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H. WOŹNIAKOWSKI (Warszawa)

MODIFICATION OF THE ALTERNATING DIRECTION IMPLICIT METHOD AND CONNECTIONS WITH VON NEUMANN'S METHOD

0. This paper is devoted to a certain modification of the alternating direction implicit (ADI) method [3] leading to an application to any region of the boundary value problem. We are interested in solving a system of linear algebraic equations of high dimension on a symmetric and positive definite matrix A which may be decomposed into a sum of symmetric, positive definite, but not necessarily commutative, matrices.

From the mathematical point of view this problem is resolved as a uniform approximation of the zero in an interval by particular rational functions with an additional normalizing condition. The obtained results show a close connection of the optimal modification of the alternating direction implicit (MADI) method with von Neumann's method [4].

1. Introduction. Let us assume that our problem lies in finding a solution of linear algebraic equations of high dimension written in the form

$$Ax = b, \tag{1}$$

where A is a symmetric positive definite matrix of dimension $n \times n$, and b denotes a given vector. By $x^* = A^{-1}b$ we denote the solution of (1).

Let a constructed sequence $\{x_k\}$ satisfy asymptotically (for $k \to \infty$) the condition

$$\|x_k - x^*\| \cong c\varkappa^k \|x_0 - x^*\|$$

for some constant c, where x_0 denotes an initial approximation of x^* and the positive number $\varkappa$ is smaller than 1. $\varkappa$ is the quotient of the geometrical sequence which is a majorant of the error of the method. The value

$$k = -\frac{1}{\ln \varkappa} t, \tag{2}$$

where t denotes the cost of performing one step in the iteration, is called *index of labouriousness of the method.* Let k_1 and k_2 be indices of labouriousness of two methods, respectively. By

$$R = k_1/k_2 \tag{3}$$

we may compare the efficiency of these methods. We may say that the method with index k_2 is R times more effective than the method with index k_1.

In the previously defined class of methods, we shall look for the most effective method, i.e., the method with the smallest index of labouriousness.

2. The von Neumann method. System (1) may be solved by von Neumann's method. It requires a transformation of system (1) to the form

$$\boldsymbol{x} = B\boldsymbol{x}+\boldsymbol{g}, \tag{4}$$

where B is a symmetric matrix of dimension $n\times n$ with spectral norm smaller than 1, whereas $\boldsymbol{g}$ is a given vector.

By a *standard transformation* of (1) into (4) we shall understand the following procedure:

Multiplying (1) by a constant γ and adding to both sides the solution $\boldsymbol{x}$, we obtain

$$\boldsymbol{x} = (I-\gamma A)\boldsymbol{x}+\gamma\boldsymbol{b}.$$

We take such a constant γ which will minimize the spectral norm of the matrix $B = I-\gamma A$, i.e., $\gamma_{\text{opt}} = 2/(\lambda_{\min}+\lambda_{\max})$, where $\lambda_{\min}$ and $\lambda_{\max}$ denote the smallest and greatest eigenvalues of A, respectively. If we do not know these eigenvalues exactly enough, we can set $\gamma = 2/(a+b)$, where $a > 0$ and $[a, b]$ denotes the narrowest known interval containing $\lambda_{\min}$ and $\lambda_{\max}$. The convergence quotient of von Neumann's method with a standard transformation equals

$$\varkappa_N = \frac{\sqrt{b/a}-1}{\sqrt{b/a}+1}.$$

3. The alternating direction implicit method (ADI). Let us assume that $A = A_1+A_2$, where the matrices A_1 and A_2 are symmetric. The constructed sequence $\{\boldsymbol{x}_k\}$ of the ADI-method is defined as

$$\begin{aligned}(I+a_kA_1)\boldsymbol{x}_{k+1/2} &= (I-a_kA_2)\boldsymbol{x}_k+a_k\boldsymbol{b},\\ (I+\beta_kA_2)\boldsymbol{x}_{k+1} &= (I-\beta_kA_1)\boldsymbol{x}_{k+1/2}+\beta_k\boldsymbol{b}\end{aligned} \tag{5}$$

for certain sequences of numbers $\{a_k\}$ and $\{\beta_k\}$. Each step of the ADI-method requires solving the system with matrices $I+a_kA_1$ and $I+\beta_kA_2$.

We must assume that the form of the matrices A_1, A_2 is relatively simple (usually they are tridiagonal). Using (5), we obtain the expression of the error of the k-th approximation of the solution $\boldsymbol{x}^*$ as

$$\boldsymbol{x}_{k+1} - \boldsymbol{x}^* = P_k(A_1, A_2)(\boldsymbol{x}_1 - \boldsymbol{x}^*),$$

where

$$P_k(x, y) = \prod_{i=k}^{1} [(1+\beta_i y)^{-1}(1-\beta_i x)(1+\alpha_i x)^{-1}(1-\alpha_i y)]$$

with the notation $\prod_{i=k}^{1} a_i = a_k \cdot a_{k-1} \cdot \ldots \cdot a_1$. The vector $\boldsymbol{x}_1$ denotes now the initial approximation.

If we put $\alpha_i = \beta_i = \alpha$ for $i = 1, 2, \ldots, k$, where $\alpha > 0$, the ADI-method is convergent (see [3], p. 306). In addition, if we assume that A_1 and A_2 are commutative, then we may, for a certain k, give the optimal sequences $\{\alpha_i\}$ and $\{\beta_i\}$, $i = 1, 2, \ldots, k$, and an approximation of the convergence quotient of the ADI-method (see [6]). Namely, if we assume that the eigenvalues of A_1 and A_2 belong to the intervals $[a_1, b_1]$ and $[a_2, b_2]$, respectively, where $a_1 + a_2 > 0$, and write

$$m = \frac{2(b_1 - a_1)(b_2 - a_2)}{(a_1 + a_2)(b_1 + b_2)}, \qquad k' = \frac{1}{1 + m + \sqrt{m(m+2)}},$$

we obtain

$$\varkappa_A \cong \exp\big(\pi^2/\ln(k'/4)\big).$$

The essential assumption of this result is the commutation of the matrices A_1 and A_2. When we consider the numerical solution of the Poisson equation and take for A_1, A_2 the matrices from a discrete approximation of the second derivatives $\partial^2/\partial x^2$, $\partial^2/\partial y^2$, respectively, we can check that the condition of commutation is satisfied if and only if the region of the boundary value problem is a rectangle (see [1]).

Regions which are not rectangles lead to non-commutative matrices and the only possibility to use the ADI-method with ensured convergence is for $\alpha_i = \beta_i = \text{const} > 0$.

4. Modification of the alternating direction implicit method (MADI). Now we assume that $A = A_1 + A_2$ which can be replaced by the identity $A = A_1' + A_2'$, where $A_1' = A_1 + aI$, and $A_2' = A_2 - aI$. Let us consider the sequence $\{\boldsymbol{z}_k\}$ defined by

$$\begin{aligned} (I + \alpha_k B_1 A_1' C)\boldsymbol{z}_{k+1/2} &= (I - \alpha_k B_1 A_2' C)\boldsymbol{z}_k + \alpha_k B_1 \boldsymbol{b}, \\ (I + \beta_k B_2 A_2' C)\boldsymbol{z}_{k+1} &= (I - \beta_k B_2 A_1' C)\boldsymbol{z}_{k+1/2} + \beta_k B_2 \boldsymbol{b}. \end{aligned} \tag{6}$$

The matrices B_1, B_2, C and the coefficients α_k, β_k ought to be defined in such a way that this process be convergent. We assume that C is a non-singular matrix and we set $\boldsymbol{u} = C^{-1}\boldsymbol{x}$.

Let $E_k = z_k - u$ be the error of the approximation z_k. Using the identities

$$I + a_i B_1 A_1' C = I - a_i B_1 A_2' C + a_i B_1 A C,$$
$$I + \beta_i B_2 A_2' C = I - \beta_i B_2 A_1' C + \beta_i B_2 A C,$$

we obtain from (6) the expressions

$$(7)\quad E_{k+1} = \prod_{i=k}^{1} [(I + \beta_i B_2 A_2' C)^{-1} (I - \beta_i B_2 A_1' C)(I + a_i B_1 A_1' C)^{-1} (I - a_i B_1 A_2' C)] E_1.$$

Obviously, we must assume that suitable inverted matrices exist. We notice that in the case $B_1 = B_2 = C = I$ formula (6) expresses as formula (5) which defines the ADI-method. We shall define the matrices B_1, B_2 and C so that the matrices in (7), i.e., the matrices

$$(8)\qquad B_2 A_2' C, \quad B_2 A_1' C, \quad B_1 A_1' C, \quad B_1 A_2' C$$

were symmetric and commutative. If we assume $A_1 A_2 = A_2 A_1$, we may put $B_1 = B_2 = C = I$, $a = 0$. The choice of the matrices B_1, B_2 and C can be connected with any given problem, i.e., with given matrices A_1 and A_2.

In the general case we define the interesting matrices with the additional condition

$$(9)\qquad B_2 A_2' C = B_1 A_1' C = I.$$

Let us write $P = B_1 A_2' C$, hence the matrices in (8) are now equal to I, P^{-1}, I, P, respectively, thus they are commutative. Moreover, we shall define the matrices B_1, B_2, C so that the matrix P will be symmetric and, for a suitable parameter a, positive or negative definite.

Let us assume that the eigenvalues of the symmetric matrices A_1, A_2 belong to the intervals $[a_1, b_1]$, $[a_2, b_2]$, respectively. Let a belong to the interval $(-a_1, +\infty)$. Matrix A_1 is positive definite now; this implies the existence of a real lower triangular matrix L such that $A_1' = LL^T$.

Matrix L depends, of course, on parameter a. Putting $B_1 = L^{-1}$, $C = (L^T)^{-1}$ and $B_2 = L^T A_2'^{-1}$, we obtain $P = L^{-1} A_2' (L^{-1})^T$.

The matrix P is symmetric and its eigenvalues are equal to the eigenvalues of the matrix $A_1'^{-1} A_2'$. We must define the parameter a so that matrix P will be non-singular or, equivalently, that matrix A_2' will be non-singular. Notice that, for a certain type of matrix A_1' (for instance, for a tridiagonal matrix), the decomposition into the product LL^T is as labourious as multiplication by matrix A_1'; thus the cost of multiplication of a given vector by matrix P is approximately equal to the cost of multiplication of a vector by the matrices A_1' and A_2'.

Taking into account the previous notation, we may present (7) in the form

(10) $$E_{k+1} = f_k(P)E_1,$$

where

$$f_k(x) = \prod_{i=k}^{1} \frac{1-\beta_i x^{-1}}{1+\beta_i}\,\frac{1-a_i x}{1+a_i} = \frac{w_{2k}(x)}{x^k},$$

and $w_{2k}(x)$ is a polynomial of degree $2k$ with normalization condition $w_{2k}(-1) = (-1)^k$.

Now, if we assume that one of the parameters a_i, β_i is zero, one of the steps of (6) will be trivial. So we can perform one of the steps of (6) at the same cost twice and as a result we obtain the expression for the k-th error,

(11) $$E_{k+1} = g_{k,r}(P)E_1,$$

where

(12) $$g_{k,r}(x) = \frac{w_{2k}(x)}{x^r}, \qquad 0 \leqslant r \leqslant 2k,$$

and $w_{2k}(x)$ denotes, as previously, a polynomial of degree $2k$ with normalization condition $w_{2k}(-1) = (-1)^r$. The parameter r denotes the number of non-zero coefficients β_i. If $r = k$, formula (11) changes into (10), which means that every step of the iteration consists of two different, not trivial, half-steps done in accordance with (6).

From equality (11) and from the definition of matrix C, we obtain

(13) $$\|(L^T)^{-1}z_{k+1} - x^*\|_2 \leqslant \sqrt{\frac{b_1+a}{a_1+a}}\,\|g_{k,r}(P)\|_2\,\|(L^T)^{-1}z_1 - x^*\|_2.$$

Let us assume that the eigenvalues of matrix P belong to set I. From the symmetry of matrix P we obtain

$$\|g_{k,r}(P)\|_2 \leqslant \|g_{k,r}\|_I = \sup_{x\in I} |g_{k,r}(x)|.$$

The coefficients a_i, β_i and also the parameter r are defined so as to obtain the best convergence of process (6), i.e., so as to minimize $\|g_{k,r}(P)\|_2$. Thus, we consider an approximation problem of finding a rational function of form (12) which minimizes $\|g_{k,r}\|_I$. Let

(14) $$\delta_{2k} = \inf_{0\leqslant r\leqslant 2k}\ \inf_{a_i\beta_i} \|g_{k,r}\|_I.$$

Depending upon parameter a, we obtain different estimations of eigenvalues of the matrix P and, consequently, different sets I. We shall consider three versions, assuming, without loss of generality, that $a_1 > 0$.

1° Case $-a_1 < a < a_2$. For a in this interval, the matrix P is positive definite and its eigenvalues $\lambda(P)$ satisfy the inequality

$$c = \frac{a_2 - a}{b_1 + a} \leqslant \lambda(P) \leqslant \frac{b_2 - a}{a_1 + a} = d.$$

Assuming that we have no additional information about the eigenvalues of matrix P, we put $I = [c, d]$.

Using the results from [5], p. 52-56, we obtain

$$\inf_{a_i, \beta_i} \|g_{k,r}\|_I \cong 2 \left[\left(\frac{\sqrt{d(c+1)} - \sqrt{c(d+1)}}{\sqrt{d+1} - \sqrt{c+1}} \right)^{2r} \left(\frac{\sqrt{d+1} - \sqrt{c+1}}{\sqrt{d+1} + \sqrt{c+1}} \right)^{2k} + \right.$$
$$\left. + \left(\frac{\sqrt{d(c+1)} + \sqrt{c(d+1)}}{\sqrt{d+1} + \sqrt{c+1}} \right)^{2r} \left(\frac{\sqrt{d+1} + \sqrt{c+1}}{\sqrt{d+1} - \sqrt{c+1}} \right)^{2k} \right]^{-1}. \tag{15}$$

Minimizing the right-hand side of (15) with respect to r, we check that for $cd \leqslant 1$ the optimal value of r is equal to zero and, for $cd > 1$, the optimal r equals $2k$. Minimizing now the results obtained with respect to a, we obtain the solution of problem (14) for $r = 0$, $a = a_2$, as

$$\delta_{2k,1} \cong 2 \left(\frac{\sqrt{(b_2 + a_1)/(a_1 + a_2)} - 1}{\sqrt{(b_2 + a_1)/(a_1 + a_2)} + 1} \right)^{2k}.$$

The parameter a may be equal to a_2, because if $r = 0$ we use in the iterative process (6) only the matrix P which may now be singular.

If $cd > 1$, we do not obtain a better result, because the optimal a equals $-a_1$, from which singularity of the matrix A_1' follows. Keeping in mind that we have $2k$ steps to perform, the convergence quotient of the optimal MADI-method in the first version is equals to

$$\varkappa_{M,1} \cong \frac{\sqrt{(b_2 + a_1)/(a_1 + a_2)} - 1}{\sqrt{(b_2 + a_1)/(a_1 + a_2)} + 1}.$$

2° Case $a > b_2$. The matrix P is now negative definite and its eigenvalues satisfy the inequalities

$$-1 < c = \frac{a_2 - a}{a_1 + a} \leqslant \lambda(P) \leqslant \frac{b_2 - a}{b_1 + a} = d < 0.$$

Let us take $I = [c, d]$. Notice that, for each $r = 0, 1, \ldots, 2k$, the inequality

$$\sup_{x \in I} \left| \frac{w_{2k}(x)}{x^r} \right| \leqslant \sup_{x \in I} |w_{2k}(x)|$$

is true, from which it follows that the optimal r in formula (14) equals zero. Now we use, as before, only matrix P, and the optimal a equals b_2; hence $\delta_{2k,2} = \delta_{2k,1}$ and $\varkappa_{M,2} = \varkappa_{M,1}$.

3° Case $a_2 < a < b_2$. Now the eigenvalues of matrix P satisfy the inequality

$$0 \leqslant \varepsilon = \frac{|\lambda_0(A_2')|}{b_1+a} \leqslant |\lambda(P)| \leqslant \frac{\max(a-a_2,\, b_2-a)}{a_1+a} = \varrho,$$

where $\lambda_0(A_2')$ denotes the absolute smallest eigenvalue of matrix A_2'. To obtain a convergent process we must assume that $\varrho < 1$ which is equivalent to

$$\max\left(a_2,\, \frac{b_2-a_1}{2}\right) < a < b_2.$$

By the same argumentation as in 2°, we come to the conclusion that the optimal r equals zero. Using results from [2], we obtain the solution of problem (14) as

$$\delta_{2k,3} \cong 2\left(\frac{\sqrt{(1-\varepsilon^2)/(1-\varrho^2)}-1}{\sqrt{(1-\varepsilon^2)/(1-\varrho^2)}+1}\right)^k.$$

If we do not know the eigenvalues of matrix A_2, we may put $\varepsilon = 0$, and taking $a = (a_2+b_2)/2$, we have

$$\delta_{2k,3} \cong 2\left(\frac{\sqrt{\beta}-1}{\sqrt{\beta}+1}\right)^k,$$

where $\beta = (b_2+a_2+2a_1)^2/[4(a_1+a_2)(a_1+b_2)]$.

It is easy to check that in this case $\delta_{2k,1} = \delta_{2k,2} = \delta_{2k,3}$ if we take in the respective formulas an equality instead of an approximate equality.

If we know the eigenvalues of matrix A_2, we may hope that the third version would lead to the best convergence; particularly, if the eigenvalues of matrix A_2 do not lie near the point $(a_2+b_2)/2$ (see chapter 7).

For an even number of steps, the convergence quotient is equal to

$$\varkappa_{M,3} = \sqrt{\frac{\sqrt{\beta}-1}{\sqrt{\beta}+1}}.$$

At last, notice that we have in inequality (13) the multiplier $\sqrt{(b_1+a)/(a_1+a)}$ which is equal to

$$\sqrt{\frac{b_1+a_2}{a_1+a_2}},\quad \sqrt{\frac{b_1+b_2}{a_1+b_2}},\quad \sqrt{\frac{2b_1+b_2+a_2}{2a_1+b_2+a_2}}$$

for the three versions, respectively. This multiplier is smallest in the second version.

COROLLARY 1. *If we do not know the eigenvalues of matrix A_2, the optimal version of the MADI-method is characterized by the convergence quotient*

$$\varkappa_M = \frac{\sqrt{(b_2+a_1)/(a_1+a_2)}-1}{\sqrt{(b_2+a_1)/(a_1+a_2)}+1}, \tag{16}$$

and we obtain

$$\|(L^T)^{-1}z_{k+1}-x^*\|_2 \leqslant 2\sqrt{\frac{b_1+b_2}{a_1+b_2}}\,\varkappa_M^k\|(L^T)^{-1}z_1-x^*\|_2.$$

If we know the eigenvalues of matrix A_2, we ought to compare $\varkappa_M$ with the convergence quotient $\varkappa_{M,3}$ (for $\varepsilon = 0$ and $a = (a_2+b_2)/2$ we have $\varkappa_M = \varkappa_{M,3}$), and if $\varkappa_{M,3} < \varkappa_M$, we may apply the third version of the MADI-method.

Note that the convergence quotient given by (16) does not depend upon b_1, i.e., upon the highest eigenvalue of matrix A_1. In certain systems of algebraic equations (which is equivalent to certain regions in the boundary value problems (see chapter 7)) this is convenient. For $a = b_2$ we obtain the matrices A_1+b_2I and A_2-b_2I, where the second one, and thus also matrix P, is singular. From that follows the independence of the convergence quotient of b_1 (case $\beta_i \equiv 0$).

To conclude our considerations in this chapter, we want to show that the optimal MADI-method is equivalent to von Neumann's method with non-standard transformation to (4).

We transform a given system $Ax = b$ changing matrix A as

$$\begin{aligned} A &= A_1+A_2 = (A_1+aI)+(A_2-aI) = LL^T+A_2' \\ &= L\big(I+L^{-1}A_2'(L^T)^{-1}\big)L^T = L(I+P)L^T. \end{aligned}$$

Writing, as previously, $u = L^Tx^*$, we obtain $(I+P)u = L^{-1}b$. Applying now the standard transformation, we infer that the convergence quotient equals $\varkappa_M$.

Summarizing, the modification of the alternating direction implicit method is equivalent to von Neumann's method with non-standard transformation to (4) under assumption (9).

5. A comparison with von Neumann's method. From chapter 2 it follows that we obtain the convergence quotient of von Neumann's method with standard trnasformation as equal to

$$\varkappa_N = \frac{\sqrt{b/a}-1}{\sqrt{b/a}+1},$$

where $[a, b]$ denotes the interval containing the eigenvalues of matrix A. If we know the intervals $[a_1, b_1]$, $[a_2, b_2]$ containing the eigenvalues of the matrices A_1, A_2, respectively, we may set $a = a_1 + a_2$ and $b = b_1 + b_2$.

Let us compare the von Neumann method with the second and third versions of the MADI-method. To do this, let us consider the quotient of the cost of one step in both methods $T = t_N/t_M$ (see (2)). If the matrix $B = I - 2/(a+b)A$ in system (4) is primitive (see [4]), this quotient T approximately equals 1, and if B is a cyclic matrix, then, remembering (see [4]) that successive steps in von Neumann's method one may have at the half cost, we obtain $T \cong 1/2$.

Thus, for the second version of the MADI-method, we obtain from (2)

$$\frac{k_N}{k_{M,2}} = \frac{\ln \varkappa_{M,2}}{\ln \varkappa_N} T \cong \frac{\sqrt{b_1+b_2}+\sqrt{a_1+a_2}}{\sqrt{b_2+a_1}+\sqrt{a_1+a_2}} T.$$

Usually we have an ill-condition system, i.e., $b_1 \gg a_1$, $b_2 \gg a_2$, hence

$$\frac{k_N}{k_{M,2}} \cong \sqrt{1+b_1/\max(b_2, a_1)}\, T.$$

If we can decompose matrix A_1 or matrix A_2 into the product LL^T, we may assume that $b_1 \geqslant b_2$, obtaining $k_N/k_{M,2} \geqslant \sqrt{2}T$.

Summarizing, we formulate

COROLLARY 2. *If B is a primitive matrix, the second version of the MADI-method is $\sqrt{1+b_1/\max(b_2, a_1)}$ times more effective than von Neumann's method with standard transformation. If B is a cyclic matrix, the use of the second version is only profitable for $b_1 > 3\max(b_2, a_1)$.*

Let us consider the third version of the MADI-method now. By the same argumentation as before, we can obtain

$$\frac{k_N}{k_{M,3}} \cong \frac{1}{2}\sqrt{\frac{b_1+b_2}{a_1+a_2}}\sqrt{\frac{1-\varrho^2}{1-\varepsilon^2}}T. \tag{17}$$

If we put $a = (b_2+a_2)/2$, we obtain

$$\varrho = \frac{b_2-a_2}{b_2+a_2+2a_1} \quad \text{and} \quad \varepsilon = \frac{2|\lambda_0(A_2')|}{b_2+a_2 2+b_1},$$

where, as always, $\lambda_0(A_2')$ denotes the absolute smallest eigenvalue of the matrix $A_2' = A_2 - (b_2+a_2)/2I$. If $\varepsilon > 0$, this version leads us to the smallest convergence quotient and formula (17) may be presented as

$$\frac{k_N}{k_{M,3}} \cong \sqrt{1+\frac{b_1}{\max(b_2, a_1)}}\,\frac{1}{\sqrt{1-\varepsilon^2}}T.$$

Finally, we obtain

COROLLARY 3. *The third version of the MADI-method for* $a = (a_2+b_2)/2$ *is* $1/\sqrt{1-\varepsilon^2}$ *times more effective than the second one, whereas it is*

$$\sqrt{1+\frac{b_1}{\max(b_2,\, a_1)}\frac{1}{\sqrt{1-\varepsilon^2}}T}$$

times more effective than von Neumann's method with standard transformation.

6. The algorithm of the MADI-method. In the second version of the MADI-method, the matrix P has eigenvalues in the interval $[d, 0]$, where $d = (a_2-b_2)/(a_1+b_2)$ (case $\beta_i \equiv 0$).

Formula (6) may now be written in the form

$$z_{k+1} = \frac{1}{1+a_k}(I-a_kP)z_k+\frac{a_k}{1+a_k}B_1\boldsymbol{b},$$

and the $(k+1)$-th error of approximation is equal to

$$\boldsymbol{E}_{k+1} = g_k(P)\boldsymbol{E}_1, \tag{18}$$

where now

$$g_k(x) = T_k\left(-\frac{2}{d}x+1\right)/T_k\left(\frac{2+d}{d}\right),$$

and $T_k(u)$ denotes the k-th Chebyshev polynomial of the first kind.

Using the recurrence formula for Chebyshev polynomials and also (18), we obtain the recurrence formula for successive members of the sequence z_k, namely

$$z_{k+1} = t_k(Bz_k+\boldsymbol{g}+t_{k-1}z_{k-1}), \qquad k = 1, 2, \ldots,$$

where $B = -4/d(P-d/2I)$, and $\boldsymbol{g} = 4/dB_1\boldsymbol{b}$.

The coefficients t_k may be computed from

$$t_0 = 0, \qquad t_k = 1/\left[2\left(\frac{2+d}{2}\right)-t_{k-1}\right].$$

In the third version of the MADI-method the eigenvalues of matrix P belong to $I = [-\varrho, -\varepsilon]\cup[\varepsilon, \varrho]$. We set the parameters σ, μ, ν to be equal to

$$\sigma^2 = \frac{\varrho^2+\varepsilon^2}{2}, \qquad \mu = \frac{(\varrho^2-\varepsilon^2)^2}{16}, \qquad \nu = 1-\sigma.$$

Using results from [2], we obtain the recurrence formulas

$$z_1 = -P\boldsymbol{x}_0+B_1\boldsymbol{b},$$

$$z_{k+1} = q_k(-Pz_k+B_1\boldsymbol{b}-z_{k-1})z+_{k-1}, \qquad k = 1, 2, \ldots,$$

where q_k is equals to

$$q_0 = 2,$$

$$q_{2k} = \frac{w_k(q_{2k-1}-1)}{q_{2k-1}(\sigma w_k+1)-w_k},$$

$$q_{2k+1} = w_k/q_{2k}.$$

The parameters w_k are equal to $w_0 = 2/\nu$, $w_k = 1/(\nu-\mu w_{k-1})$.

7. Numerical examples. In order to illustrate our previous considerations let us investigate the equation

$$\beta\frac{\partial^2 u}{\partial x^2}(x,y)+\frac{\partial^2 u}{\partial y^2}(x,y) = f(x,y) \tag{19}$$

for (x, y) belonging to the set D presented in Fig. 1.

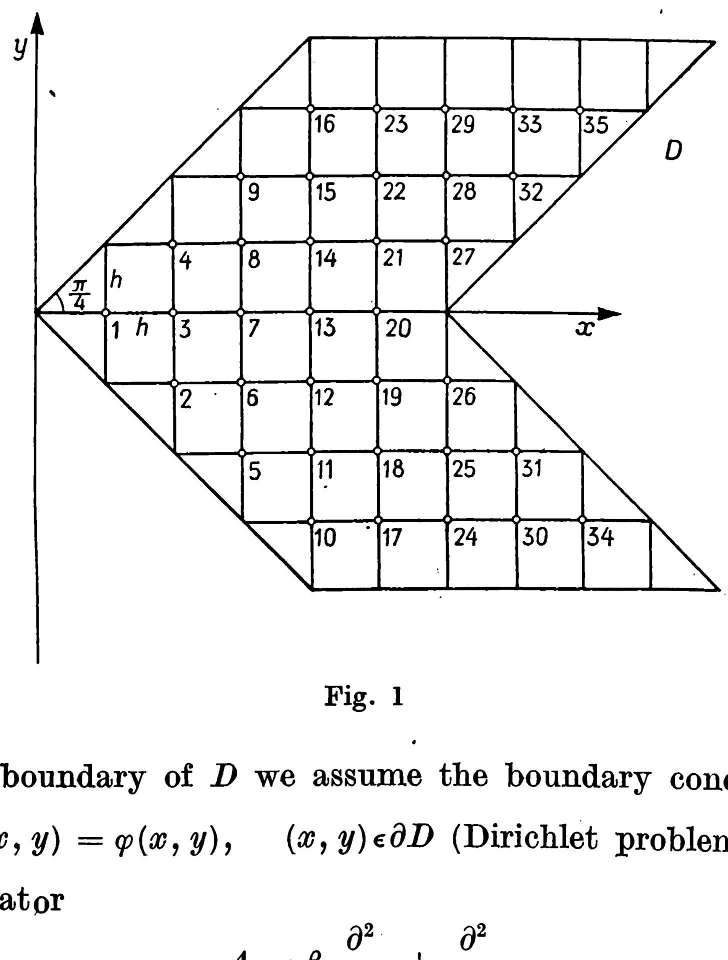

Fig. 1

For the boundary of D we assume the boundary condition

$$u(x,y) = \varphi(x,y), \quad (x,y)\in\partial D \text{ (Dirichlet problem)}.$$

The operator

$$\Delta \equiv \beta\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$$

is approximated by a five-point difference approximation for any point belonging to the square grid on D, i.e.,

$$\Delta u_j^i \cong -\left(\beta\,\frac{-u_{j+1}^i+2u_j^i-u_{j-1}^i}{h^2}+\frac{-u_j^{i+1}+2u_j^i-u_j^{i-1}}{h^2}\right),$$

where $u_j^i = u(x_j, y_i)$, $x_j = x_0+jh$, $y_i = y_0+ih$.

Let p denote the maximal number of internal points on the vertical lines, r — the number of internal points on each horizontal line, and s — the number of vertical lines with p internal points. In Fig. 1 we have $p = 7$, $r = 5$, $s = 2$. The total number of unknowns, i.e., of values of the solution in internal points, is equals to $p(2s+p-1)/2$. We take as matrices A_1, A_2 matrices from discrete approximation of the second derivatives with respect to y, x, respectively (with numeration presented in Fig. 1).

Let S_i be the dimension matrix i in the form

$$S_i = \beta\begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & & & \\ & & & \ddots & & \\ & & & & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix}.$$

Matrix A consists of the matrices S_i being on the diagonal in the order

$$S_1, S_3, S_5, \ldots, S_{p-2}, \underbrace{S_p, \ldots, S_p}_{s \text{ times}}, S_{(p-1)/2},$$

$$S_{(p-1)/2}, S_{(p-3)/2}, S_{(p-3)/2}, \ldots, S_2, S_2, S_1, S_1.$$

The form of matrix A_2 is more complicated, but also in this case the natural fragmentation corresponding to horizontal lines may be observed.

The eigenvalues of A_1, A_2 relevant for the convergence of the MADI--method are equal to

$$a_1 = 4\beta\sin^2\left(\frac{\pi}{2}\,\frac{1}{p+1}\right), \quad b_1 = 4\beta\sin^2\left(\frac{\pi}{2}\,\frac{p}{p+1}\right),$$

$$a_2 = 4\sin^2\left(\frac{\pi}{2}\,\frac{1}{r+1}\right), \quad b_2 = 4\sin^2\left(\frac{\pi}{2}\,\frac{r}{r+1}\right).$$

The MADI-method requires the introduction of the matrix $A_1' = A_1+$ $+a_2I$ in the form LL^T. Matrix L has a form analogous to that of matrix A_1, namely, matrix L consists of matrices L_i being on the diagonal in the same order as matrices S_i, where

$$L_i = \begin{bmatrix} l_1 & & & & \\ -\dfrac{\beta}{l_1} & l_2 & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & -\dfrac{\beta}{l_{i-1}} & l_i \end{bmatrix}$$

with $l_1 = \sqrt{c}$, $l_j = \sqrt{c-\beta^2/l_{j-1}^2}$, $j = 2, 3, \ldots, i$, where $c = 2\beta + a_2$. In practice one needs only to store the numbers l_j.

Let us compare the MADI-method with von Neumann's method for our problem. Because by a different numeration of the grid-points we obtain a cyclic matrix, we have the cost quotient $T \cong 1/2$. Using the considerations from chapter 5, we obtain, for the second version of the MADI-method,

$$\frac{k_N}{k_{M,2}} \cong \frac{1}{2}\sqrt{1+\frac{b_1}{b_2+a_1}}$$

$$\cong \begin{cases} \dfrac{1}{2}\sqrt{1+\beta} & \text{for } \beta \sin^2 \dfrac{1}{p+1} \ll 1, \\[2ex] \dfrac{1}{2}\sqrt{1+\operatorname{ctg}^2\left(\dfrac{\pi}{2}\dfrac{1}{p+1}\right)} & \text{for } \beta \sin^2 \dfrac{1}{p+1} \gg 1. \end{cases}$$

If $\beta \sin^2 1/(p+1) \ll 1$, the use of this version of the MADI-method is reasonable for $\beta > 3$.

If we want to apply the third version of the MADI-method, we notice that the different eigenvalues of matrix A_2 are equal to

$$\lambda_i(A_2) = 4\sin^2\left(\frac{\pi}{2}\frac{i}{r+1}\right), \quad i = 1, 2, \ldots, r.$$

For r even, $\lambda_i(A_2) \neq (a_2+b_2)/2 = 2$, and putting

$$\varepsilon = \frac{4\sin\dfrac{\pi}{4}\dfrac{1}{r+1}\sin\dfrac{\pi}{2}\dfrac{r+\frac{1}{2}}{r+1}}{b_1+2} \cong \frac{\sin\dfrac{\pi}{4}\dfrac{1}{r+1}}{\beta+\frac{1}{2}}$$

for $\beta \sin^2 1/(p+1) \ll 1$ we obtain

$$\frac{k_N}{k_{M,3}} \cong \frac{1}{2}\sqrt{1+\beta}\,\frac{1}{\sqrt{1-\varepsilon^2}}.$$

We want to indicate that usually $1/\sqrt{1-\varepsilon^2} \cong 1$, so we do not obtain an essential advantage with respect to the second version of the MADI-method.

Equation (19), for $\beta = 1$, has also been considered on a triangle and a regular hexagon with a hexagonal grid. The matrix in this system is primitive, so $T \cong 1$, and comparing the second version of the MADI-method with von Neumann's method, we obtain

$$\frac{k_N}{k_{M,2}} \cong \sqrt{\frac{3}{2}}.$$

All the numerical computations were done on the Gier computer in the Numerical Centre of Warsaw University.

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INSTITUTE OF MATHEMATICAL MACHINES
UNIVERSITY OF WARSAW

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H. WOŹNIAKOWSKI (Warszawa)

MODYFIKACJA METODY NAPRZEMIENNYCH KIERUNKÓW I JEJ ZWIĄZEK Z METODĄ VON NEUMANNA

STRESZCZENIE

Praca zawiera pewną modyfikację metody ADI, rozwiązującą układ równań algebraicznych wysokiego stopnia $(A_1+A_2)x = \mathbf{0}$, o symetrycznej dodatnio określonej macierzy A_1+A_2. Nie zakłada się, że macierze A_1 i A_2 są przemienne, można zatem używać tej metody, zwanej w skrócie MADI, do dowolnych obszarów, niekoniecznie będących prostokątami.

Metoda MADI wymaga rozłożenia macierzy A_1 na LL^T, gdzie L jest macierzą dolną trójkątną o elementach rzeczywistych. W pracy pokazano, że metoda MADI jest $\varepsilon\sqrt{1+(b_1-a_1)/(b_2+a_1)}$ razy efektywniejsza od metody von Neumanna (a_1, b_1 i a_2, b_2 oznaczają tu odpowiednio najmniejsze i największe wartości własne macierzy A_1 i A_2). Stała ε jest równa 1, jeśli macierz

$$B = I - \frac{2}{a_1+a_2+b_1+b_2}(A_1+A_2)$$

jest prymitywna, natomiast równa się $1/2$, jeśli B jest macierzą cykliczną.