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ESTIMATION OF RELIABILITY IN THE EXPONENTIAL CASE (I)

1. Introduction and summary. One of the most important characteristics in the reliability theory is the *reliability function*, i.e. the probability that the life-time of element is not less than a given time t. If the element has the one-parameter exponential life-time distribution with probability density function (pdf)

(1)
$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geqslant 0, \\ 0 & \text{for } x < 0, \end{cases}$$

then the reliability function is of the form

(2)
$$R(t) = P\{X \geqslant t\} = e^{-\lambda t}.$$

There are many papers concerning the statistical estimation of this function for various designs of life test. The most often applied life designs are the following:

(a) with replacement of the failed items and with duration of the observation until the moment of the r-th failure ($r \leqslant N$, where N is the number of all tested items);

(b) with replacement — until the fixed moment T;

(c) without replacement — until the moment of the r-th failure;

(d) without replacement — until the fixed moment T.

These designs are described in detail in the book [2].

The most popular estimation in the reliability theory is the classical minimum variance unbiased estimation based on the sufficient and complete statistics. Many authors have given minimum variance unbiased estimators of the function $R(t)$ in cases (a), (b) (see, for example, Gnedenko et al. [2]) and (c) (see Basu [1], Laurant [5], Pugh [6] and Tate [7]). These estimators have usually been obtained in two ways: using the Rao-Blackwell-Lehmann-Scheffe theorem and using the method of integral transforms. In the present note we derive the unbiased estimator of the function $R(t)$ based on the sufficient statistic in case (d).

Let N identical items having the life-time distribution with pdf (1) be placed simultaneously on life test according to case (d). Let $D(t)$ denote the number of failures until the moment t and let $x_1, x_2, \ldots, x_{D(T)}$ be the moments of failures until the moment T. The joint pdf of the vector $(X_1, X_2, \ldots, X_{D(T)}, D(T))$ is of the form

$$p(x_1, \ldots, x_d, d) = \frac{N!}{d!(N-d)!} \lambda^d \exp\left\{-\lambda\left[\sum_{i=1}^{d} x_i + (N-d)T\right]\right\},$$

$$d = 0, 1, \ldots, N, \quad 0 \leqslant x_i \leqslant T, \quad i = 1, 2, \ldots, d.$$

The factorization theorem implies that the sufficient statistic for the parameter λ is the vector $(D(T), S(T))$, where

$$S(T) = \sum_{i=1}^{D(T)} X_i + (N - D(T))T$$

is the accumulated observed life-time of all items.

The estimator

(3)
$$\tilde{R}(t) = \begin{cases} 1 - \dfrac{D(t)}{N} & \text{for } 0 \leqslant t \leqslant T, \\ \left(1 - \dfrac{D(T)}{N}\right)\left(1 - \dfrac{D(T)}{N-1}\right) \cdots \left(1 - \dfrac{D(t-pT)}{N-p}\right) & \text{for } pT < t \leqslant (p+1)T, \end{cases}$$

where $p = 1, 2, \ldots, N-1$, is an unbiased estimator of $R(t)$ based on the observations which have been obtained in life design (d), but it is not a sufficient statistic function. (The unbiasedness of $\tilde{R}(t)$ will be proved in the next section.)

Taking the conditional expectation

(4) $E\{\tilde{R}(t) \mid D(T), S(T)\} = \hat{R}(t),$

we obtain a new unbiased estimator of $R(t)$. By virtue of the Rao-Blackwell theorem, this estimator has a variance smaller than $\tilde{R}(t)$, more generally, it is even better than $\tilde{R}(t)$ under the assumption of a strictly convex loss function. Unfortunately, we cannot state that this estimator is unbiased and of minimum variance since the statistic $(D(T), S(T))$, as it will be proved in section 4, is not complete.

2. Unbiasedness of the estimator $\tilde{R}(t)$.

THEOREM 1. $\tilde{R}(t)$ given by (3) is an unbiased estimator for the function $R(t)$.

The proof of this theorem is based on the following lemmas which are easy to verify:

LEMMA 1. *The random variable $D(T)$ has the binomial distribution* $b(N, d; \; p_0)$ *with* $p_0 = 1 - e^{-\lambda T}$.

LEMMA 2. *If $t \leqslant T$, then the joint distribution of $\big(D(t), \; D(T)\big)$ is of the form*

$$(5) \quad P\{D(t) = k, \; D(T) = d\}$$

$$= \frac{N!}{k!(d-k)!(N-d)!} \, (1 - e^{-\lambda t})^k (e^{-\lambda t} - e^{-\lambda T})^{d-k} e^{-\lambda T(N-d)}, \quad 0 \leqslant k \leqslant d \leqslant N.$$

LEMMA 3. *If $t \leqslant T$, then the conditional distribution of $D(t)$, given* $D(T) = d$, *is the binomial $b(d, k; \; p_0)$ with*

$$p_0 = (1 - e^{-\lambda t})/(1 - e^{-\lambda T}).$$

Proof of theorem 1. If $t \leqslant T$, then the theorem follows immediately from lemma 1.

Let us write

$$(6) \quad \Phi_{N,p}(D) = \left(1 - \frac{D(T)}{N}\right)\left(1 - \frac{D(T)}{N-1}\right) \cdots \left(1 - \frac{D(T)}{N-p+1}\right).$$

If $pT < t \leqslant (p+1)T$, $p = 1, 2, \ldots, N-1$, we have

$$(7) \quad E\tilde{R}(t) = E\,\Phi_{N,p}(D) - E\left\{\Phi_{N,p}(D)\frac{D(t^*)}{N-p}\right\},$$

where $t^* = t - pT \epsilon (0, T]$. It is easy to see that

$$E\,\Phi_{N,p}(D) = \frac{1}{N(N-1)\ldots(N-p+1)}\,\mu_{[p]},$$

where $\mu_{[p]}$ denotes the p-th factorial moment of the random variable $V(T) = N - D(T)$. The random variable $V(T)$ has a binomial distribution $b(N, k; \; e^{-\lambda T})$, and whence (see Kendall and Stuart [4], p. 99)

$$\mu_{[p]} = N(N-1)\ldots(N-p+1)e^{-\lambda pT}.$$

This yields

$$(8) \quad E\,\Phi_{N,p}(D) = e^{-\lambda pT}.$$

Now we evaluate the second expectation in (7). From lemmas 1-3 we obtain

$$E\left\{\Phi_{N,p}(D)\frac{D(t^*)}{N-p}\right\} = E\left\{\Phi_{N,p}(D)E\left[\frac{D(t^*)}{N-p}\,\Big|\,D(T)\right]\right\}$$

$$= \frac{1 - e^{-\lambda t^*}}{1 - e^{-\lambda T}}\,E\left\{\Phi_{N,p}(D)\frac{D(T)}{N-p}\right\}.$$

Since, according to (6),

$$\mathrm{E}\left\{\varPhi_{N,p}(D)\,\frac{D(T)}{N-p}\right\} = \mathrm{E}\,\varPhi_{N,p}(D) - \mathrm{E}\,\varPhi_{N,p+1}(D),$$

we get

$$\mathrm{E}\,\tilde{R}(t) \;=\; \frac{1}{1-e^{-\lambda T}}\,[(e^{-\lambda t^*} - e^{-\lambda T})\,\mathrm{E}\,\varPhi_{N,p}(D) + (1-e^{-\lambda t^*})\,\mathrm{E}\,\varPhi_{N,p+1}(D)]$$

$$= e^{-\lambda(pT+t^*)} = e^{-\lambda t}.$$

3. Main result.

THEOREM 2. *The estimator*

$$\hat{R}(t) \;=\; \mathrm{E}\,[\tilde{R}(t)\mid D(T),\,S(T)]$$

has the form

(9)

$$\hat{R}(t) \;=\; \begin{cases} 1 - \dfrac{R(d,\,s;\,t)}{N} & \textit{for } 0 \leqslant t \leqslant T, \\[2ex] \left(1 - \dfrac{d}{N}\right)\!\left(1 - \dfrac{d}{N-1}\right)\cdots\left(1 - \dfrac{d}{N-p+1}\right)\!\left(1 - \dfrac{R(d,\,s;\,t-pT)}{N-p}\right) \\[1ex] \qquad\qquad \textit{for } pT < t \leqslant (p+1)T,\; p = 1,2,\ldots,N-1, \end{cases}$$

where

(10) $R(d,\,s;\,t)$

$$= \begin{cases} 0 & \textit{for } d = 0, \\[2ex] \displaystyle\sum_{k=0}^{d} k\binom{d}{k}\,\frac{\displaystyle\sum_{i=0}^{d-k}\sum_{j=0}^{k}(-1)^{i+j}\binom{k}{j}\binom{d-k}{i}[s-(d-k+j-i)t-(N-d+i)T]_+^{d-1}}{\displaystyle\sum_{j=0}^{d}(-1)^{j}\binom{d}{j}[s-(N-d+j)T]_+^{d-1}} \\[2ex] & \textit{for } d > 0, \end{cases}$$

and

$$a_+^m = [\max(0,\,a)]^m \quad \textit{for } m > 0$$

and

$$a_+^0 = \begin{cases} 0 & \textit{for } a \leqslant 0, \\ 1 & \textit{for } a > 0. \end{cases}$$

Before the proof of this theorem we state the following lemmas.

LEMMA 4. *The conditional pdf of* $S(T)$, *given* $D(T) = d$, *is of the form*

(11) $f_T^{d^*}(s-(N-d)T)$

$$= \frac{\lambda^d \exp[-\lambda(s-(N-d)T)]}{(d-1)!\,(1-e^{-\lambda T})^d}\,\sum_{j=0}^{d}(-1)^{j}\binom{d}{j}[s-(N-d+j)T]_+^{d-1} \quad \textit{for } d > 0$$

and

$$f_T^{0*}(s - NT) = \delta_{NT}(s) \quad \text{for } d = 0,$$

where $\delta_{NT}(s)$ stands for pdf of the random variable values NT with probability 1.

Proof. It is easy to see that the conditional distribution of random variable $S(T) - [N - D(T)]T$, given $D(T) = d$, is the same as the unconditional distribution of the sum of d independent identically, exponentially distributed random variables truncated at the point T. Pdf of this sum is of the form $f_T^{d*}(s)$ and has been derived by Hoem [3].

LEMMA 5. *The joint pdf of the sufficient statistic $(D(T), S(T))$ is of the form*

$$(12) \qquad g(s, d) = \frac{\lambda^d \binom{N}{d} e^{-\lambda s}}{(d-1)!} \sum_{j=0}^{d} (-1)^j \binom{d}{j} [s - (N - d + j)T]_+^{d-1}$$

$$\text{for } d = 1, 2, \ldots, N$$

and

$$g(s, 0) = \delta_{NT}(s) e^{-\lambda TN}.$$

Proof. This lemma follows immediately from lemmas 1 and 4.

LEMMA 6. *If $Y_1, \ldots, Y_m$ are independent, identically distributed random variables with pdf*

$$f_{(t,T)}(y) = \begin{cases} \dfrac{\lambda e^{-\lambda y}}{e^{-\lambda t} - e^{-\lambda T}} & \text{for } t \leqslant y \leqslant T, \\ 0 & \text{otherwise,} \end{cases}$$

then the random variable $U = Y_1 + Y_2 + \ldots + Y_m$ has pdf of the form

$$(13) \quad f_{(t,T)}^{m*}(y) = \frac{\lambda^m e^{-\lambda y}}{(m-1)!(e^{-\lambda t} - e^{-\lambda T})^m} \sum_{j=0}^{m} (-1)^j \binom{m}{j} [y - (m-j)t - jT]_+^{m-1}.$$

We set, in addition,

$$f_{(t,T)}^{0*}(y) = \delta_0(y).$$

Proof. The probability density function (13) can be obtained from the density $f_{T-t}^{m*}(s)$ (see lemma 4) by the translation $y = s - mt$.

LEMMA 7. *The conditional pdf of $S(T)$, given $D(t) = k$ and $D(T) = d$ for $t \leqslant T$, is of the form*

$$(14) \quad h(s \mid k, d) = \frac{\lambda^d e^{-\lambda(s - (N-d)T)}}{(d-1)!(1 - e^{-\lambda t})^k (e^{-\lambda t} - e^{-\lambda T})^{d-k}} \times$$

$$\times \sum_{i=0}^{d-k} \sum_{j=0}^{k} (-1)^{i+j} \binom{k}{j} \binom{d-k}{i} [s - (d - k + j - i)t - (N - d + i)T]_+^{d-1}$$

$$\text{for } 0 \leqslant k \leqslant d \text{ and } d > 0,$$

and

(15) $h(s \mid 0, 0) = \delta_{NT}(s).$

Proof. If $D(t) = k$ and $D(T) = d$ $(t \leqslant T, 0 < k < d)$, then the random variable $S(T) - (N - d)T$ is the sum of life-times of those k elements which failed in the interval $(0, t)$ and the life-times of $d - k$ elements which failed in the interval $[t, T)$. Hence, the conditional pdf of $Y(T) = S(T) - (N - d)T$, given $D(t) = k$ and $D(T) = d$, is of the form

(16) $p(y \mid k, d) = \int\limits_{-\infty}^{\infty} f_t^{k*}(y - v) f_{(t,T)}^{(d-k)*}(v) \, dv,$

where the functions f_t^{k*} and $f_{(t,T)}^{(d-k)*}$ are defined by (11) and (13). Substituting into (16) expressions (11) and (13), we obtain

(17) $p(y \mid k, d) = C(y) \sum\limits_{i=0}^{d-k} \sum\limits_{j=0}^{k} (-1)^{i+j} \binom{k}{j}\binom{d-k}{i} \times$

$$\times \int\limits_{-\infty}^{\infty} [y - v - jt]_+^{k-1} [v - (d - k - i)t - iT]_+^{d-k-1} dv,$$

where

$$C(y) = \frac{\lambda^d e^{-\lambda y}}{(k-1)!(d-k-1)!(1 - e^{-\lambda t})^k (e^{-\lambda t} - e^{-\lambda T})^{d-k}}.$$

Next, we evaluate the integral

$$I = \int\limits_{-\infty}^{\infty} [y - v - jt]_+^{k-1} [v - (d - k - i)t - iT]_+^{d-k-1} dv.$$

It is easy to verify that

$$I = \int\limits_{a}^{\max(a,b)} (b - v)^{k-1}(v - a)^{d-k-1} dv,$$

where $a = (d - k)t + i(T - t)$ and $b = y - jt$. After the change of variables $z = v - a$ and substituting $b - a = c$ we have

$$I = \int\limits_{0}^{c_+} (c_+ - z)^{k-1} z^{d-k-1} dz = c_+^{d-2} \int\limits_{0}^{c_+} \left(1 - \frac{z}{c_+}\right)^{k-1} \left(\frac{z}{c_+}\right)^{d-k-1} dz$$

$$= c_+^{d-1} \int\limits_{0}^{1} (1 - u)^{k-1} u^{d-k-1} du = c_+^{d-1} B(k, d - k),$$

where $B(k, d - k)$ is the beta-function.

Returning to the previous notations and substituting $y = s - (N - d)T$, we obtain, from (17), pdf (14).

It is easy to see that formula (14) is also valid in cases $k = 0$ and $d = k$ $(d > 0)$. In case $d = 0$ it is obvious that $S(T) = NT$, and whence (15) is true.

LEMMA 8. *If* $t \leqslant T$, *then the conditional distribution* $D(t)$, *given* $D(T) = d$ *and* $S(T) = s$, *is of the form*

(18) $\quad \mathrm{P}\{D(t) = k \mid D(T) = d, S(T) = s\}$

$$= \binom{d}{k} \frac{\displaystyle\sum_{i=0}^{d-k}\sum_{j=0}^{k}(-1)^{i+j}\binom{k}{j}\binom{d-k}{i}[s-(d-k+j-i)t-(N-d+i)T]_+^{d-1}}{\displaystyle\sum_{j=0}^{d}(-1)^j\binom{d}{j}[s-(N-d+j)T]_+^{d-1}}$$

$$\text{for } d > 0 \text{ and } 0 \leqslant k \leqslant d,$$

and

$$\mathrm{P}\{D(t) = 0 \mid D(T) = 0, S(T) = NT\} = 1.$$

Proof. This lemma follows from the fact that

$$\mathrm{P}\{D(t) = k \mid D(T) = d, S(T) = s\} = \frac{h(s \mid k, d)\mathrm{P}\{D(t) = k, D(T) = d\}}{g(s, d)}$$

and from lemmas 2, 5 and 7.

Proof of theorem 2. In order to derive the estimator $\hat{R}(t)$ it is sufficient to calculate the conditional expectation

$$\mathrm{E}[D(t) \mid D(T), S(T)] = R(d, s; t) \quad \text{for } t \leqslant T.$$

Taking the expectation of distribution (18), we obtain $R(d, s; t)$, and whence $\hat{R}(t)$.

4. Uncompleteness of the sufficient statistic $(D(T), S(T))$.

THEOREM 3. *The statistic* $(D(T), S(T))$ *is not complete except for the trivial case* $N = 1$.

Proof. In order to prove the completeness of the statistic $(D(T), S(T))$ it is necessary to show that, for any function $\varphi(d, s)$, the condition

(19) $\quad\quad\quad \mathrm{E}\varphi(D, S) \equiv 0 \quad \text{for every } \lambda > 0$

implies $\varphi(d, s) = 0$ with probability one.

Expression (19) is of the form

$$\mathrm{E}\varphi(D, S) = \sum_{d=1}^{N}\binom{N}{d}\frac{\lambda^d}{(d-1)!}\int_{(N-d)T}^{NT}e^{-\lambda s}\varphi(d, s) \times$$

$$\times \sum_{j=0}^{d}(-1)^j\binom{d}{j}[s-(N-d+j)T]_+^{d-1}+\varphi(0, NT)e^{-\lambda NT} \equiv 0 \quad \text{for every } \lambda > 0.$$

After easy modifications we get

(20) $\displaystyle\sum_{d=1}^{N} \frac{\binom{N}{d}}{(d-1)!} \lambda^d \sum_{j=0}^{d-1} (-1)^j \binom{d}{j} e^{\lambda(d-j)T} \times$

$\displaystyle\times \int_0^{(d-j)T} \varphi\big(d, s+(N-d+j)T\big) s^{d-1} e^{-\lambda s} \equiv C = -\varphi(0, NT).$

If $N = 1$, then from (20) we have

$$\lambda e^{\lambda T} \int_0^T \varphi(1, s) e^{-\lambda s}\, ds \equiv C,$$

and whence

(21) $\displaystyle\int_0^\infty \tilde{\varphi}(1, s) e^{-\lambda s}\, ds \equiv \frac{C}{\lambda} e^{-\lambda T},$

where

$$\tilde{\varphi}(1, s) = \begin{cases} \varphi(1, s) & \text{for } 0 \leqslant s < T, \\ 0 & \text{otherwise}. \end{cases}$$

The right-hand side of (21) is the Laplace transform of the function

$$\psi(1, s) = \begin{cases} 0 & \text{for } 0 \leqslant s < T, \\ C & \text{for } s \geqslant T. \end{cases}$$

It follows from the uniqueness of the Laplace transform that $\varphi(1, s) = 0$ almost everywhere (a.e.) for $0 \leqslant s < T$, and also $C = -\varphi(0, T) = 0$. This proves the completeness for the case $N = 1$.

For clarity reasons, we prove the uncompleteness of the statistic $\big(D(T), S(T)\big)$ only in the case $N = 2$. In generality, the proof can be done similarly.

For $N = 2$, expression (20) has the form

$$2\lambda e^{\lambda T} \int_0^T \varphi(1, s+T) e^{-\lambda s}\, ds +$$

$$+ \lambda^2 \Big[e^{2\lambda T} \int_0^{2T} \varphi(2, s) s e^{-\lambda s}\, ds - 2 e^{\lambda T} \int_0^T \varphi(2, s+T) s e^{-\lambda s}\, ds \Big] \equiv C.$$

By easy transformations we get

$$\frac{1}{\lambda} \int_0^{2T} \varphi(1, s) e^{-\lambda s}\, ds + \frac{1}{2} \int_0^{2T} \varphi(2, s) s e^{-\lambda s}\, ds - \int_T^{2T} \varphi(2, s) s e^{-\lambda s}\, ds +$$

$$+ T \int_T^{2T} \varphi(2, s) e^{-\lambda s}\, ds \equiv \frac{C}{2\lambda^2} e^{-2\lambda T}, \quad \text{where } \varphi(1, s) = 0 \text{ for } s \notin (T, 2T].$$

By elementary properties of Laplace transforms this can be written in the form

(22) $$\int\limits_0^\infty \tilde{\varphi}(s)e^{-\lambda s}ds \equiv \frac{C}{2\lambda^2}e^{-2\lambda T},$$

where the function $\tilde{\varphi}(s)$ is defined as

$$\tilde{\varphi}(s) = \begin{cases} \dfrac{\varphi(2,s)}{2}\,s & \text{for } 0 \leqslant s < T, \\[2ex] \int\limits_0^s \varphi(1,t)dt + \dfrac{\varphi(2,s)}{2}\,(2T-s) & \text{for } T \leqslant s < 2T, \\[2ex] \int\limits_0^{2T} \varphi(1,t)dt & \text{for } s \geqslant 2T. \end{cases}$$

The right-hand side of (22) is the Laplace transform of the function

$$\psi(s) = \begin{cases} 0 & \text{for } 0 \leqslant s < 2T, \\[1ex] \dfrac{C}{2}(s-2T) & \text{for } s \geqslant 2T. \end{cases}$$

Hence we obtain $\varphi(2,s) = 0$ a.e. for $0 \leqslant s < T$, but also

$$\varphi(2,s) = \frac{2}{s-2T}\int\limits_0^s \varphi(1,t)dt \qquad \text{for } T \leqslant s < 2T$$

and

$$\frac{C(s-2T)}{2} = \int\limits_T^{2T} \varphi(1,t)dt.$$

This proves that the statistic $(D(T),\, S(T))$ is not complete in the case $N = 2$.

If $N > 2$, in (20) we can set $\varphi(d,s) \equiv 0$ for $d > 2$ and the proof runs similarly.

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ESTYMACJA NIEZAWODNOŚCI W PRZYPADKU WYKŁADNICZYM (I)

STRESZCZENIE

W określaniu niezawodności stosuje się najczęściej cztery plany badania (a) z odnową uszkodzonych elementów i czasem trwania obserwacji do momentu r-tej awarii, (b) z odnową — do ustalonego momentu T, (c) bez odnowy — do momentu r-tej awarii, (d) bez odnowy — do ustalonego momentu T. W literaturze znane są liczne prace poświęcone nieobciążonej estymacji z minimalną wariancją funkcji wykładniczej niezawodności $R(t) = e^{-\lambda t}$ dla trzech pierwszych planów badania. Estymatory te są funkcjami statystyk dostatecznych i zupełnych. W tej pracy podano nieobciążony estymator funkcji $R(t)$, oparty na statystyce dostatecznej dla czwartego z wymienionych planów badania. Nie można jednak twierdzić, że jest to estymator o minimalnej wariancji, ponieważ statystyka dostateczna nie jest zupełna.