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EVALUATION OF COEFFICIENTS OF A DOUBLE CHEBYSHEV SERIES OF THE FUNCTION $g(p(x+y))$

Abstract. In the paper we give an algorithm for computing estimations of the coefficients of the double Chebyshev series of the function $g(p(x+y))$ of two variables by means of the Chebyshev coefficients of the function $g(x)$ of one variable.

1. Introduction. Let the function $g(x)$ defined in $[-1, 1]$ have there a uniformly convergent Chebyshev series

(1.1) $$\sum_{n=0}^{\infty}{}' b_n T_n(x),$$

where $\sum'$ means that the first term of the sum is taken with factor $\frac{1}{2}$. For a real number p, $p \neq 0$, we consider a function of two variables x, y in the form

(1.2) $$f(x, y) = g(p(x+y)), \quad -1 \leqslant x, y \leqslant 1, |p| < 0.5,$$

and its double Chebyshev series

(1.3) $$\sum_{k=0}^{\infty}{}' \sum_{l=0}^{\infty}{}' a_{kl} T_k(x) T_l(y),$$

where

(1.4) $$a_{kl} = a_{kl}[f] = \frac{4}{\pi^2} \int_{-1}^{1} \int_{-1}^{1} \frac{T_k(x) T_l(y) f(x, y)}{(1-x^2)^{1/2}(1-y^2)^{1/2}} dx\, dy.$$

In this paper we give an algorithm for computing estimations of the coefficients a_{kl} by means of the coefficients b_n.

2. Connection between coefficients a_{kl} and b_n. We introduce the notation

$$T_{kl} \equiv T_k(x) T_l(y), \quad a_{kl}^{(n)} \equiv a_{kl}[T_n(p(x+y))]$$

and, in accordance with general usage, we put

(2.1) $$a_{-k,-l} = a_{k,-l} = a_{-k,l} = a_{kl}.$$

In [5] we gave formulae for the Chebyshev coefficients $a_{kl}^{(n)}$ of the function $T_n(p(x+y))$ and some recurrence relations for them. We now express the coefficients (1.4) in terms of the coefficients b_n and $a_{kl}^{(n)}$. From (1.1) we have

$$\text{(2.2)} \quad \sum_{n=0}^{\infty}{}' b_n T_n(p(x+y)) = \sum_{n=0}^{\infty}{}' b_n \sum_{k=0}^{\infty}{}' \sum_{l=0}^{\infty}{}' a_{kl}^{(n)} T_{kl}$$
$$= \sum_{n=0}^{\infty}{}' b_n \sum_{k=0}^{n}{}' \sum_{l=0}^{n-k}{}' a_{kl}^{(n)} T_{kl} = \sum_{k=0}^{\infty}{}' \sum_{l=0}^{\infty}{}' \Big(\sum_{n=k+l}^{\infty}{}^{\circ} b_n a_{kl}^{(n)} \Big) T_{kl},$$

where $a_{kl}^{(n)} = 0$ for $k+l > n$ and $\sum^{\circ}$ means that the term in which $k+l = 0$ is taken with factor 1/2. We consider the N-th partial sum of the series (1.1) for the argument $p(x+y)$:

$$\text{(2.3)} \quad \sum_{n=0}^{N}{}' b_n T_n(p(x+y)) = \sum_{k=0}^{N}{}' \sum_{l=0}^{N-k}{}' \sum_{n=k+l}^{N}{}^{\circ} b_n a_{kl}^{(n)} T_{kl}.$$

Let

$$\text{(2.4)} \quad c_{kl} = \sum_{n=k+l}^{N}{}^{\circ} b_n a_{kl}^{(n)}.$$

From (1.3) and (2.2) it follows that

$$a_{kl} = \sum_{n=k+l}^{\infty}{}^{\circ} b_n a_{kl}^{(n)} = c_{kl} + \sum_{n=N+1}^{\infty} b_n a_{kl}^{(n)}.$$

Thus the coefficients c_{kl} of the double Chebyshev series of the partial sum (2.3) form an estimation of the Chebyshev coefficients a_{kl} of the function (1.2). It is known that $a_{kl}^{(n)} = a_{lk}^{(n)}$. Thus from (2.4) it follows that $c_{kl} = c_{lk}$.

We can use the algorithm given by Basu [1] for computing the partial sum of the double Chebyshev series. Basu considered also some examples of the application of these series (see also [3]).

3. Algorithm for the evaluation of the coefficients c_{kl}. Now we give an algorithm for computing the coefficients c_{kl}. This algorithm is a consequence of Paszkowski's method of a transformation of the sum

$$\sum_{n=0}^{N}{}' b_n T_n(qx+r)$$

with given coefficients b_n and given constants q and r into the sum

$$\sum_{n=0}^{N}{}' d_n T_n(x)$$

which is equivalent to the first one. The algorithm of Paszkowski follows (see [4]) from Clenshaw's algorithm for computing values of a linear combination of Chebyshev polynomials. We now recall Clenshaw's algorithm (see, e.g., [2], p. 56, and [3], p. 275).

Clenshaw's algorithm. For given numbers N, b_0, $b_1, \ldots, b_N$ we compute

$$\sigma_{N+2} = 0, \quad \sigma_{N+1} = 0,$$

(3.1) $$\sigma_m = b_m + 2x\sigma_{m+1} - \sigma_{m+2} \quad (m = N, N-1, \ldots, 1),$$

$$\sigma_0 = \tfrac{1}{2} b_0 + x\sigma_1 - \sigma_2.$$

Then

$$\sigma_0 = \sum_{n=0}^{N}{}' b_n T_n(x).$$

We introduce the auxiliary polynomials $S_m(x, y)$ of variables x and y defined by the formulae (compare (3.1))

$$S_{N+2}(x, y) = 0, \quad S_{N+1}(x, y) = 0,$$

(3.2)

$$S_m(x, y) = b_m + 2p(x+y)S_{m+1}(x, y) - S_{m+2}(x, y) \quad (m = N, N-1, \ldots, 1),$$

$$S_0(x, y) = \tfrac{1}{2} b_0 + p(x+y)S_1(x, y) - S_2(x, y).$$

Thus $S_0(x, y)$ is equal to the partial sum (2.3). Let

$$d_{kl}^{(m)} \equiv a_{kl}[S_m(x, y)]$$

be the coefficients of the double Chebyshev series of the polynomial $S_m(x, y)$. From (3.2) it follows that for all indices k and l the relation

$$d_{kl}^{(N+1)} = d_{kl}^{(N+2)} = 0$$

holds. We give recurrence relations from which we can compute c_{kl}.

Theorem. *The Chebyshev coefficients c_{kl} of the partial sum* (2.3) *are expressed by the Chebyshev coefficients b_n of the function $g(x)$ by means of the following recurrence formulae*:

$$d_{kl}^{(m)} = 4\delta_{kl} b_m + p(d_{k-1,l}^{(m+1)} + d_{k+1,l}^{(m+1)} + d_{k,l-1}^{(m+1)} + d_{k,l+1}^{(m+1)}) - d_{kl}^{(m+2)}$$

$$(m = N, N-1, \ldots, 1;\ k = 0, 1, \ldots, N-m;\ l = k, k+1, \ldots, N-m-k),$$

(3.3)

$$c_{kl} \equiv d_{kl}^{(0)} = 2\delta_{kl} b_0 + \tfrac{1}{2} p(d_{k-1,l}^{(1)} + d_{k+1,l}^{(1)} + d_{k,l-1}^{(1)} + d_{k,l+1}^{(1)}) - d_{kl}^{(2)}$$

$$(k = 0, 1, \ldots, N;\ l = k, k+1, \ldots, N-k),$$

where

$$d_{kl}^{(m)} = d_{lk}^{(m)}, \quad c_{kl} = c_{lk},$$

$$d_{kl}^{(m)} = 0 \quad \textit{for } |k|+|l| > N-m,$$

$$\delta_{kl} = \begin{cases} 1 & \textit{for } k = l = 0, \\ 0 & \textit{for } |k|+|l| > 0. \end{cases}$$

Proof. From (2.1) we have

$$S_m(x, y) = \frac{1}{4} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} d_{kl}^{(m)} T_{kl}.$$

Therefore, by means of the recurrence formulae

$$2xT_n(x) = T_{n+1}(x) + T_{n-1}(x)$$

after easy evaluations we immediately receive (3.3) from (3.2).

Remark. Formulae (3.3) can be easily generalized for the case of computing the estimation of the Chebyshev coefficients of the function $g(p(x+y)^2)$.

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