = \frac{1}{\mu}\Big(\int_0^t P(x)\,dx - \int_0^{t-\mu} P(x)\,dx\Big) \leqslant \exp\Big(-\frac{t-\mu}{\mu}\Big).$$

The upper bound for $t > \mu$ is attained by

$$H(x) = \begin{cases} \exp\Big(-\dfrac{x}{\mu}\Big) & \text{for } x < t-\mu, \\ \exp\Big(-\dfrac{t-\mu}{\mu}\Big) & \text{for } t-\mu \leqslant x < t, \\ 0 & \text{for } x \geqslant t, \end{cases}$$

which is NBUE with mean μ. The degenerate reliability function concentrating at μ provides the upper bound for $t \leqslant \mu$.

THEOREM 3. *If P is NWUE (NWU) with mean μ, then*

$$0 \leqslant P(t) \leqslant \frac{\mu}{\mu+t} \quad \textit{for all } t \geqslant 0.$$

The bounds are sharp.

Proof. Let P be NWUE with mean μ. For a fixed t we have

$$P(t) \leqslant \frac{1}{t} \int_0^t P(x)\,dx$$

and from (2), by reversing the inequality, we obtain

$$P(t) \leqslant 1 - \frac{1}{\mu} \int_0^t P(x)\,dx.$$

Thus $P(t)$ is not greater than

$$\min\Big(1 - \frac{1}{\mu} \int_0^t P(x)\,dx,\ \frac{1}{t} \int_0^t P(x)\,dx\Big).$$

This expression, as a function of $\int_0^t P(x)\,dx$, has its maximum $\mu/(\mu+t)$ for

$$\int_0^t P(x)\,dx = \frac{\mu t}{\mu+t}.$$

We define the function H by

$$H(x) = \Big(\frac{\mu}{\mu+t}\Big)^k \quad \text{for } (k-1)t \leqslant x < kt,\ k = 1, 2, \ldots$$

It is not difficult to verify that $H(x)$ is NWU with mean μ and reaches at t its maximum $\mu/(\mu+t)$. Since NWU is a subclass of NWUE, $H(x)$ reaches the maximum of $P(t)$ also in the NWU class.

To show the lower bounds in both NWU and NWUE classes it is sufficient to take the function

$$H(x) = \varepsilon^k \quad \text{for } (k-1)t \leqslant x < kt,\ k = 1, 2, \ldots,$$

where $\varepsilon > 0$ is arbitrarily small and t is so great that $\int_0^\infty H(x)\,dx = \mu$.

Reference

[1] A. W. Marshall and F. Proschan, *Classes of distributions applicable in replacement with renewal theory implications*, Proc. Sixth Berkeley Symp. 1 (1970), p. 395-415.

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A. KORZENIOWSKI (Wrocław) i A. OPAWSKI (Warszawa)

OSZACOWANIA NIEZAWODNOŚCI W KLASACH NBU, NWU I NBUE, NWUE

STRESZCZENIE

Dla funkcji niezawodności $P(t)$ $(t \geqslant 0)$ i danego $\mu = \int_0^\infty P(x)\,dx$ znajduje się górne i dolne oszacowania $P(t)$ w każdej z klas NBU, NWU i NBUE, NWUE. Dowodzi się, że znalezione oszacowania są najmocniejsze.
