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ON A CLASS OF TWO-DIMENSIONAL FLOWS OF A DIPOLAR INCOMPRESSIBLE FLUID

Bleustein and Green [1] examined in detail the continuum theory of dipolar fluids, the simplest of multipolar fluids introduced by Green and Rivlin [2] based on the conventional kinematics. A simple dipolar fluid admits of monopolar and simple dipolar fluid body forces (f_i and F_{ij}) and non-symmetric monopolar and simple dipolar stresses (σ_{ij} and Σ_{ijk}). Further, these stresses are linear functions of the velocity gradients of higher orders. The equations of motion of a simple dipolar incompressible fluid can be written in the simplified form

$$(1)\quad \mu(1-l^2\partial_k\partial_k)v_{i,jj}+\varrho F_i^*-p_{,i}=\varrho(1-d^2\partial_k\partial_k)\dot{v}_i+\varrho d^2(v_{i,kj}v_{k,j}+v_{k,jj}v_{i,k})$$

together with the continuity equation

$$(2)\qquad v_{i,i}=0,$$

where $F_i^*=f_i-F_{ji,j}$. In these equations, v_k is the velocity vector in the direction of the coordinate x_k, p the dipolar (modified) pressure, ϱ the fluid density, μ the coefficient of viscosity, l and d are material constants, having the dimensions of length and which characterize the dipolar nature of the liquid.

The theory of dipolar fluids predicts a non-parabolic velocity profile and a considerably reduced mass flux in isothermal Poiseuille flow in capillary tubes [1] and an increase in the surface drag on a sphere placed in a uniform streaming, the fluid inertia being neglected [3].

The non-linearity of equations characterizing the motion of dipolar fluids is of a greater degree than that in the Newtonian theory. It is, therefore, understandable that these equations admit very few exact solutions to yield the velocity and pressure distributions which fit into the differential equations without any sort of simplification either by approximations or truncations. The aim of the present paper is to exhibit a class of exact solutions of a special type, when the flow is two-dimensional and the external force field is introduced in a suitable manner. The method of obtaining the solution rests mainly on the possibility of balancing the non-linear inertial and dipolar terms with the pressure contingent on the suitably introduced external force. It is of interest to note that the solution is just similar to those obtained in the classical viscous case [4] and in the visco-elastic case [5] but, for a suitable modification of viscosity parameter due to dipolar material constants d and l. Special cases of (i) impulsive force, (ii) steady force and (iii) sinusoidal force have been treated in detail.

A class of exact solutions of equations (1). Employing plane rectangular coordinates x and y, we obtain the solution of (1) and (2) for the given force field $\boldsymbol{F}^*$ with the components expressible in cellular form with single Fourier space components (F^*_x, F^*_y),

$$\begin{aligned} F^*_x &= f_0(t) + f_c(t)\cos ky + f_s(t)\sin ky, \\ F^*_y &= g_0(t) + g_c(t)\cos kx + g_s(t)\sin kx, \end{aligned} \tag{3}$$

with the conditions $f_0 = f_c = f_s = g_0 = g_c = g_s = 0$, $t < 0$ and $k \neq 0$. Further, it is supposed that the liquid starts from rest. Also, the functions f's and g's are arbitrary but of C^2-class.

We notice that, for such a force field, the velocity which is the solution of (1) has the components u and v expressed as

$$\begin{aligned} u &= a_0(t) + a_c(t)\cos ky + a_s(t)\sin ky, \\ v &= b_0(t) + b_c(t)\cos kx + b_s(t)\sin kx, \end{aligned} \tag{4}$$

which evidently satisfy the continuity equation. The functions a's and b's are of C^2-class and are determined from equations (1) under the assumption that non-linear terms arising out of the inertial and dipolar parts of (1) balance with the pressure terms.

Substituting (3) and (4) in (1) and comparing, we infer that

$$\begin{aligned} \dot{a}_0 &= f_0, \\ \dot{a}_c + \eta k^2 a_c + k b_0 a_s &= f_c/(1 + k^2 d^2), \\ \dot{a}_s + \eta k^2 a_s - k b_0 a_c &= f_s/(1 + k^2 d^2), \\ \dot{b}_0 &= g_0, \\ \dot{b}_c + \eta k^2 b_c + k a_0 b_s &= g_c/(1 + k^2 d^2), \\ \dot{b}_s + \eta k^2 b_s - k a_0 b_c &= g_s/(1 + k^2 d^2), \end{aligned} \tag{5}$$

and

$$\frac{\partial p}{\partial x} = -\varrho k(1+2k^2 d^2)\,(b_c \cos kx + b_s \sin kx)\,(-a_c \sin ky + a_s \cos ky),$$

$$\frac{\partial p}{\partial y} = -\varrho k(1+2k^2 d^2)\,(a_c \cos ky + a_s \sin ky)\,(-b_c \sin kx + b_s \cos kx),$$

where $\eta = \mu(1+l^2k^2)/\varrho(1+k^2d^2)$.

The initial conditions require that all the a's and b's vanish for $t < 0$. These equations are structurally the same as those obtained for a classical viscous liquid [4] and for a visco-elastic liquid [5] but for a modification in the viscosity parameter due to the dipolar material constants l and d.

Equations (5) yield the solution

$$a_c(t) = \int_0^t \exp(-\eta k^2 p)\Big\{f_c(t-p)\cos \int_{t-p}^{t} kb_0(t^1)\,dt^1 - -f_s(t-p)\sin \int_{t-p}^{t} kb_0(t^1)\,dt^1\Big\}\,dp/(1+d^2k^2),$$

$$a_s(t) = \int_0^t \exp(-\eta k^2 p)\Big\{f_c(t-p)\sin \int_{t-p}^{t} kb_0(t^1)\,dt^1 + +f_s(t-p)\cos \int_{t-p}^{t} kb_0(t^1)\,dt^1\Big\}\,dp/(1+d^2k^2)$$

and two similar expressions for $b_c(t)$ and $b_s(t)$. From these results we have the velocity distribution

$$u(y,t) = a_0(t) + \int_0^t \exp(-\eta k^2 p)\Big\{f_c(t-p)\cos k\Big[y - \int_{t-p}^{t} b_0(t^1)\,dt^1\Big] + +f_s(t-p)\sin k\Big[y - \int_{t-p}^{t} b_0(t^1)\,dt^1\Big]\Big\}\,dp/(1+d^2k^2),$$

$$v(x,t) = b_0(t) + \int_0^t \exp(-\eta k^2 p)\Big\{g_c(t-p)\cos k\Big[x - \int_{t-p}^{t} a_0(t^1)\,dt^1\Big] + +g_s(t-p)\sin k\Big[x - \int_{t-p}^{t} b_0(t^1)\,dt^1\Big]\Big\}\,dp/(1+d^2k^2),$$

and hence the pressure

$$p = \frac{\varrho(1+2d^2k^2)}{(1+d^2k^2)^2}\int_0^t \exp(-\eta k^2 p)\Big\{f_c(t-p)\sin k\Big[y - \int_{t-p}^{t} b_0(t^1)\,dt^1\Big] -$$

$$-f_s(t-p)\cos k\left[y-\int_{t-p}^{t} b_0(t^1)\,dt^1\right]\Bigg\}dp\times$$

$$\times\int_0^t \exp(-\eta k^2 p)\left\{g_c(t-p)\sin k\left[x-\int_{t-p}^{t} a_0(t^1)\,dt^1\right]-\right.$$

$$\left.-g_s(t-p)\cos k\left[x-\int_{t-p}^{t} a_0(t^1)\,dt^1\right]\right\}dp$$

but for an additive constant of integration.

Special cases.

(i) *Impulsive force.* $f_c(t)=f_s(t)=U\delta(t-t_0)$ and $g_c(t)=g_s(t)=V\delta(t-t_0)$, $\delta(t-t_0)$ being the Dirac unit impulse function, and $a_0=a$, $b_0=b$ are constants.

In this case, we obtain the velocity

$$u(y,t)=a+2^{1/2}U\exp\{-\eta k^2(t-t_0)\}\sin k\left[y-b(t-t_0)+\frac{\pi}{4k}\right]/(1+d^2k^2)$$

and

$$v(x,t)=b+2^{1/2}V\exp\{-\eta k^2(t-t_0)\}\sin k\left[x-a(t-t_0)+\frac{\pi}{4k}\right]/(1+d^2k^2)$$

together with the pressure

$$\frac{p-p_0}{\varrho}=\frac{2UV(1+2d^2k^2)}{(1+k^2d^2)^2}\exp\{-2\eta k^2(t-t_0)\}\cos k\left[y-b(t-t_0)+\frac{\pi}{4k}\right]\times$$

$$\times\cos k\left[x-a(t-t_0)+\frac{\pi}{4k}\right].$$

The initial cellular structure of the velocity is blown down the stream by the mean flow with components (a,b) and decays exponentially with the characteristic time $1/\eta k^2$. The pressure, being quadratic in velocity components, decays at twice the rate of the velocity.

(ii) *Steady force.* $f_c(t)=f_s(t)=X$ and $g_c(t)=g_s(t)=Y$, where X, Y and $a_0(t)=a$, $b_0(t)=b$ are constants.

In this case, we obtain the velocity

$$u(y,t)=a+2X\{\cos k(y-\varepsilon)-\exp(-\eta k^2t)\cos k(y-\varepsilon-bt)\}/kR(1+d^2k^2)$$

and

$$v(x,t)=b+2Y\{\cos k(x-\theta)-\exp(-\eta k^2t)\cos k(x-\theta-at)\}/kS(1+d^2k^2)$$

together with the pressure

$$\frac{p-p_0}{\varrho}=\frac{4XY(1+2d^2k^2)}{k^2RS(1+d^2k^2)^2}\{\sin k(y-\varepsilon)-\exp(-\eta k^2t)\sin k(y-\varepsilon-bt)\}\times$$
$$\times\{\sin k(x-\theta)-\exp(-\eta k^2t)\sin k(x-\theta-at)\},$$

where $\eta k-b=R\cos k\varepsilon$, $\eta k+b=R\sin k\varepsilon$, $\eta k-a=S\cos k\theta$ and $\eta k+a=R\sin k\theta$. In this case, the solution consists of a transient part which decays exponentially as it is blown down the stream and a steady part.

(iii) *Sinusoidal force.* The system at rest is agitated by the force

$$F_x(y,t)=A\cos k(y-Ut)\quad\text{and}\quad F_y(x,t)=B\cos k(x-Vt).$$

In this case, we obtain the velocity

$$u(y,t)=A\{\cos k(y+\varepsilon-Ut)-\exp(-\eta k^2t)\cos k(y+\varepsilon)\}/kR(1+d^2k^2)$$

and

$$v(x,t)=B\{\cos k(x+\theta-Vt)-\exp(-\eta k^2t)\cos k(x+\theta)\}/kS(1+d^2k^2)$$

together with the pressure

$$\frac{p-p_0}{\varrho}=\frac{AB(1+2d^2k^2)}{k^2RS(1+d^2k^2)^2}\{\sin k(y-\varepsilon-Ut)-\exp(-\eta k^2t)\sin k(y+\varepsilon)\}\times$$
$$\times\{\sin k(x+\theta-Vt)-\exp(-\eta k^2t)\sin k(x+\theta)\},$$

where $\eta k=R\cos k\theta$, $U=R\sin k\theta$, $\eta k=S\cos k\theta$ and $V=S\sin k\theta$.

References

[1] J. L. Bleustein and A. E. Green, *Dipolar fluids*, Int. J. Eng. Sci. 5 (1967), p. 323-340.

[2] A. E. Green and R. S. Rivlin, *Multipolar continuum mechanics*, Archs. Ration. Mech. Analysis 17 (1964), p. 113-147.

[3] R. N. Hills, *The slow flow of a dipolar fluid past a sphere*, Int. J. Eng. Sci. 5 (1967), p. 957-967.

[4] W. C. Meecham, *Some exact solutions of the Navier Stokes' and the hydro-mechanic equations*, Phys. Fluids 2 (1959), p. 121-124.

[5] N. Ch. Pattabhi Ramacharyulu, *Problems in the study of non-Newtonian fluid dynamics*, Ph. D. Thesis, Osmania University 1967.

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O KLASIE PRZEPŁYWÓW DWUWYMIAROWYCH DIPOLOWEJ CIECZY NIEŚCIŚLIWEJ

STRESZCZENIE

W pracy znaleziono dokładne rozwiązanie równań ruchu dwuwymiarowego cieczy dipolowej w przypadku „komórkowego" typu ruchu. Zastosowana metoda oparta jest głównie na możliwości zrównoważenia nieliniowych wyrazów bezwładnościowych i dipolowych przez ciśnienie w przypadku odpowiednio dobranych sił zewnętrznych. Dokładnie rozpatrzono trzy rodzaje sił: (i) impulsu, (ii) stałej oraz (iii) sinusoidalnej.