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REGULAR POSITIVE BASES

1. Introduction. The present paper is a continuation of Romanowicz's work [8], so we shall use the same notation and apply the results obtained there.

Let B be a positive basis for a finite-dimensional linear space L. For each element $x \neq 0$ in L there exists a family of sets $D \subset B$ such that

$$(1.1) \qquad x \in \operatorname{relint} \operatorname{pos} D \quad \text{and} \quad x \notin \operatorname{relint} \operatorname{pos} D' \text{ for all } D' \subsetneqq D.$$

The union of all such sets augmented with the origin is denoted by $B(x)$. Moreover, we assume that $B(0) = \{0\}$. The set

$$B^0(x) = \operatorname{relint} \operatorname{pos} B(x)$$

is called a *basis cone for* x. Let us notice that the basis cone for x can be defined also for an arbitrary set A in L such that $\operatorname{card} A < \infty$ and $x \in \operatorname{pos} A$.

Using Romanowicz's partition theorem for a positive basis [8] it is possible to find a recurrence formula for $B(x)$. It is easy to see that each $x \in L$ has a unique representation in $B = B_1 \cup (\Delta + c)$:

$$x = p_1(x) + p_2(x),$$

where

$$(1.2) \qquad \begin{aligned} p_1(x) &= x_1 - \sum_{a \in \Delta(x_2)} \alpha_a c, \\ p_2(x) &= x_2 + \sum_{a \in \Delta(x_2)} \alpha_a c = \sum_{a \in \Delta(x_2)} \alpha_a (a + c). \end{aligned}$$

Here

$$x = x_1 + x_2, \quad x_1 \in L_1 = \operatorname{lin} B_1, \quad x_2 \in L_2 = \operatorname{lin} \Delta,$$

$$x_2 = \sum_{a \in \Delta(x_2)} \alpha_a a, \quad \alpha_a > 0,$$

$$L_1 \cup L_2 = L, \quad L_1 \cap L_2 = \{0\}.$$

By (1.2) one can show that

$$(1.3)\quad B(x) = \begin{cases} [(B_1 \cup \{c\})(p_1(x)) \setminus \{c\}] \cup (\Delta + c) & \text{if } c \in (B_1 \cup \{c\})(p_1(x)), \\ B_1(p_1(x)) \cup (\Delta + c)(p_2(x)) & \text{if } c \notin (B_1 \cup \{c\})(p_1(x)). \end{cases}$$

In the case of a strong positive basis (cf. [3] and [5]–[7]), formula (1.3) takes the form

$$B(x) = \Delta_1(x_1) \cup \Delta_2(x_2) \cup \ldots \cup \Delta_k(x_k),$$

whence

$$B^0(x) = \Delta_1^0(x_1) + \Delta_2^0(x_2) + \ldots + \Delta_k^0(x_k),$$

where

$$x = x_1 + x_2 + \ldots + x_k, \qquad x_i \in L_i = \operatorname{lin} \Delta_i, \qquad B = \bigcup_{i=1}^{k} \Delta_i,$$

$$L = L_1 + L_2 + \ldots + L_k,$$

$$L_i \cap L_j = \{0\}, \qquad i \neq j,\ i, j \in \{1, 2, \ldots, k\}.$$

For an arbitrary positive basis of L, however, the basis cones can contain the whole line or even can coincide with the entire space L. It is sufficient to consider the following example of the basis $B = \{b_1, b_2, \ldots, b_6\}$ for L, $\dim L = 4$, where

$$b_1 = (1, 0, 0, 0), \qquad b_4 = (-1, -1, -1, 0),$$
$$b_2 = (0, 1, 0, 0), \qquad b_5 = (\tfrac{1}{2}, 1, 1, 1),$$
$$b_3 = (0, 0, 1, 0), \qquad b_6 = (\tfrac{1}{2}, 1, 1, -1).$$

Clearly,

$$B = B_1 \cup (\Delta + c), \qquad B_1 = \{b_1, b_2, b_3, b_4\},$$
$$c = (\tfrac{1}{2}, 1, 1, 0), \qquad \Delta = \{b_5 - c, b_6 - c\}$$

and for $x = (\frac{1}{2}, 2, 2, 0)$ we have $B(x) = B \cup \{0\}$ by (1.3).

Thus there appears an interesting problem of determining positive bases for which none of the basis cones contains the whole line. These bases are called *regular*. The problem of characterizing regular bases was formulated by Romanowicz [8]. The present paper is mainly devoted to this problem.

Note that simplicial bases and strong positive bases are the examples of regular bases. Let us remark also that the concept of regularity can be extended to that for positively spanning sets.

The set $C \subset L$ is called *strongly critical* in a regular basis B if $C \cup B$ is a regular set. In our considerations we shall confine ourselves to sets C which satisfy the following conditions:

(1.4a) $\lambda b \notin C$ for $b \in B$ and $\lambda > 0$;

(1.4b) if $c_1, c_2 \in C$, $c_1 \neq c_2$, then $c_1 \neq \lambda c_2$, $\lambda > 0$.

An element of a strongly critical set is called a *strongly critical vector*. The strongly critical vectors are of the same weight in the partition of regular bases as critical vectors in Romanowicz's partition theorem for positive bases.

In Section 2, we prove that a positive basis $B = B_1 \cup (\Delta + c)$ is regular iff B_1 is a regular basis for the subspace L_1 and c is a strongly critical vector in B_1.

The usefulness of this theorem is apparently connected with the possibility of finding strongly critical vectors (as well as strongly critical sets) in a regular basis. A method for determining strongly critical vectors is given in Section 3.

The class of regular positive bases is of great importance since the closed full-dimensional basis cones K_i $(1 \leqslant i \leqslant m)$ in a regular positive basis B of L have the following properties (Section 4):

(i) $\operatorname{pos} K_i = K_i$;

(ii) $\operatorname{int} K_i \cap \operatorname{int} K_j = \emptyset$ for all i, j $(i \neq j)$;

(iii) no K_i contains the whole line;

(iv) $K_i \cup K_j$ contains the whole line for all i, j $(i \neq j)$;

(v) $\bigcup_{i=1}^{m} K_i = L$;

(vi) $\operatorname{card}\{K_i\colon 1 \leqslant i \leqslant m\} \leqslant 2^n$, $n = \dim L$.

From these properties it follows that the regular positive bases can be applied to find a partial solution of two unsolved problems.

The first one, formulated by Bàràny [1], is to determine the number m of elements in the family of cones which satisfy properties (i)–(iv). Evidently, for the family of closed full-dimensional basis cones, Bàràny's hypothesis that $m \leqslant 2^n$ is correct.

The second problem, known as the "inner illumination problem", is to determine the minimal number s of the illuminating directions for an arbitrary convex body. Hadwiger's hypothesis is $s \leqslant 2^n$. The application of regular bases allows us to prove (Section 4) that some nontrivial class of polyhedrons (dual to polyhedrons generated by regular positive basis) satisfies Hadwiger's hypothesis.

2. Theorem on the geometrical structure of regular bases. Let us notice first a few lemmas resulting from the properties of convex bodies and the definition of basis cones.

LEMMA 2.1. *Let B be a positively spanning set in L and $x \in L \setminus \{0\}$. If a set $A \subset B$ satisfies the conditions*

(i) $x \in \operatorname{relint} \operatorname{pos} A$,

(ii) $\operatorname{pos} A$ *does not contain the whole line*,

then $A \subset B(x)$ *and* $\operatorname{relint} \operatorname{pos} A \subset B^0(x)$.

Lemma 2.2. *If* B *is a regular positively spanning set and* $y \in B^0(x)$, *then* $B(y) = B(x)$.

Lemma 2.3. *If* B *is a regular positively spanning set and* $y \in \operatorname{pos} B(x)$, *then* $B(y) \subset B(x)$.

Lemma 2.3 allows us to prove

Lemma 2.4. *C is a strongly critical set for a regular basis B iff*

$$\operatorname{pos}(B \cup C)(x) = \operatorname{pos} B(x) \quad \textit{for all } x \in L. \tag{2.1}$$

Proof. Let us assume that C is a strongly critical set for a basis B. Then, for each x, the cone $\operatorname{pos}(B \cup C)(x)$ does not contain the whole line. By Lemma 2.3 we have

$$B(c) \subset (B \cup C)(c) \subset (B \cup C)(x)$$

provided that $c \in C \cap (B \cup C)(x)$. Using (1.4a) and (1.4b) we obtain

$$\operatorname{pos}(B \cup C)(x) = \operatorname{pos}((B \cup C)(x) \cap B),$$

whence $\operatorname{pos}(B \cup C)(x) \subset \operatorname{pos} B(x)$ and due to $B(x) \subset (B \cup C)(x)$ we have (2.1).

The sufficiency of condition (2.1) can be easily shown by virtue of regularity of the positive basis B.

By Lemmas 2.3 and 2.4 we have now

Corollary 2.1. *C is a strongly critical set for the regular basis B iff, for an arbitrary set D such that $D \subset B \cup C$ and $\operatorname{pos} D$ does not contain the whole line, the condition $x \in \operatorname{relint} \operatorname{pos} D$ implies*

$$D \subset \operatorname{pos} B(x).$$

From Lemma 2.4 we have also

Corollary 2.2. *Every strongly critical set for the regular basis B is also a critical one for B.*

One can assume that not all critical vectors are strongly critical ones.

For example, let us consider the basis $B = \{b_1, b_2, b_3, b_4\}$ for L, $\dim L = 3$, where

$$b_1 = (1, 0, 0), \quad b_3 = (0, 0, 1),$$
$$b_2 = (0, 1, 0), \quad b_4 = (-1, -1, -1).$$

Clearly, $c = (\frac{1}{2}, 1, 1)$ is a critical vector for B but not a strongly critical one.

Theorem 2.1. *A positive basis $B = B_1 \cup (\Delta + c)$ is a regular one iff B_1 is a regular basis for the subspace $L_1 = \operatorname{lin} B_1$ and c is a strongly critical vector in B_1.*

Proof. Let us assume that B is a regular basis. Then B_1 as a subbasis of B is also a regular one. Let $c \neq 0$ ($c = 0$ is a trivial case), $x \in L_1$ and let $D \subset B_1 \cup \{c\}$ be determined by condition (1.1). If $c \notin D$, then $D \subset B_1(x)$ by Lemma 2.1. Thus

$$D \subset \operatorname{pos} B_1(x). \tag{2.2}$$

If $c \in D$, then $c \in (B_1 \cup \{c\})(x)$ and, by (1.3), $\Delta + c \subset B(x)$. Hence $c \in \operatorname{pos} B(x)$, and using Lemma 2.3 we get

$$B(c) \subset B(x).$$

Putting now $x = c$ into (1.3), we obtain

$$B(c) = B_1(c) \cup (\Delta + c)$$

and

$$B_1(c) \subset (B_1 \cup \{c\})(x) \setminus \{c\}.$$

Further

$$c \in \operatorname{pos}((B_1 \cup \{c\})(x) \setminus \{c\}).$$

The cone $\operatorname{pos}((B_1 \cup \{c\})(x) \setminus \{c\})$ does not contain the whole line since it is a subset of the cone $\operatorname{pos} B(x)$.

Moreover, $x \in \operatorname{relint} \operatorname{pos}((B_1 \cup \{c\})(x) \setminus \{c\})$ and, by Lemma 2.1,

$$(B_1 \cup \{c\})(x) \setminus \{c\} \subset \operatorname{pos} B_1(x)$$

and

$$c \in \operatorname{pos} B_1(x).$$

Two last relations give finally (2.2), which together with Corollary 2.1 completes the proof in one direction.

Let now B_1 be a regular subbasis of B, c be a strongly critical vector for B_1, and let $x \in L$. If

$$c \notin (B_1 \cup \{c\})(p_1(x)),$$

then it is easy to see from (1.3) that the cone $\operatorname{pos} B(x)$ does not contain the whole line. Similarly, if

$$c \in (B_1 \cup \{c\})(p_1(x)),$$

then due to Corollary 2.1 and formula (3.1) the cone $\operatorname{pos} B(x)$ also does not contain the whole line, which completes the proof.

Theorem 2.1 allows us to partition a regular basis B (a nonsimplicial one) as follows:

$$B = \Delta_1 \cup (\Delta_2 + c_1) \cup \ldots \cup (\Delta_k + c_{k-1}),$$

where Δ_i is a simplicial basis of the subspace L_i for $i = 1, 2, \ldots, r$, $L_i \cap L_j = \{0\}$ for $i \neq j$, $L = L_1 + L_2 + \ldots + L_r$, whereas c_j is a strongly critical vector for the basis

$$B_j = \Delta_1 \cup (\Delta_2 + c_1) \cup \ldots \cup (\Delta_j + c_{j-1})$$

of the subspace $L_1 + L_2 + \ldots + L_j$, where $B_1 = \Delta_1$ and $j = 1, 2, \ldots, r-1$.

Clearly, all subspaces B_j are also regular bases.

Theorem 2.1 together with Romanowicz's theorem about a simplicial decomposition [8] gives us a complete characterization of regular bases. For full clearness it is necessary to determine additionally the set of strongly critical vectors for a regular basis.

3. Strongly critical sets for regular basis. Let $C^*(B)$ denote the set of all strongly critical vectors for B. Then, according to Corollary 2.2, $C^*(B) \subset C(B)$, where $C(B)$ is the set of critical vectors for B.

LEMMA 3.1. *If B is a regular basis and $z \in C^*(B)$, then*

$$B(-z) \cap B(z) = \{0\}.$$

Proof. It is sufficient to confine our considerations to the case $z \neq 0$. Suppose that $B(-z) \cap B(z) \neq \{0\}$ and let D denote a subset of B such that

$$-z \in \operatorname{relint} \operatorname{pos} D,$$

$$-z \notin \operatorname{relint} \operatorname{pos} D' \text{ for all } D' \subsetneq D, \quad D \cap B(z) \neq \emptyset.$$

Clearly, $0 \notin D$, $-z \in \operatorname{pos} D$ and $D \subset B(-z)$. Let

$$D_1 = D \cap B(z) \quad \text{and} \quad D_2 = D \setminus D_1.$$

Since the cone $\operatorname{pos} B(z)$ does not contain the whole line, $D_2 \neq \emptyset$ and $-z \notin \operatorname{pos} D_2$. Furthermore, $\operatorname{pos}(D_2 \cup \{z\})$ does not contain the whole line and by Corollary 2.1 we have, for $x \in \operatorname{relint} \operatorname{pos}(D_2 \cup \{z\})$,

$$D_2 \cup \{z\} \subset \operatorname{pos} B(x).$$

Hence, by Lemma 2.3 we obtain $D_2 \cup B(z) \subset B(x)$, and further $D \subset B(x)$. In this case, however, $z \in \operatorname{pos} B(x)$ and $-z \in \operatorname{pos} B(x)$, $z \neq 0$, which contradicts the assumption of regularity of the basis B.

Note that a critical vector z in a regular basis B which satisfies the condition $B(z) \cap B(-z) = \{0\}$ may not be a strongly critical one in this basis. As an example, one can consider the basis $B = \{b_1, b_2, \ldots, b_6\}$ for L, $\dim L = 4$, where

$$b_1 = (1, 0, 0, 0), \qquad b_2 = (0, 1, 0, 0),$$
$$b_3 = (0, 0, 1, 0), \qquad b_4 = (-1, -1, -1, 0),$$
$$b_5 = (-1, 0, 0, 1), \qquad b_6 = (-1, 0, 0, -1)$$

and $z = -b_4$.

Nevertheless, the following lemma holds:

LEMMA 3.2. *If Δ is a simplicial basis, then $z \in C^*(\Delta)$ iff $z \in C(\Delta)$ and $\Delta(z) \cap \Delta(-z) = \{0\}$.*

Proof. By Lemma 3.1 it is sufficient to show that if $z \in C(\Delta)$ and $\Delta(z) \cap \Delta(-z) = \{0\}$, then $z \in C^*(\Delta)$. Let $z \neq 0$. Suppose that $D \subset \Delta$ is a simplex such that the cone $\operatorname{pos}(D \cup \{z\})$ does not contain the whole line and let

$$x \in \operatorname{relint} \operatorname{pos}(D \cup \{z\}).$$

If $D \subset \Delta(z)$, then $\Delta(x) = \Delta(z)$ and $D \cup \{z\} \subset \operatorname{pos} \Delta(x)$. In the opposite case we have $D \cap \Delta(-z) \neq \emptyset$ since $\Delta = \Delta(z) \cup \Delta(-z)$ for each $z \in L$, $z \neq 0$. Moreover, $\Delta(-z) \setminus D \neq \{0\}$, since the cone $\operatorname{pos}(D \cup \{z\})$ does not contain the whole line. Thus, from the above and the assumption $\Delta(-z) \cap \Delta(z) = \{0\}$ it follows that $D \cup (\Delta(z) \setminus \{0\})$ is a simplex and

$$x \in \operatorname{relint} \operatorname{pos}\big(D \cup (\Delta(z) \setminus \{0\})\big).$$

Therefore, we have also $D \cup \{z\} \subset \operatorname{pos} \Delta(x)$, which completes the proof.

COROLLARY 3.1. *Let $\Delta = \{a_0, a_1, \ldots, a_n\}$ be a simplex basis and let*

$$0 = \sum_{i \in N} \xi_i a_i, \quad \xi_i > 0, \ \sum_{i \in N} \xi_i = 1, \ N = \{0, 1, \ldots, n\}.$$

Then $z \in C^*(\Delta)$ iff

$$z = \lambda \sum_{i \in I} \xi_i a_i,$$

where $\lambda \geqslant 0$, $I \subsetneqq N$, card $I \geqslant 2$.

The set I is called a *support* of vector z.

Note that the set

$$E(\Delta) = \{e(I) = (\sum_{i \in I} \xi_i)^{-1} \sum_{i \in I} \xi_i a_i \colon I \in \mathscr{P}(\Delta)\},$$

where

$$\mathscr{P}(\Delta) = \{I \colon I \subset N, \text{ card } I \geqslant 2\}$$

(i.e., the set of all centres of proper simplicial faces with dimension $n \geqslant 1$) can be considered instead of $C^*(\Delta)$.

Remark 3.1. The set $E(\Delta)$ is finite for an arbitrary simplicial basis of finite dimension.

Let $\mathscr{C}^*(\Delta)$ denote the family of strongly critical sets for a simplicial basis Δ. Now we present an additional characterization of strongly critical sets for simplicial bases, which will be useful in further considerations

LEMMA 3.3. *The set $C \subset E(\Delta)$ is a strongly critical one for the basis Δ iff for every set $D \subset \Delta \cup C$ such that*

$$\bigcup_{d \in D} \Delta(d) \setminus \{0\} = \Delta \tag{3.1}$$

we have

$$0 \in \operatorname{conv} D. \tag{3.2}$$

Proof. Assume that C is a strongly critical set for Δ and consider the set $D \subset \Delta \cup C$ satisfying (3.1). If $0 \notin \operatorname{conv} D$, then the cone $\operatorname{pos} D$ does not contain the whole line and, by Corollary 2.1, $\Delta \subset \Delta(x)$, where $x \in \operatorname{relint} \operatorname{conv} D$. This contradicts the fact that Δ is a regular basis.

Suppose now that condition (3.2) holds for each $D \subset \Delta \cup C$ satisfying (3.1). Let $\bar{D} \subset \Delta \cup C$ be an arbitrary set such that the cone $\operatorname{pos} \bar{D}$ does not contain the whole line. Then $0 \notin \operatorname{conv} \bar{D}$ and by assumption we have

$$\bar{A} = \bigcup_{d \in \bar{D}} \Delta(d) \setminus \{0\} \neq \Delta.$$

The set $\bar{A}$ is a proper side of the simplex Δ, and if $x \in \operatorname{relint} \operatorname{conv} \bar{D}$, then

$$x \in \operatorname{relint} \operatorname{conv} \bar{A}.$$

Thus $\bar{A} = \Delta(x)$ and, by Corollary 2.1, $C \in \mathscr{C}^*(\Delta)$.

Further, instead of $p_2 C^*(\Delta)$ (cf. (1.2)) defined as

$$p_2 C^*(\Delta) = \{z\colon\ z = \lambda \sum_{i \in I} \xi_i (a_i + c),\ I \subsetneq N,\ \operatorname{card} I \geqslant 2,\ \lambda > 0\},$$

we write $C^*(\Delta + c)$. A vector $z \in C^*(\Delta + c)$ is called a *strongly critical vector* for $\Delta + c$. Analogously, instead of $p_2 \mathscr{C}^*(\Delta)$ we write $\mathscr{C}^*(\Delta + c)$. Now, by Lemma 3.3 we have ($\partial(\Delta + c)$ denotes the set $\bigcup_{a \in \Delta} \operatorname{pos}((\Delta \setminus \{a\}) + c)$)

LEMMA 3.4. *A set $C \subset \partial(\Delta + c)$ is strongly critical for $\Delta + c$, $c \neq 0$, iff for arbitrary $D \subset (\Delta + c) \cup C$ the condition*

$$\bigcup_{d \in D} (\Delta + c)(d) = \Delta + c \tag{3.3}$$

implies

$$c \in \operatorname{pos} D.$$

We shall now present a characterization of the family $\mathscr{C}^*(B)$ whose elements are strongly critical sets for regular bases $B = B_1 \cup (\Delta + c)$. By $S_B^*(C)$, $C \in \mathscr{C}^*(B)$, we denote the set of elements z such that

$$z \in C^*(B) \quad \text{and} \quad C \cup \{z\} \in \mathscr{C}^*(B).$$

$S^*(C)$ is called a *star* of strongly critical vectors for the basis B, generated by the strongly critical set C. Analogously, we put

$$S^*_{\Delta+c}(C) = \{z \in C^*(\Delta+c)\colon\ C \cup \{z\} \in \mathscr{C}^*(\Delta+c)\}.$$

Let us prove first

LEMMA 3.5. *If z is a strongly critical vector for the regular basis $B = B_1 \cup (\Delta+c)$, then $z \in L_1$ or $z \in \partial(\Delta+c)$.*

Proof. Let $z \in C^*(B)$, $z = p_1(z)+p_2(z)$ (cf. (1.2)). We have to show that $p_1(z) = 0$ or $p_2(z) = 0$.

Let us assume that $p_1(z) \neq 0$ and $p_2 \neq 0$. Then the cone $\operatorname{pos} D$, $D = \{z\} \cup B_1(-p_1(z))$, does not contain the whole line. It is clear because using Theorem 2.1 we see that B_1 is a regular basis and the cone $\operatorname{pos} B_1(-p_1(z))$, $z \notin L_1$, does not contain the whole line. Thus, by the regularity of $B \cup \{z\}$ in L and Corollary 2.1 we obtain

$$D \subset \operatorname{pos} B(x),$$

where $x \in \operatorname{relint} \operatorname{pos} D$. Further, by virtue of Lemma 2.3,

$$B(z) \cup B_1(-p_1(z)) \subset \operatorname{pos} B(x).$$

Moreover, using Theorem 2.1 as well as formula (1.3) for the regular basis we have

$$B_1(p_1(z)) \subset \operatorname{pos} B(z) \subset \operatorname{pos} B(x).$$

Hence $p_1(z) \in \operatorname{pos} B(x)$ and $-p_1(z) \in \operatorname{pos} B(x)$, which contradicts the assumption.

From Lemma 3.5 it follows that a strongly critical set for the regular basis $B = B_1 \cup (\Delta+c)$ has the partition

$$C = C_1 \cup C_\Delta,$$

where

$$C_1 \subset L_1, \quad C_\Delta \subset \partial(\Delta+c), \quad C_1 \cap C_\Delta \subset \{0\}.$$

THEOREM 3.1. *A finite set C is strongly critical for the regular basis $B = B_1 \cup (\Delta+c)$ iff C is of the form $C_1 \cup C_\Delta$, where*

$$C_1 \cup \{c\} \in \mathscr{C}^*(B_1), \quad C_\Delta \in \mathscr{C}^*(\Delta+c).$$

Proof. Let $C \in \mathscr{C}^*(B)$. By Lemma 3.5 we have

$$C = C_1 \cup C_\Delta, \quad C_1 \subset L_1 \text{ and } C_\Delta \subset \partial(\Delta+c).$$

We prove first that $C_1 \cup \{c\} \in \mathscr{C}^*(B_1)$.

The set $B \cup C_1$ is strongly critical in $L = \operatorname{lin} B$. Let

$$B_1^* = B_1 \cup C_1 \cup \{c\}.$$

Note that if $x \in L_1$, then

$$\operatorname{pos} B_1^*(x) \subset \operatorname{pos}(B \cup C_1)(x).$$

Hence, using again the regularity of the set $B \cup C_1$ we infer that for an arbitrary $x \in L_1$ the cone $\operatorname{pos} B_1^*(x)$ does not contain the whole line. This means that $C_1 \cup \{c\} \in \mathscr{C}^*(B_1)$.

Now, we will show that $C_\Delta \in \mathscr{C}^*(\Delta + c)$. The set $B \cup C_\Delta$ is regular in $L = \operatorname{lin} B$. Moreover, $C_\Delta \subset \partial(\Delta + c)$. If $c = 0$, then $C_\Delta \in \mathscr{C}^*(\Delta)$. Suppose that $c \neq 0$. Assume that the set $D \subset (\Delta + c) \cup C_\Delta$ satisfies condition (3.3). Then there exists

$$x \in \operatorname{relint} \operatorname{pos} D \cap \operatorname{relint} \operatorname{pos}(\Delta + c).$$

Suppose that $c \notin \operatorname{pos} D$ and consider the vector $x - c$. The cone $\operatorname{pos}(D \cup \{c\})$ does not contain the whole line and

$$x - c \in \operatorname{relint} \operatorname{pos}(D \cup \{-c\}).$$

By Lemma 2.1 we have

$$D \cup \{-c\} \subset \operatorname{pos}(B \cup C_\Delta)(x - c),$$

and thus $x \in \operatorname{pos}(B \cup C_\Delta)(x - c)$. Due to Lemma 2.3 we have also

$$\Delta + c \subset \operatorname{pos}(B \cup C_\Delta)(x - c),$$

and further $c \in \operatorname{pos}(B \cup C_\Delta)(x - c)$. It is, however, impossible, because by the regularity of the set $B \cup C_\Delta$ it follows that the cone $\operatorname{pos}(B \cup C_\Delta)(x - c)$ does not contain the whole line. Hence, it cannot contain nonzero vectors c and $-c$. Thus $c \in \operatorname{pos} D$ and by Lemma 3.4 we obtain $C_\Delta \in \mathscr{C}^*(\Delta + c)$.

Suppose that $C = C_1 \cup C_\Delta$, where $C_1 \cup \{c\} \in \mathscr{C}^*(B_1)$ and $C_\Delta \in \mathscr{C}^*(\Delta + c)$.

Let the set $D \subset B \cup C$, $D = D_1 \cup D_\Delta$, where

$$D_1 \subset B_1 \cup C_1 \quad \text{and} \quad D_\Delta \subset (\Delta + c) \cup C_\Delta,$$

satisfy condition (1.1).

If $c = 0$, then $x = x_1 + x_2$, $x_1 \in L_1 = \operatorname{lin} B_1$, $x_2 \in L_2 = \operatorname{lin} \Delta$, and by the assumption we obtain

$$D_1 \subset \operatorname{pos} B_1(x_1) \quad \text{and} \quad D_\Delta \subset \operatorname{pos} \Delta(x_2).$$

Hence, by (1.3),

$$D \subset \operatorname{pos} B(x) = \operatorname{pos}\big(B_1(x_1) \cup \Delta(x_2)\big).$$

Due to the arbitrariness of the choice of the set D and the regularity of the basis B, the cone $\operatorname{pos}(B \cup C)(x)$ does not contain the whole line.

If $c \neq 0$, then

$$x = p_1(x) + p_2(x), \qquad p_1(x) \in L_1,\ p_2(x) \in \partial(\Delta + c).$$

Since $x \in \operatorname{relint} \operatorname{pos} D$, we have

$$x = \bar{x}_1 + \bar{x}_\Delta,$$

where

$$\bar{x}_1 \in \operatorname{relint} \operatorname{pos} D_1 \quad \text{and} \quad \bar{x}_\Delta \in \operatorname{relint} \operatorname{pos} D_\Delta.$$

If $\bar{x}_\Delta \in \partial(\Delta + c)$, then $\bar{x}_1 = p_1(x)$ and $\bar{x}_\Delta = p_2(x)$, and by the assumption we have

$$D_1 \subset \operatorname{pos} B_1(p_1(x))$$

and

$$D_\Delta \subset \operatorname{pos}(\Delta + c)(p_2(x)) = \operatorname{pos}(\Delta(x_2) + c).$$

Using formula (1.3) we obtain

(3.4) $$D \subset \operatorname{pos} B(x).$$

If $\bar{x}_\Delta \in \operatorname{relint} \operatorname{pos}(\Delta + c)$, then

$$\bigcup_{d \in D_\Delta} (\Delta + c)(d) = \Delta + c$$

and due to Lemma 3.4

(3.5) $$c \in \operatorname{pos} D_\Delta.$$

Note that there exists a $\lambda > 0$ such that

$$\bar{x}_\Delta - \lambda c \in \partial(\Delta + c).$$

Then $x = (\bar{x}_1 + \lambda c) + (\bar{x}_\Delta - \lambda c)$ and taking into account the uniqueness of the partition $x = p_1(x) + p_2(x)$, we have $p_1(x) = \bar{x}_1 + \lambda c$ and $p_2(x) = \bar{x}_\Delta - \lambda c$. Moreover, the cone $\operatorname{pos}(D_1 \cup \{c\})$ does not contain the whole line and

$$p_1(x) \in \operatorname{relint} \operatorname{pos}(D_1 \cup \{c\})$$

by (3.5) and since the set D satisfies condition (1.1).

Hence $C_1 \in \mathscr{C}^*(B_1)$ implies

$$D_1 \cup \{c\} \subset \operatorname{pos} B_1(p_1(x)).$$

Since in that case $c \in \operatorname{pos} B_1(p_1(x))$, by (1.3) we have

$$B(x) = B_1(p_1(x)) \cup (\Delta + c),$$

and consequently

$$D \subset \operatorname{pos} B(x).$$

Thus using (3.4) we obtain

$$(B \cup C)(x) \subset \operatorname{pos} B(x),$$

which proves the regularity of the set $B \cup C$ and completes the proof.

As a consequence of Theorem 3.1 we have

THEOREM 3.2. *If $B = B_1 \cup (\Delta + c)$ is a regular basis, then*

$$C^*(B) = S^*_{B_1}(\{c\}) \cup C^*(\Delta + c).$$

Using Romanowicz's partition theorem for a positive basis and Remark 3.1 we can immediately show

COROLLARY 3.2. *Every regular positive basis of a finite-dimensional space admits a finite number of strongly critical directions.*

From Theorem 2.1 we obtain also

THEOREM 3.3. *If $B = B_1 \cup (\Delta + c)$ is a regular basis and $C = C_1 \cup C_\Delta$ is in $\mathscr{C}^*(B)$, then*

$$S^*_B(C) = S^*_{B_1}(C_1 \cup \{c\}) \cup S^*_{\Delta+c}(C_2).$$

Theorems 3.2 and 3.3 give us a recurrence method for determining strongly critical sets for an arbitrary regular basis

$$B = \Delta_1 \cup (\Delta_2 + c_1) \cup \ldots \cup (\Delta_k + c_{k-1})$$

by means of the union of the appropriate stars for simplex bases.

THEOREM 3.4. *Each maximal set in the family $\mathscr{C}^*(\Delta)$, $\Delta = \{a_0, a_1, \ldots, a_n\}$, is uniquely determined by the following* (i) *or* (ii):

(i) *the choice of the partition $I_1 \cup I_2 \cup \ldots \cup I_k$ $(k \geqslant 2)$ of the set N, where $N = \{0, 1, \ldots, n\}$,* card $I_j \geqslant 2$, $I_j \cap I_i = \emptyset$, $i, j \in \{1, 2, \ldots, k\}$, *and the choice of the maximal chain*

$$I_j^1 \subset I_j^2 \subset \ldots \subset I_j^{r_j} = I_j,$$

where card $I_j^1 = 2$ *and* $j = 1, 2, \ldots, k$;

(ii) *the choice of a single chain*

$$I_1^1 \subset I_1^2 \subset \ldots \subset I_1^{r_1},$$

where card $I_1^1 = 2$, card $I_1^{r_1} = n-1$, $n = \dim \Delta$.

Theorem 3.4 is a consequence of the following

LEMMA 3.6. *A set $C \subset C^*_E(\Delta)$ satisfies the condition $C \in \mathscr{C}^*_E(\Delta)$ iff the following implication is true*:

$$I_1, I_2 \in \mathscr{P}(C),\ I_1 \neq I_2 \Rightarrow I_1 \cap I_2 = \emptyset \text{ or } I_1 \subset I_2 \text{ or } I_2 \subset I_1.$$

4. Some related results and problems. Recently Bàràny [1] has formulated the problem of determining the number of cones which satisfy conditions (i)–(iv) from Section 1.

It turns out that this number is not bounded by 2^n (see, e.g., an example by Vrečica [2]). It is easy to see that the family of full-dimensional basis cones for regular basis satisfies the above-mentioned conditions. This fact follows from Lemmas 2.1–2.3 and from the following simple

LEMMA 4.1. *If M is a maximal subset of a positive basis B such that the cone* pos M *does not contain the whole line, then*

(1) $-(B\setminus M)\subset \operatorname{pos} M$;

(2) $\operatorname{relint}\operatorname{pos}(-(B\setminus M))\subset \operatorname{int}\operatorname{pos} M$;

(3) *for each proper subset Z of $B\setminus M$,*

$$\operatorname{relint}\operatorname{pos}(-Z)\cap \operatorname{int}\operatorname{pos} M=\emptyset;$$

(4) *if $y\in \operatorname{relint}\operatorname{pos}(B\setminus M)$, then $B(y)=(B\setminus M)\cup\{0\}$.*

Let us remark that the union of all full-dimensional basis cones coincides with the entire space (see (v) from Section 1). The remaining question is whether the number of cones which satisfy conditions (i)–(v) from Section 1 is bounded by 2^n. Here we shall show that for the full-dimensional basis cones for a regular basis the answer to this question is affirmative.

Let us remark that by Theorem 2.1 and formula (1.3) one can obtain

COROLLARY 4.1. *If $B=B_1\cup(\Delta+c)$ is a regular basis and $c\neq 0$, then the basis cones for B are of the form*

$$K_1+K_2,$$

where K_1 is a basis cone for B_1 such that

$$c\notin \operatorname{cl} K_1,\qquad K_2=\operatorname{relint}\operatorname{pos}(D+c),\ D\subsetneqq \Delta,$$

or of the form

$$K_1+\operatorname{relint}\operatorname{pos}(\Delta+c),$$

where K_1 is a basis cone for B_1 such that $c\in\operatorname{cl} K_1$.

Corollary 4.1 together with formula (1.3) for $c=0$ lead to the inequality

$$s(B)\leqslant s(B_1)\operatorname{card}\Delta,$$

where $s(B)$ ($s(B_1)$) denotes the number of full-dimensional basis cones for the regular basis B (B_1).

By Romanowicz's partition theorem for a positive basis we have

$$s(B)\leqslant \operatorname{card}\Delta_1\operatorname{card}\Delta_2\ldots\operatorname{card}\Delta_k.$$

The equality holds iff $c_1=c_2=\ldots=c_{k-1}=0$ (i.e., iff B is a strong positive basis).

Note also that

$$\operatorname{card}\Delta_1+\operatorname{card}\Delta_2+\ldots+\operatorname{card}\Delta_k=\operatorname{card} B$$

and, by the obvious inequality $r+1\leqslant 2^r$ for all $r\in N$, we obtain

THEOREM 4.1. *If B is a regular positive basis of a linear n-dimensional space L, then*

$$n+1\leqslant s(B)\leqslant 2^n.$$

The equality $s(B) = n+1$ *is attained only for a simplicial basis while the equality* $s(B) = 2^n$ *only for the maximal basis.*

For an arbitrary positive basis B of L, the set $W(B) = \operatorname{conv} B$ is called a *basis polyhedron*. The positive basis B is called a *proper basis* provided that for each proper face K of $W(B)$ the set relint pos K coincides with a certain basis cone. Note that every proper basis is regular.

LEMMA 4.2. *A positive basis* B *is a proper one iff* $\operatorname{conv} D \subset \partial W(B)$ *for an arbitrary set* $D \subset B$ *for which*

$$x \in \operatorname{relint} \operatorname{pos} D \quad \textit{and} \quad x \notin \operatorname{relint} \operatorname{pos} D' \textit{ for all } D' \subsetneq D.$$

LEMMA 4.3. *Let* W_1 *and* W_2 *be convex polyhedrons in* L *and let*

$$\operatorname{aff} W_1 \cap \operatorname{aff} W_2 = \{c\}, \qquad c \in \operatorname{relint} W_1, \; c \in \partial W_2,$$
$$W = \operatorname{conv}(W_1 \cup W_2).$$

Then

(i) K *is a proper face of the polyhedron* W, $c \in K$, *iff*

$$K = \operatorname{conv}(K_1 \cup W_2),$$

where K_1 *is a face of the polyhedron* W_1, $c \in K_1$.

(ii) K *is a proper face of the polyhedron* W, $c \notin K$, *iff*

$$K = \operatorname{conv}(K_1 \cup K_2),$$

where K_1 *is a face of the polyhedron* W_1, $c \notin K_1$, *and* K_2 *is a proper face of* W_2 (*note that* $K_1 = \emptyset$ *or* $K_2 = \emptyset$ *can happen*).

Let us remark that if the set $B = \{b_1, b_2, \ldots, b_m\}$ is a positive basis of L, then the set

$$B^* = \{\beta_1 b_1, \beta_2 b_2, \ldots, \beta_m b_m\}, \qquad \beta_i > 0,$$

is also a positive basis of L. The bases B and B^* are called *equivalent* ones.

THEOREM 4.2. *For every positive basis* B *there exists a proper basis* B^* *equivalent to* B.

The proof of this theorem is carried out by induction on $n = \dim \operatorname{lin} B$ taking into account Corollary 4.1 and Lemma 4.2.

Using Theorem 4.2 and Lemma 4.2 it is easy to prove the following

THEOREM 4.3. *The basis* $B = B_1 \cup (\Delta + c)$ *is proper iff* B_1 *is a proper basis of the subspace* $L_1 = \operatorname{lin} B_1$ *and* c *is a strongly critical vector for* B_1, $c \in \partial W(B_1)$ *or* $c = 0$.

Theorem 4.3 gives us a simple method for constructing proper bases. Especially, for $c_1 = c_2 = \ldots = c_k = 0$ the basis B is strongly positive and also proper.

Let now V be a full-dimensional convex polyhedron and let $S_1, S_2, \ldots, S_k$ be its faces. By $\mathscr{W}^*$ we denote the family of polyhedrons which satisfy the condition

$$0 \in \operatorname{int} V \cap \bigcap_{i \neq j} \operatorname{conv}(S_i \cup S_j).$$

Note that for an arbitrary proper basis B we have $W(B) \in \mathscr{W}^*$ (cf. (iv) from Section 1 and Lemma 4.1).

We say that the direction k *illuminates* a convex body S of the n-dimensional space L at point $x \in \partial S$ if

$$\{x + tk \colon t \geqslant 0\} \cap \operatorname{int} S \neq \emptyset.$$

It is clear that if w is a vertex of the polyhedron W, then the set of directions illuminating W at the point w is equivalent to $\operatorname{int} C_w$. In our notation, $C_w + w$ means the cone supporting the polyhedron W at the vertex w. On the other hand,

$$C_w = \{x \in L \colon x = t(y - w),\ t \geqslant 0,\ y \in W\}.$$

Two different vertices w_1 and w_2 of the polyhedron W are called *independent* with respect to illumination if

$$\operatorname{int} C_{w_1} \cap \operatorname{int} C_{w_2} = \emptyset.$$

Let us denote by $\mathscr{W}$ the class of polyhedrons for which every pair of different vertices is independent with respect to illumination.

Let W be a full-dimensional convex polyhedron in L and let $0 \in \operatorname{int} W$. The set

$$W^* = \{x \in L \colon (x, y) \leqslant 1 \text{ for } y \in W\}$$

is also a full-dimensional convex polyhedron in L and it is called the *dual polyhedron* of W. For W^* we have also $0 \in \operatorname{int} W^*$.

It is easy to prove the following

THEOREM 4.4. *$W \in \mathscr{W}$ iff $W = V^* + p$, where $V \in \mathscr{W}^*$ and $p \in L$.*

COROLLARY 4.2. *If B is a proper basis, then $W^*(B) \in \mathscr{W}$.*

One can assume that the family of polyhedrons dual to the basis polyhedrons for the proper basis is not equivalent to $\mathscr{W}$.

Finally, let us remark that the class $\mathscr{W}$ has some other interesting properties.

Let $\mathscr{W}_1$ be the class of polyhedrons such that each pair of their vertices is antipodal [4]. Let $\mathscr{W}_2$ denote the class of polyhedrons such that for each three of their vertices we have

$$\operatorname{int}(W + w_1 - w_3) \cap \operatorname{int}(W + w_2 - w_3) = \emptyset.$$

It is easy to prove that $\mathscr{W} = \mathscr{W}_1 = \mathscr{W}_2$ (for the proof of the last equality see [4]).

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