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A NOTE ON OPTIMAL $C(\alpha)$-TESTS FOR HOMOGENEITY OF THE POISSON DISTRIBUTION

1. Introduction. Neyman and Scott have shown [2] that the test for homogeneity of the Poisson distribution, based on the classical dispersion coefficient (for short, test Z), is an optimal $C(\alpha)$-test for a vast family of sequences of alternatives converging to a Poisson distribution, each sequence specifying a different character of possible inhomogeneity.

In this paper*, necessary and sufficient conditions for an optimal $C(\alpha)$-test for homogeneity of the Poisson distribution to be the same as the test Z are given. These conditions show that the test Z is an optimal $C(\alpha)$-test for a much larger family of sequences of alternatives than it has been pointed out by Neyman and Scott. Also, two examples of optimal $C(\alpha)$-tests for homogeneity of the Poisson distribution, which are not equivalent to the test Z, are given. The last section contains some remarks concerning the performance of the derived optimal $C(\alpha)$-tests for testing the homogeneity of the Poisson distribution.

2. General expression for optimal $C(\alpha)$-tests for homogeneity. Consider N experimental units $U_1, \ldots, U_N$ and suppose that, for every unit U_i, $i = 1, \ldots, N$, a Poisson random variable X_i is observed. It is assumed that the expected value, say Λ, can vary from one experimental unit to the next and is subject to a probability distribution, say $F(\lambda)$. In that case the unconditional probability distribution of X_i is

$$p(x) = \frac{1}{x!}\int_0^\infty e^{-\lambda}\lambda^x dF(\lambda).$$

The random variables $X_1, \ldots, X_N$ are assumed to be independent. The problem is to test the hypothesis that all the experimental units

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are characterized by the same, though unknown, value, say λ_0, or, equivalently, that $\Lambda = \lambda_0$ with probability one.

In order to bring the problem into the general framework in which the theory of $C(\alpha)$-tests is applicable, one has, as indicated by Neyman and Scott [2], to postulate a family of distributions $\{F_\xi(\lambda)\}$, $\xi \in (0, a]$, such that $\{F_\xi(\lambda)\}$ converges as $\xi \to 0$ to a distribution degenerated at $\Lambda = \lambda_0$, say $\varepsilon(\lambda - \lambda_0)$, so that the corresponding sequence of unconditional probability distributions of X_i, say $\{p(x|\xi)\}$, converges to the Poisson distribution with expected value λ_0. Clearly, there are infinitely many such families of distributions. Let $p(x, 0) = e^{-\lambda_0}\lambda_0^x/x!$.

In this setup the problem of testing the homogeneity of the Poisson distribution reduces to the problem of testing the hypothesis H_0: $\xi = 0$ against the alternative H_1: $\xi > 0$, while λ_0 is a nuisance parameter. Clearly, with appropriate restrictions of regularity of the functions involved, this problem can be tested by the optimal $C(\alpha)$-test.

In this paper considerations are restricted to families of distributions $\{F_\xi(\lambda)\}$ satisfying the following conditions (conditions A, for short):

(a) $F_\xi(\lambda) = 0$ for $\lambda < 0$ and $\xi \in (0, a]$;

(b) $F_\xi(\lambda) \to \varepsilon(\lambda - \lambda_0)$ as $\xi \to 0$;

(c) for all $k \geqslant 1$, the limits

$$c_k = \lim_{\xi \to 0} \frac{\mathrm{E}(\Lambda_\xi - \lambda_0)^k}{\xi} \tag{1}$$

exist and are finite;

(d) there exists a value ξ_0, $0 < \xi_0 < a$, such that (5) can be integrated termwise for all ξ, $0 \leqslant \xi \leqslant \xi_0$, and that the order of lim and $\sum$ can be subsequently reversed.

The following theorem gives a general expression for test functions of optimal $C(\alpha)$-tests with respect to families of probability distribution functions which satisfy conditions A:

THEOREM 1. *Let $\{F_\xi(\lambda)\}$, $\xi \in (0, a]$, be a family of distributions satisfying conditions* A. *Then the corresponding optimal $C(\alpha)$-test criterion for testing H_0: $\xi = 0$ against H_1: $\xi > 0$ has the form*

$$Z_N = \frac{\sum_{i=1}^{N} g(X_i, \hat{\lambda}_0)}{\sqrt{N\sigma^2(\hat{\lambda}_0)}}, \tag{2}$$

where

$$g(x, \lambda) = \frac{1}{e^{-\lambda}\lambda^x} \sum_{k=2}^{\infty} \frac{c_k}{k!} \frac{d^k(e^{-\lambda}\lambda^x)}{d\lambda^k} \quad \text{and} \quad \sigma^2(\lambda) = \operatorname{Var} g(X_i, \lambda),$$

while $\hat{\lambda}_0$ stands for a locally root N consistent estimator for λ_0.

Proof. As it is well known (cf. [1] and [2]), the test function $g = g(x, \lambda)$ of the optimal $C(\alpha)$-criterion has the form

$$g = \varphi_\xi - a\varphi_{\lambda_0},$$

where

$$\varphi_\xi = \frac{\partial \log p(x|\xi)}{\partial \xi}\bigg|_{\xi=0}, \quad \varphi_{\lambda_0} = \frac{\partial \log p(x|0)}{\partial \lambda}\bigg|_{\lambda=\lambda_0},$$

and

$$a = \frac{\operatorname{cov}(\varphi_\xi, \varphi_{\lambda_0})}{\operatorname{Var}\varphi_{\lambda_0}}.$$

Because

(3) $$p(x|0) = \frac{1}{x!} e^{-\lambda_0}\lambda_0^x,$$

it is easily seen that

(4) $$\varphi_\xi = \frac{1}{x!} \lim_{\xi\to 0} \frac{1}{\xi} \int_0^\infty (e^{-\lambda}\lambda^x - e^{-\lambda_0}\lambda_0^x)\, dF_\xi(\lambda).$$

Replacing $e^{-\lambda}\lambda^x$ by its expansion around $\lambda = \lambda_0$,

$$e^{-\lambda}\lambda^x = e^{-\lambda_0}\lambda_0^x + \sum_{k=1}^\infty \frac{d^k(e^{-\lambda}\lambda^x)}{d\lambda^k}\bigg|_{\lambda=\lambda_0} \frac{(\lambda-\lambda_0)^k}{k!}$$

reduces (4) to

(5) $$\varphi_\xi = \frac{1}{x!} \lim_{\xi\to 0} \frac{1}{\xi} \int_0^\infty \sum_{k=1}^\infty \frac{1}{k!} \frac{d^k(e^{-\lambda}\lambda^x)}{d\lambda^k}\bigg|_{\lambda=\lambda_0} (\lambda-\lambda_0)^k dF_\xi(\lambda).$$

Finally, in view of assumptions A,

(6) $$\varphi_\xi = \frac{1}{x!} \sum_{k=1}^\infty \frac{c_k}{k!} \frac{d^k(e^{-\lambda}\lambda^x)}{d\lambda^k}\bigg|_{\lambda=\lambda_0}.$$

Using (3), it is easily seen that

(7) $$\varphi_{\lambda_0} = \frac{x}{\lambda_0} - 1.$$

To find $\operatorname{cov}(\varphi_\xi, \varphi_{\lambda_0})$, first note that

$$\mathrm{E}\frac{X_i}{\lambda^{X_i}}\frac{d^k(e^{-\lambda}\lambda^{X_i})}{d\lambda^k} = e^{-\lambda}\frac{d^k\lambda}{d\lambda^k} \quad \text{for } k \geqslant 1,$$

and, therefore,

$$\mathrm{E}\frac{X_i}{\lambda^{X_i}}\frac{d^k(e^{-\lambda}\lambda^{X_i})}{d\lambda^k} = \begin{cases} e^{-\lambda} & \text{if } k = 1, \\ 0 & \text{if } k > 1. \end{cases}$$

An application of this formula gives

$$\operatorname{cov}(\varphi_\xi, \varphi_{\lambda_0}) = \mathrm{E}\frac{1}{e^{-\lambda_0}\lambda^{X_i}}\sum_{k=1}^{\infty}\frac{c_k}{k!}\frac{d^k(e^{-\lambda}\lambda^{X_i})}{d\lambda^k}\bigg|_{\lambda=\lambda_0}\left(\frac{X_i}{\lambda_0}-1\right) \tag{8}$$

$$= \frac{c_1}{e^{-\lambda_0}\lambda_0}\mathrm{E}\frac{X_i}{\lambda_0^{X_i}}\frac{d(e^{-\lambda}\lambda^{X_i})}{d\lambda}\bigg|_{\lambda=\lambda_0} = \frac{c_1}{\lambda_0}.$$

Using (6), (7) and (8), it is easily seen that

$$g = \frac{1}{e^{-\lambda_0}\lambda_0^x}\sum_{k=2}^{\infty}\frac{c_k}{k!}\frac{d^k(e^{-\lambda}\lambda^x)}{d\lambda^k}\bigg|_{\lambda=\lambda_0}.$$

This completes the proof of theorem 1.

COROLLARY. *Let* $\{F_\xi(\lambda)\}$, $\xi\in(0, a]$, *be a family of probability distributions satisfying conditions* A. *Then the optimal* $C(\alpha)$*-test criterion of the hypothesis* H_0: $\xi = 0$ *against the alternative* H_1: $\xi > 0$ *has the form*

$$Z_N = \frac{\sum_{i=1}^{N}[(X_i-\hat{\lambda}_0)^2-X_i]}{\sqrt{2N}\,\hat{\lambda}_0} \tag{9}$$

if and only if $c_2 > 0$, *while* $c_k = 0$ *for* $k \geqslant 3$. *Here* $\hat{\lambda}_0$ *stands for any root* N *consistent estimator of the nuisance parameter* λ_0.

Proof. The formula for Z_N follows immediately from theorem 1. In fact, simple arithmetic calculations show that

$$\operatorname{Var} g(X_i, \lambda_0) = \frac{c_2^2}{4\lambda_0^4}\mathrm{E}[(X_i-\lambda_0)^2-X_i]^2 = \frac{c_2^2}{2\lambda_0^2},$$

which combined with (2) leads to (9). On the other hand, if Z_N is of the form as given by (9), it can be easily shown that $c_k = 0$ for $k \geqslant 3$.

3. A theorem on the class of sequences $\{c_k\}$. In view of theorem 1, the optimal $C(\alpha)$-test with respect to the family $\{F_\xi(\lambda)\}$, $\xi\in(0, a]$, is uniquely determined by the sequence $\{c_k\}$ of limits defined by (1). This being the case, the question arises how to characterize the class of those sequences of finite numbers $\{c_k\}$ for which there exist families of distributions $\{F_\xi(\lambda)\}$, $\xi \in (0, a]$, satisfying conditions A. A partial answer gives the following theorem:

THEOREM 2. *Let $\{F_\xi(\lambda)\}$, $\xi \in (0, a]$, be a family of distributions satisfying conditions* A. *If $c_2 > 0$ and if there exists an even integer $k_1 > 2$ such that $c_{k_1} = 0$, then $c_k = 0$ for all $k > 2$. If $c_2 = 0$, then $c_k = 0$ for all $k \geqslant 2$.*

The proof of theorem 2 will be divided in lemmas 1 and 2. It is assumed that assumptions A hold.

LEMMA 1. *If k_1 and k_2 are even, $k_1 < k_2$ and $c_{k_2} = 0$, while $c_{k_1} > 0$, then $c_k = 0$ for $k_1 < k \leqslant k_2$.*

Proof. Without loss of generality, it can be assumed that $\mathrm{E}\Lambda_\xi = 0$. Let $\varepsilon > 0$ and let $k_1 < k \leqslant k_2$. Now

$$\mathrm{E}\Lambda_\xi^k = \int\limits_{|\lambda|\leqslant\varepsilon} \lambda^k dF_\xi(\lambda) + \int\limits_{|\lambda|>\varepsilon} \lambda^k dF_\xi(\lambda).$$

Because $\lambda^k < \varepsilon^{k-k_2}\lambda^{k_2}$ for $|\lambda| > \varepsilon$, it is easily seen that

$$\int\limits_{|\lambda|>\varepsilon} \lambda^k dF_\xi(\lambda) \leqslant \varepsilon^{k-k_2} \int\limits_{|\lambda|>\varepsilon} \lambda^{k_2} dF_\xi(\lambda) \leqslant \varepsilon^{k-k_2} \int \lambda^{k_2} dF_\xi(\lambda).$$

Thus, for any $0 < \xi \leqslant a$,

$$\mathrm{E}\,\frac{\Lambda_\xi^k}{\xi} \leqslant \int\limits_{|\lambda|\leqslant\varepsilon} \frac{\lambda^k}{\xi}\, dF_\xi(\lambda) + \varepsilon^{k-k_2} \int \frac{\lambda^{k_2}}{\xi}\, dF_\xi(\lambda)$$

and, therefore,

$$\mathrm{E}\left|\frac{\Lambda_\xi^k}{\xi}\right| \leqslant \varepsilon^{k-k_1} \int\limits_{|\lambda|\leqslant\varepsilon} \frac{\lambda^{k_1}}{\xi}\, dF_\xi(\lambda) + \varepsilon^{k-k_2} \int \frac{\lambda^{k_2}}{\xi}\, dF_\xi(\lambda)$$

since $|\lambda|^k \leqslant \varepsilon^{k-k_1}\lambda^{k_1}$ for $|\lambda| \leqslant \varepsilon$. Hence, in view of the assumptions,

$$|c_k| \leqslant \varepsilon^{k-k_1} \varlimsup_{\xi\to 0} \int\limits_{|\lambda|\leqslant\varepsilon} \frac{\lambda^{k_1}}{\xi}\, dF_\xi(\lambda) \leqslant \varepsilon^{k-k_1} \lim_{\xi\to 0} \int \frac{\lambda^{k_1}}{\xi}\, dF_\xi(\lambda)$$

or

$$|c_k| \leqslant \varepsilon^{k-k_1} c_{k_1}.$$

This implies that $c_k = 0$, since $c_{k_1} \geqslant 0$ and ε can be chosen arbitrarily small.

LEMMA 2. *If k_1 and k_2 are even, $k_2 > k_1$, and $c_{k_1} = 0$, then $c_{k_2} = 0$.*

Proof. Similarly as in the proof of lemma 1, it is assumed that

$\mathrm{E}\Lambda_\xi = 0$. Suppose that $c_{k_2} > 0$. For any $\varepsilon > 0$

$$\mathrm{E}\Lambda_\xi^{k_1} \geqslant \int\limits_{|\lambda| \leqslant \varepsilon} \lambda^{k_1} dF_\xi(\lambda).$$

Since $\lambda^{k_1} > \varepsilon^{k_1-k_2}\lambda^{k_2}$ for $|\lambda| \leqslant \varepsilon$, it follows that

$$\mathrm{E}\Lambda_\xi^{k_1} \geqslant \varepsilon^{k_1-k_2} \int\limits_{|\lambda| \leqslant \varepsilon} \lambda^{k} dF_\xi(\lambda).$$

Hence

$$c_{k_1} \geqslant \varepsilon^{k_1-k_2} \varlimsup_{\xi\to 0} \int\limits_{|\lambda| \leqslant \varepsilon} \frac{\lambda^{k_2}}{\xi} dF_\xi(\lambda).$$

To complete the proof of lemma 2, it is sufficient to note that by assumptions A there exists $\varepsilon_0 > 0$ such that

$$c_{k_2}(\varepsilon_0) = \varlimsup_{\xi\to 0} \int\limits_{|\lambda| \leqslant \varepsilon} \frac{\lambda^{k_2}}{\xi} dF_\xi(\lambda) > 0,$$

since $c_{k_2}(\varepsilon_0) > 0$ implies that $c_{k_1} > 0$.

The proof of theorem 2 can be now completed as follows:

Proof of theorem 2. In view of lemma 2, the relation $c_{k_1} = 0$ implies that $c_{k_2} = 0$ for all even $k_2 > k_1$. From lemma 1 it then follows that if $k_2 > 2$ is even and if $c_{k_2} = 0$ while $c_2 > 0$, then $c_k = 0$ for $2 < k \leqslant k_2$. Thus the first assertion of theorem 2 is proved. To prove the second assertion, note that, in view of lemma 2, the assumption $c_2 = 0$ implies that $c_k = 0$ for all even k. However, this implies that $c_k = 0$ for all $k \geqslant 2$.

4. Examples. We shall present three different test criteria for homogeneity, each of them being an optimal $C(\alpha)$-test criterion with respect to a particular family of probability distributions characterizing the inhomogeneity of the experimental units.

a. First, consider a family of random variables Λ_ξ, $\xi \in (0,1]$, distributed as follows:

For every $\xi \in (0,1]$, let

$$\mathrm{P}(\Lambda_\xi = \xi r) = e^{-\lambda_0/\xi}\frac{(\lambda_0/\xi)^r}{r!}, \tag{10}$$

where $r = 0, 1, \ldots$ Then, whatever $\xi > 0$ is, the random variable X_i has Neyman's A-distribution. It is easily seen that the corresponding family of distributions $\{F_\xi(\lambda)\}$ satisfies conditions A. Moreover, since $\mathrm{E}[\Lambda_\xi - \lambda_0] = 0$, $\mathrm{E}[\Lambda_\xi - \lambda_0]^2 = \lambda_0\xi$ and $\mathrm{E}[\Lambda_\xi - \lambda_0]^k = o(\xi)$ for $k \geqslant 3$, it follows that $c_1 = 0$, $c_2 = \lambda_0$ and $c_k = 0$ for $k \geqslant 3$. Consequently, in view of the corollary, the, optimal $C(\alpha)$-test statistic is given by (9) and represents an optimal $C(\alpha)$-test criterion with respect to the family given by (10). Using the maximum likelihood estimate $\hat{\lambda}_0 = \bar{X}$, it reduces to

$$Z_{1N} = \sqrt{\frac{N}{2}}\left(\frac{S^2}{\bar{X}} - 1\right), \quad \text{where } S^2 = \frac{1}{N}\sum_{i=1}^{N}(X_i - \bar{X})^2.$$

This is the classical dispersion coefficient.

b. Next, consider a family of random variables $\Lambda_\xi = \lambda_0 + R_\xi$, $\xi \in (0, 1]$, where $\lambda_0 > 0$ is an unknown constant and R_ξ stands for the Poisson random variable with expectation equal to ξ. This implies that

$$\mathrm{P}(\Lambda_\xi = \lambda_0 + r) = e^{-\xi}\frac{\xi^r}{r!},$$

where $r = 0, 1, \ldots$ The probability generating function corresponding to the distribution of X_i has the following form:

$$G_{X_i}(u|\xi) = \exp\{\xi(e^{1-u} - 1) + \lambda_0(1-u)(1-\xi)\}.$$

Clearly, the family of distributions $\{F_\xi(\lambda)\}$ of the random variables Λ_ξ, $\xi \in (0, 1]$, satisfies conditions A. However, $c_k = 1$ for $k = 1, 2, \ldots$, and the optimal $C(\alpha)$-test function with respect to the considered family $\{F_\xi(\lambda)\}$ has the following form:

$$Z_{2N} = \frac{\sum_{i=1}^{N}\left[e^{-1}\left((1+\hat{\lambda}_0)/\hat{\lambda}_0\right)^{X_i} - X_i/\hat{\lambda}_0\right]}{\sqrt{N(e^{1/\hat{\lambda}_0} - 1 - 1/\hat{\lambda}_0)}}.$$

With the use of the estimate $\hat{\lambda}_0 = \bar{X}$, this reduces to

$$Z_{2N} = \frac{\sum_{i=1}^{N}\left[e^{-1}\left((1+\bar{X})/\bar{X}\right)^{X_i} - 1\right]}{\sqrt{N(e^{1/\bar{X}} - 1 - 1/\bar{X})}}.$$

This is also an optimal $C(\alpha)$-test function with respect to the family Λ_ξ, $\xi \in (0, 1]$, where Λ_ξ assumes only two values, say λ_0 and $\lambda_0 + 1$, with probabilities $1-\xi$ and ξ, respectively.

c. Finally, consider the family of random variables $\xi \in (0, 1/2]$, where Λ_ξ assumes the values $\lambda_0 - 1$, λ_0 and $\lambda_0 + 1$ with probabilities $\xi/2$, $1-\xi$ and $\xi/2$, respectively. Also in this case assumptions A are satisfied, and easy calculations show that $c_{2k} = 1$ and $c_{2k+1} = 0$ for $k = 1, 2, \ldots$ An application of (2) and the use of $\hat{\lambda}_0 = \bar{X}$ lead to the following optimal

$C(\alpha)$-criterion:

$$Z_{3N} = \frac{\sum_{i=1}^{N}\left[e\left((\bar{X}-1)/\bar{X}\right)^{X_i}+e^{-1}\left((\bar{X}+1)/\bar{X}\right)^{X_i}-2\right]}{\sqrt{2N\left(e^{1/\bar{X}}+e^{-1/\bar{X}}-2\right)}}.$$

5. Comments on the performance of optimal $C(\alpha)$-tests for homogeneity. The weakness of any asymptotic optimal test is connected with the question how large should the number N of observations be that would insure a reasonable approximation to the limiting properties of the tests. Comparing the three optimal $C(\alpha)$-tests presented in this paper, there is no doubt that the number N of observations insuring a reasonable approximation for Z_{1N} is substantially smaller than the number of observations needed to insure the same approximation for Z_{2N} and Z_{3N}. Moreover, in view of theorem 1 and theorem 2, one is willing to conclude that the test statistic Z_{1N} converges to the limiting normal distribution faster than any other $C(\alpha)$-optimal test statistic.

Monte Carlo methods seem to support these conjectures with respect to Z_{1N}, Z_{2N} and Z_{3N}. Under the null hypothesis the empirical distribution of Z_{1N} was always closer to the normal distribution than there were the empirical distributions of Z_{2N} and Z_{3N}. Also, under alternatives, such as considered in examples b and c, the empirical powers of Z_{2N} and Z_{3N} never exceeded the power of Z_{1N}. Only in a few cases, where N was as large as 400, the empirical powers of Z_{2N} and Z_{3N} seemed to coincide with those of Z_{1N} for some particular values of λ_0 and ξ.

Neyman and Scott [2] pointed out that Z_{1N} has a remarkable property of optimality, which they call ***robustness of optimality***, namely that Z_{1N} is an optimal $C(\alpha)$-test criterion with respect to the family of random variables

$$\Lambda_\xi = \lambda_0 e^{\Lambda\sqrt{\xi}}, \qquad \xi \in (0, 1],$$

no matter what the distribution of Λ is, provided that it has a finite support and $\mathrm{E}\Lambda = \mathrm{E}\Lambda^3 = 0$. From the results presented in this paper, it is seen that the classical test of homogeneity for the Poisson distribution has – besides the property mentioned by Neyman and Scott – some further remarkable properties of optimality.

References

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O OPTYMALNYCH TESTACH $C(\alpha)$ DLA WERYFIKACJI HIPOTEZY O JEDNORODNOŚCI ROZKŁADU POISSONA

STRESZCZENIE

Pokazuje się, że dla weryfikacji hipotezy o jednorodności rozkładu Poissona test Z, oparty na klasycznym współczynniku zmienności, jest optymalnym testem $C(\alpha)$ dla szerszej niż to wskazali Neyman i Scott [2] klasy rodzin alternatyw nieskończenie bliskich hipotezie o jednorodności. Podane są zarazem rodziny alternatyw (nieskończenie bliskich hipotezie o jednorodności), dla których test Z nie jest optymalnym testem $C(\alpha)$. Głównym wynikiem jest twierdzenie podające warunki konieczne i dostateczne na to, aby optymalny względem danej rodziny alternatyw test $C(\alpha)$ był identyczny z testem Z. Omówione są też krótko wyniki obliczeń, wykonanych na maszynie cyfrowej, w celu porównania mocy testu Z z dwoma innymi optymalnymi testami $C(\alpha)$. Sugerują one, że dla weryfikacji hipotezy o jednorodności rozkładu Poissona test Z jest lepszy niż inne optymalne testy $C(\alpha)$.