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J. BARTOSZEWICZ (Wrocław)

ON THE MOST BIAS-ROBUST LINEAR ESTIMATES OF THE SCALE PARAMETER OF THE EXPONENTIAL DISTRIBUTION

1. Introduction. Zieliński [3] has considered the statistical model $M_1 = (R_1^+, \mathscr{B}_1^+, \{P_{\lambda,1}, \lambda > 0\})$, where R_1^+ is the real half-line, $\mathscr{B}_1^+$ is the family of Borel subsets of R_1^+, and $P_{\lambda,1}$ is the exponential distribution with probability density function (pdf)

$$f_{\lambda,1}(x) = \frac{1}{\lambda}\exp\left(-\frac{x}{\lambda}\right), \quad x > 0, \tag{1}$$

and its extension $M_{p_1,p_2} = (R_1^+, \mathscr{B}_1^+, \{P_{\lambda,p}, \lambda > 0, p_1 \leqslant p \leqslant p_2\})$, where $0 < p_1 \leqslant 1 \leqslant p_2 \leqslant 2.16$ and $P_{\lambda,p}$ is the exponential power distribution with pdf

$$f_{\lambda,p}(x) = \frac{1}{\lambda\Gamma(1+1/p)}\exp\left[-\left(\frac{x}{\lambda}\right)^p\right], \quad x > 0. \tag{2}$$

For that model Zieliński [3] has studied the problem of robustness with respect to bias of estimates of the scale parameter λ. He has considered linear estimates of λ of the form

$$T_n(a) = \sum_{j=1}^{n} a_j X_j^{(n)}, \tag{3}$$

where $X_1^{(n)}, X_2^{(n)}, \ldots, X_n^{(n)}$ are order statistics of a sample of size n from a distribution $P_{\lambda,p}$ and $a = (a_1, a_2, \ldots, a_n) \in R_n^+$ satisfies the condition

$$\sum_{j=1}^{n} a_j \mathrm{E}_{1,1}(X_j^{(n)}) = 1 \tag{4}$$

(i.e. $T_n(a)$ is the unbiased estimate of λ in the model M_1). Zieliński [3] has proved that for each extension M_{p_1,p_2} $(0 < p_1 \leqslant 1 \leqslant p_2 \leqslant 2.16)$ the estimate $T_n^0 = nX_1^{(n)}$ is the most bias-robust estimate of λ in the class of estimates $T_n(a)$ satisfying (3) and (4), i.e. $b_{T_n^0}(\lambda) \leqslant b_{T_n(a)}(\lambda)$ for each $\lambda > 0$ and every $T_n(a)$ satisfying (3) and (4), where

$$b_{T_n(a)}(\lambda) = \sup_{p_1 \leqslant p \leqslant p_2} [\mathrm{E}_{\lambda,p}(T_n) - \lambda] - \inf_{p_1 \leqslant p \leqslant p_2} [\mathrm{E}_{\lambda,p}(T_n) - \lambda]$$

is the so-called *function of bias-robustness of estimate* $T_n(a)$ (see [2]).

In this note we generalize Zieliński's result considering wider extensions of the model M_1 in which families of distributions consist of distributions with monotone failure rate. The problem of robustness in such models is very important from the reliability point of view.

2. Results. For brevity, in the sequel we use the same notation as in [3]. Let us consider the extension

$$M_{p_1,p_2} = (R_1^+, \mathscr{B}_1^+, \{P_{\lambda,p}, \lambda > 0, p_1 \leqslant p \leqslant p_2\})$$

of the model M_1, where $0 < p_1 \leqslant 1 \leqslant p_2$ and $[p_1, p_2] \subset \Omega \subset R_1^+$. We assume that

(a) $P_{\lambda,1}$ is the exponential distribution with pdf (1);

(b) $P_{\lambda,p}$ has the pdf of the form

$$f_{\lambda,p}(x) = \frac{1}{\lambda} f_p\left(\frac{x}{\lambda}\right);$$

(c) $\{P_{1,p}, p \in \Omega\}$ is a stochastically decreasing family of distributions, and the expected value $\mathrm{E}_{1,p}(X)$ is finite for every $p \in \Omega$;

(d) $P_{1,p}$ is a DFRA distribution for $p \leqslant 1$ and an IFRA distribution for $p \geqslant 1$ (see [1]).

Let us consider also the second extension

$$M^*_{p_1,p_2} = (R_1^+, \mathscr{B}_1^+, \{P^*_{\lambda,p}, \lambda > 0, p_1 \leqslant p \leqslant p_2\})$$

of the model M_1, where $0 < p_1 \leqslant 1 \leqslant p_2$ and $[p_1, p_2] \subset \Omega^* \subset R_1^+$. We assume that

(a*) $P^*_{\lambda,1}$ is the exponential distribution with pdf (1);

(b*) $P^*_{\lambda,p}$ has the pdf of the form

$$f^*_{\lambda,p}(x) = \frac{1}{\lambda} f^*_p\left(\frac{x}{\lambda}\right);$$

(c*) $\{P^*_{1,p}, p \in \Omega^*\}$ is a stochastically increasing family of distributions, and the expected value $\mathrm{E}_{1,p}(X)$ is finite for every $p \in \Omega^*$;

(d*) $P^*_{1,p}$ is a DFRA distribution for $p \leqslant 1$ and an IFRA distribution for $p \geqslant 1$.

We can formulate the following propositions:

PROPOSITION 1. *The estimate* $T^0_n = nX^{(n)}_1$ *is the uniformly most bias-robust estimate of* λ *in every extension* M_{p_1,p_2} $(0 < p_1 \leqslant 1 \leqslant p_2,\ [p_1, p_2] \subset \Omega)$ *of the model* M_1 *in the class of all linear estimates* $T_n(a)$, $a \in R^+_n$, *which are unbiased in* M_1.

PROPOSITION 2. *The estimate*

$$T^*_n = \frac{X^{(n)}_n}{1+1/2+\ldots+1/n}$$

is the uniformly most bias-robust estimate of λ *in every extension* $M^*_{p_1,p_2}$ $(0 < p_1 \leqslant 1 \leqslant p_2,\ [p_1, p_2] \subset \Omega^*)$ *of the model* M_1 *in the class of all linear estimates* $T_n(a)$, $a \in R^+_n$, *which are unbiased in* M_1.

The proof of Proposition 1 is quite similar to that of Zieliński [3]. We indicate only these parts of it in which the more general assumptions are used. The essential step of Zieliński's proof is that the expected values $\mathrm{E}_{1,p}(X^{(n)}_j)$ are monotone functions with respect to p for each (j, n). This is implied by assumption (c). If a family of distributions of a random variable X is stochastically decreasing (increasing), then the family of distributions of every order statistic $X^{(n)}_j$ is also stochastically decreasing (increasing). Furthermore, it is well known that if a family of distributions $\{P_\theta,\ \theta \in \Theta\}$ of a random variable X is stochastically decreasing (increasing), then the expected value $\mathrm{E}_\theta(X)$ is a decreasing (increasing) function of $\theta \in \Theta$.

The second essential element of Zieliński's proof is to show that the function $x/s_p(x)$ is increasing in x for $p \leqslant 1$ and decreasing for $p \geqslant 1$, where

$$s_p(x) = F^{-1}_{1,1}\big(F_{1,p}(x)\big) \quad \text{and} \quad F_{1,p}(x) = \int_0^x f_{1,p}(v)\,dv.$$

It is easy to see that

$$s_p(x) = -\log[1-F_{1,p}(x)] = \int_0^x r_{1,p}(u)\,du,$$

where $r_{1,p}$ is the failure rate function of the distribution $P_{1,p}$ (see [1]). Therefore, we have

$$\frac{x}{s_p(x)} = \left[\frac{1}{x}\int_0^x r_{1,p}(u)\,du\right]^{-1},$$

and if $P_{1,p}$ is a DFRA (IFRA) distribution for $p \leqslant 1$ $(p \geqslant 1)$, then $x/s_p(x)$ is an increasing (decreasing) function in x.

The proof of Proposition 2 is analogous up to the change of monotonicity of the expected values $\mathrm{E}_{1,p}(X_j^{(n)})$.

3. Examples. A. Let us consider Zieliński's problem [3]. The family of distributions $\{P_{1,p},\ 0 < p \leqslant 2.16\}$, where $P_{1,p}$ has the pdf (2) with $\lambda = 1$, is stochastically decreasing and the distribution $P_{1,p}$ is DFR (hence also DFRA) for $p \leqslant 1$ and IFR (hence also IFRA) for $p \geqslant 1$. It follows from Proposition 1 that the statistic T_n^0 is the uniformly most bias-robust estimate of λ in every extension M_{p_1,p_2} of the model M_1 described in Section 1 in the class of all linear estimates satisfying (3) and (4).

B. Consider the model M_1 and its extension

$$\tilde{M}_{p_1,p_2} = (R_1^+, \mathscr{B}_1^+, \{\tilde{P}_{\lambda,p},\ \lambda > 0,\ p_1 \leqslant p \leqslant p_2\}),$$

where $[p_1, p_2] \subset R_1^+ \setminus \{0\}$ and $\tilde{P}_{\lambda,p}$ is the gamma distribution with pdf

$$\tilde{f}_{\lambda,p}(x) = \frac{x^{p-1}e^{-x/\lambda}}{\lambda^p \Gamma(p)}, \quad x > 0.$$

The family of distributions $\{\tilde{P}_{1,p},\ 0 < p < \infty\}$ is stochastically increasing and $\tilde{P}_{1,p}$ is a DFR distribution for $p \leqslant 1$ and an IFR distribution for $p \geqslant 1$. Proposition 2 implies that the statistic T_n^* is the uniformly most bias-robust estimate of λ in every extension $\tilde{M}_{p_1,p_2}$ $(0 < p_1 \leqslant 1 \leqslant p_2 < \infty)$ of the model M_1 in the class of all linear estimates satisfying (3) and (4).

C. Consider the model M_1 and its extension

$$\bar{M}_{p_1,1} = (R_1^+, \mathscr{B}_1^+, \{\bar{P}_{\lambda,p},\ \lambda > 0,\ p_1 \leqslant p \leqslant 1\}),$$

where $p_1 \in (0, 1)$ and $\bar{P}_{\lambda,p}$ is a distribution with pdf

$$\bar{f}_{\lambda,p}(x) = p\,\frac{1}{\lambda}\,e^{-x/\lambda} + (1-p)\,\frac{x^{\beta-1}e^{-x/\lambda}}{\lambda^\beta \Gamma(\beta)}, \quad x > 0,$$

where β is a fixed number from the interval $(0, 1)$. The family of distributions $\{\bar{P}_{1,p},\ p \in (0, 1]\}$ is stochastically increasing, and $\bar{P}_{1,p}$ is a DFR distribution (as a mixture of two DFR distributions, see [1]) for $0 \leqslant p \leqslant 1$. Therefore, by Proposition 2, the statistic T_n^* is the uniformly most bias-robust estimate of λ in every extension $\bar{M}_{p_1,1}$ $(0 < p_1 < 1)$ of the model M_1 in the class of all linear estimates satisfying (3) and (4).

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MATHEMATICAL INSTITUTE
UNIVERSITY OF WROCŁAW
50-384 WROCŁAW

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