ZASTOSOWANIA MATEMATYKI
APPLICATIONES MATHEMATICAE
Hugo Steinhaus Jubilee Volume
X (1969)

K. URBANIK (Wrocław)

SELF-DECOMPOSABLE PROBABILITY DISTRIBUTIONS ON R^m

Let $X_{nk}(k = 1, 2, \ldots, k_n; n = 1, 2, \ldots)$ be an array of random variables, whose values are vectors in R^m. In each row $X_{n1}, X_{n2}, \ldots, X_{nk_n}$ we assume the variables to be independent and uniformly asymptotically negligible, i.e. for every $\varepsilon > 0$

$$\lim_{n\to\infty} \sup_{1\leqslant k\leqslant k_n} P(|X_{nk}| > \varepsilon) = 0,$$

where $|\ |$ denotes the norm in R^m. Consider sums

$$\sum_{k=1}^{k_n} X_{nk} - a_n \qquad (n = 1, 2, \ldots), \tag{1}$$

where a_n are constant vectors in R^m. It is well known that the class of all possible limit distributions of (1) coincides with the class of infinitely divisible distributions. Moreover, a complex-valued function φ on R^m is the characteristic function of an infinitely divisible distribution if and only if it has a Lévy-Khintchine representation

$$\varphi(z) = \exp\left\{i(a, z) - \frac{1}{2}(Az, z) + \int_{R^m} \left(e^{i(z,u)} - 1 - \frac{i(z, u)}{1+|u|^2}\right) \frac{1+|u|^2}{|u|^2} \mu(du)\right\}, \tag{2}$$

where a is a vector from R^m, A is a symmetric non-negative operator in R^m, μ is a finite Borel measure on R^m vanishing at the origin and (z, u) denotes the inner product in R^m. Further, the function φ determines a, A and μ uniquely (see [1], [6] and [7]).

Let $X_1, X_2, \ldots$ be a sequence of independent random vectors in R^m. Consider normed sums

$$\frac{1}{c_n} \sum_{k=1}^{n} X_k - a_n \qquad (n = 1, 2, \ldots), \tag{3}$$

where a_n are vectors from R^m, c_n are positive numbers and the random variables $X_{nk} = \dfrac{X_k}{c_n}$ $(k = 1, 2, \ldots, n;\ n = 1, 2, \ldots)$ are uniformly asymptotically negligible. It is clear that the limit distribution of (3) is infinitely divisible provided it exists. Moreover, the class of all possible limit distributions of sequences (3) coincides with the class of self-decomposable probability distributions in R^m. We say that a probability distribution in R^m with the characteristic function ψ is self-decomposable if, for every number c satisfying the inequality $0 < c < 1$, there exists a characteristic function ψ_c such that, for every $z \in R^m$, $\psi(z) = \psi(cz)\psi_c(z)$ (see [7], p. 322). For real-valued random variables P. Lévy established some necessary and sufficient conditions for self-decomposability in terms of the Khintchine measure μ in the representation formula (2) ([6], p. 196; [7], p. 324). The aim of this note is to prove an analogue of the Lévy-Khintchine formula for self-decomposable distributions on R^m. Namely, we shall prove the following theorem.

THEOREM. *A function φ on R^m is the characteristic function of a self-decomposable distribution on R^m if and only if*

$$(4)\qquad \varphi(z) = \exp\Big\{i(a, z) - \frac{1}{2}(Az, z) + \int_{R^m}\Big(\int_0^{(z,u)} \frac{e^{it}-1}{t}\,dt - i\,\frac{(z, u)}{|u|}\arctan|u|\Big)\frac{1}{\log(1+|u|^2)}\,\nu(du)\Big\},$$

where a is a vector from R^m, A is a symmetric non-negative operator in R^m (the same as in the Lévy-Khintchine representation) and ν is a finite Borel measure on R^m which assignes zero mass to the origin. Moreover, the function φ determines a, A and ν uniquely.

For the one-dimensional case this representation formula has been proved in [8]. The method of proof, stimulated by results of Kendall [4] and Johansen [3], consists in finding the extreme points of a certain convex set formed by Khintchine measures μ corresponding to self-decomposable distributions. Once the extreme points are found one can apply a theorem by Choquet on representation of the points of a compact convex set as barycenters of the extreme points.

Before proving the theorem we shall prove some lemmas. Let S_m be the m-dimensional unit sphere and $[0, \infty]$ the compactified positive half-line. Put $Q^m = [0, \infty] \times S_{m-1}$. Further, for every $x \in R^m \setminus \{0\}$ we put $h(x) = (|x|, x/|x|)$. It is clear that h is a one-to-one continuous mapping from $R^m \setminus \{0\}$ onto $(0, \infty) \times S_{m-1}$. In what follows we shall identify $R^m \setminus \{0\}$ and $(0, \infty) \times S_{m-1}$. Consequently, Q^m can be regarded as a compactification of $R^m \setminus \{0\}$. Let q be a positive number. Given $r \in [0, \infty]$ and $w \in S_{m-1}$, we put $q(r, w) = (qr, w)$. It is easy to verify that for every $x \in R^m \setminus \{0\}$ the formula $h(qx) = qh(x)$ holds.

Given a finite Borel measure μ on Q^m and a positive number c we define a set-function μ_c by the formula

$$\mu_c(E) = \mu(E) - c^2 \int\limits_{c^{-1}E} \frac{1+|x|^2}{1+c^2|x|^2}\,\mu(dx),$$

where E is an arbitrary Borel subset of Q^m, $c^{-1}E = \{c^{-1}x\colon x\in E\}$ and the integrand is defined as its limiting value c^{-2} when $x\in Q^m\setminus R^m$. Let M_m be the set of all finite Borel measures μ on Q^m such that, for every number c satisfying the inequality $0 < c < 1$, the set-functions μ_c are measures, i.e. are non-negative. Further, by M_m^0 we shall denote the subset of M_m consisting of measures concentrated on $R^m\setminus\{0\}$. Let K_m be the subset of M_m consisting of probability measures and let $K_m^0 = K_m \cap M_m^0$. It is obvious that all sets M_m, M_m^0, K_m and K_m^0 are convex. The space of all probability measures on Q^m with the weak convergence (i.e. weak topology) is a metrizable compact space. We consider the induced topology on K_m.

LEMMA 1. *The set K_m is compact.*

Proof. Of course, it suffices to prove that the set K_m is closed in the space of all probability measures on Q^m. Suppose that $\mu^{(n)}\in K_m$ $(n = 1, 2, \ldots)$ and the sequence $\mu^{(1)}, \mu^{(2)}, \ldots$ is weakly convergent to a probability measure μ. Then, for each positive number c, μ_c is the weak limit of the sequence $\mu_c^{(1)}, \mu_c^{(2)}, \ldots$. Hence it follows that for c satisfying the condition $0 < c < 1$, μ_c is a measure. Consequently, $\mu\in K_m$. Thus the set K_m is closed which completes the proof.

We know that self-decomposable distributions are infinitely divisible. A description of those infinitely divisible distributions which are self-decomposable is given by the following lemma.

LEMMA 2. *An infinitely divisible distribution on R^m is self-decomposable if and only if its Khintchine measure from the Lévy-Khintchine representation formula belongs to M_m^0.*

Proof. Let φ be the characteristic function of a self-decomposable distribution on R^m. By a, A and μ we shall denote the corresponding parameters from the Lévy-Khintchine representation of φ. Let $0 < c < 1$ and let φ_c be the characteristic function satisfying the equation

$$\varphi(z) = \varphi(cz)\varphi_c(z) \qquad (z\in R^m). \tag{5}$$

It is known that φ_c is the characteristic function of an infinitely divisible distribution (see [7], p. 323). Denoting by a^c, A^c and μ^c the parameters from the Lévy-Khintchine representation of φ_c, we have, by virtue of (2), (5) and the uniqueness of the Lévy-Khintchine representation, the formulas

(6)
$$a^c = (1-c)a+(c^3-c)\int\limits_{R^m}\frac{x}{1+c^2|x|^2}\,\mu(dx), \qquad A^c = (1-c^2)A, \qquad \mu^c = \mu_c.$$

Hence, in particular, it follows that μ_c is a measure concentrated on $R^m\setminus\{0\}$. Since the measure μ is concentrated on $R^m\setminus\{0\}$, we infer that $\mu\in M_m^0$.

Suppose now that $\mu\in M_m^0$. Let a be an arbitrary vector from R^m and A an arbitrary symmetric non-negative operator on R^m. Then the triplet a, A and μ determines, by means of the Lévy-Khintchine formula (2), the characteristic function φ of an infinitely divisible distribution on R^m. Furthermore, for every c satisfying the condition $0<c<1$, the set function μ_c is a measure on R^m vanishing at the origin. Thus the parameters a^c, A^c and μ^c defined by (6) determine, according to (2), the characteristic function φ_c of an infinitely divisible distribution. It is easy to verify that φ and φ_c satisfy condition (5). Consequently, φ is the characteristic function of a self-decomposable distribution which completes the proof.

Let W be a Borel subset of S_{m-1} and $F_1=\{0\}\times W$, $F_2=\{\infty\}\times W$ and $F_3=(0,\infty)\times W$. Taking into account the definition of the multiplication of elements of Q^m by positive numbers, we infer that the sets F_1, F_2 and F_3 are invariant under this multiplication. Hence and from the definition of the set-function μ_c $(c>0)$ it follows that $(\mu|F_j)_c=\mu_c|F_j$ $(j=1,2,3)$, where $\mu|F$ denotes the restriction of the measure μ to the set F. Consequently, if $\mu\in M_m$, then $\mu|F_j\in M_m$ $(j=1,2,3)$. Hence we get the following lemma.

LEMMA 3. *The extreme points of the set K_m are measures concentrated on one of the following sets*: $\{0\}\times\{w\}$, $\{\infty\}\times\{w\}$, $(0,\infty)\times\{w\}$, *where* $w\in S_{m-1}$.

By λ_u for $u\in Q^m\setminus(R^m\setminus\{0\})$ we shall denote the probability measures concentrated at the point u. Further, for $u\in R^m\setminus\{0\}$ we put

$$\lambda_u(E) = \frac{2}{\log(1+|u|^2)}\int\limits_{E\cap I_u}\frac{|x|}{1+|x|^2}\,q_u(dx),$$

where $I_u=\{cu: 0<c\leqslant 1\}$ and q_u is the Lebesgue measure on the interval I_u. It is evident that λ_u are probability measures. Moreover, by simple computation we obtain the formula $(\lambda_u)_c=0$ for $0<c<1$. Thus $\lambda_u\in K_m$ for all $u\in Q^m$. Moreover, $\lambda_u\in K_m^0$ for $u\in R^m\setminus\{0\}$. By $e(K_m)$ we shall denote the set of extreme points of K_m.

LEMMA 4. $e(K_m)=\{\lambda_u: u\in Q^m\}$.

Proof. It is obvious that the measures λ_u for $u \in Q^m \setminus (R^m \setminus \{0\})$ are extreme points of K_m. Moreover, by Lemma 3, they are the only extreme measures which are not concentrated on $R^m \setminus \{0\}$. Consequently, the remaining extreme points of K_m are extreme points of K_m^0 and, by Lemma 3, they are measures concentrated on half-lines. We note that a measure μ concentrated on a real line L is an extreme point of K_m^0 if and only if its restriction to L is an extreme point of K_1^0 on L. In [8] all extreme points of K_1^0, i.e. extreme points of the set of probability measures which are the Khintchine measures of one-dimensional self-decomposable distributions were found. From this result it follows that the measures λ_u with $u \in R^m \setminus \{0\}$ are the only extreme points of K_m^0. The Lemma is thus proved.

Proof of the Theorem. By Lemma 4 $e(K_m) = \{\lambda_u \colon u \in Q^m\}$. Further, the mapping $\lambda_u \to u$ is a homeomorphism between $e(K_m)$ and Q^m. Now we can apply the theorem of Choquet [2], which in this case is a corollary of the Krein-Milman theorem [5]. We then get that for every measure $\mu \in K_m$ there exists a probability measure ω on Q^m such that for all continuous functions f on Q^m we have

$$\int_{Q^m} f(x)\mu(dx) = \int_{Q^m} \Big(\int_{Q^m} f(x)\lambda_u(dx) \Big) \omega(du). \tag{7}$$

Moreover, the measure ω will assign zero mass to the set $Q^m \setminus (R^m \setminus \{0\})$ if and only if μ does so. Further, the formula (7) holds for all bounded continuous functions on $R^m \setminus \{0\}$ whenever $\mu \in K_m^0$. Hence we get the following statement: $\mu \in M_m^0$ if and only if there exists a finite Borel measure ν on $R^m \setminus \{0\}$ such that

$$\int_{R^m \setminus \{0\}} f(x)\mu(dx) = \frac{1}{2} \int_{R^m \setminus \{0\}} \Big(\int_{R^m \setminus \{0\}} f(x)\lambda_u(dx) \Big) \nu(du) \tag{8}$$

for all continuous bounded functions f on $R^m \setminus \{0\}$. Setting

$$f_z(x) = \left(e^{i(z,x)} - 1 - \frac{i(z,x)}{1+|x|^2} \right) \frac{1+|x|^2}{|x|^2} \qquad (z \in R^m) \tag{9}$$

into (8), we obtain the formula

$$\int_{R^m \setminus \{0\}} f_z(x)\mu(dx) = \int_{R^m \setminus \{0\}} \left(\int_0^{(z,u)} \frac{e^{it}-1}{t}\, dt - i\frac{(z,u)}{|u|} \arctan |u| \right) \frac{1}{\log(1+|u|^2)} \nu(du) \tag{10}$$

which implies, in view of Lemma 2, the representation (4).

We note that the formula (8) establishes a one-to-one correspondence between the measures μ and ν. In fact, it is obvious that the measure ν determines the measure μ uniquely. Further, let $0 \leqslant r < \infty$, $w \in S_{m-1}$ and let $h(r, w)$ be an arbitrary continuously differentiable with respect to r function with a compact carrier. Put $g(x) = h(|x|, x/|x|)$ for $x \in R^m \setminus \{0\}$. Then the function

$$f^*(x) = 2g(x) + \frac{1+|x|^2}{|x|} \log(1+|x|^2) \frac{\partial g(x)}{\partial |x|}$$

is continuous and bounded on $R^m \setminus \{0\}$. Moreover, it is easy to verify that

$$\int_{R^m \setminus \{0\}} f^*(x) \lambda_u(dx) = 2g(u) \qquad (u \in R^m \setminus \{0\}).$$

Hence and from (8) we get the formula

$$\int_{R^m \setminus \{0\}} f^*(x) \mu(dx) = \int_{R^m \setminus \{0\}} g(u) \nu(du)$$

which shows that the measure ν is uniquely determined by the measure μ.

Now we shall prove that the characteristic function φ determines parameters a, A and ν in (4) uniquely. Suppose that the function φ has two representations (4) with the triplets (a_1, A_1, ν_1) and (a_2, A_2, ν_2) respectively. Then, denoting by μ_1 and μ_2 the measures corresponding in (8) to ν_1 and ν_2 respectively, we have, by (4) and (10), the formula

$$\varphi(z) = \exp\left\{i(a_1, z) - \frac{1}{2}(A_1 z, z) + \int_{R^m \setminus \{0\}} f_z(x) \mu_1(dx)\right\}$$
$$= \exp\left\{i(a_2, z) - \frac{1}{2}(A_2 z, z) + \int_{R^m \setminus \{0\}} f_z(x) \mu_2(dx)\right\}.$$

Hence, by (9) and the uniqueness of the Lévy-Khintchine representation, we obtain the equalities $a_1 = a_2$, $A_1 = A_2$ and $\mu_1 = \mu_2$. Since the measure ν_j is uniquely determined by the measure μ_j in formula (8), we have the equality $\nu_1 = \nu_2$ which completes the proof.

References

[1] S. Bochner, *Harmonic analysis and the theory of probability*, Berkeley and Los Angeles 1955.

[2] G. Choquet, *Le théorème de représentation intégrale dans les ensembles convexes compacts*, Ann. Inst. Fourier 10 (1960), pp. 333-344.

[3] S. Johansen, *An application of extreme point methods to the representation of infinitely divisible distributions*, Zeitschrift für Wahrscheinlichkeitstheorie und verw. Gebiete 5 (1966), pp. 304-316.

[4] D. G. Kendall, *Extreme-point methods in stochastic analysis*, Zeitschrift für Wahrscheinlichkeitstheorie und verw. Gebiete 1 (1963), pp. 295-300.

[5] M. G. Krein and D. P. Milman, *On extreme points of regularly convex sets*, Studia Math. 9 (1940), pp. 133-138.

[6] P. Lévy, *Théorie de l'addition des variables aléatoires*, Paris 1954.

[7] M. Loève, *Probability theory*, New York 1950.

[8] K. Urbanik, *A representation of self-decomposable distributions*, Bull. Acad. Pol. Sci., série des sci. math., astr. et phys. 16 (1968), pp. 196-204.

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES,
INSTITUTE OF MATHEMATICS, UNIVERSITY OF WROCŁAW

Received on 22. 12. 1967