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APPLICATIONES MATHEMATICAE
20,4 (1990), pp. 531–542

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SOME PROPERTIES OF TRUNCATED DISTRIBUTIONS CONNECTED WITH log-CONCAVITY OF DISTRIBUTION FUNCTIONS

Abstract. Important properties such as unimodality, dispersivity, etc. have already been proved with the assumption of log-concavity of distributions (e.g., [7] and [8]). In this work we give results concerning truncated distributions, especially translation and central absolute moments. They may be connected with those about monotone failure rate distributions in the theory of reliability (e.g., [1]). Some of them generalize outcomes of [2] related to the gaussian case. Especially it is shown that the assumption of log-concavity of density functions and distribution functions leads to remarkable monotonicity properties.

1. Introduction. Let F be the left continuous distribution function (d.f.) of some real random variable (r.v.) and

$$S = \{a \in \mathbf{R}\colon\ F(a) > 0\}, \quad d = \inf S.$$

DEFINITION 1. We define the *right truncated distribution* at a point a of S to be the d.f.

$$F_a(x) = \begin{cases} \dfrac{F(x)}{F(a)} & \forall x \leqslant a, \\ 1 & \forall x > a. \end{cases}$$

Such distributions can usefully be developed as probabilistic models in various fields: biology, physics, psychological sciences, etc.

In the following an r.v. with d.f. F_a will be denoted by X_a.

Let

$$M(a) = \mathrm{E}(X_a) = \int_{-\infty}^{+\infty} x\,dF_a(x)$$

and

$$M_n(a) = \int_{-\infty}^{+\infty} |x - M(a)|^n\,dF_a(x), \quad n \in N - \{0\},$$

be the expectation and absolute central moment of order n, respectively.

It is clear that the set $\mathscr{F} = \{F_a\colon a \in S\}$ is stochastically increasing, that is,

$$\forall (a, a') \in S \times S, \quad a < a', \ F_a \geqslant F_{a'}.$$

This property implies that M is an increasing function. One of the purposes of this paper is to give sufficient conditions on F to have an analogous outcome for M_n. So we shall generalize a result of Brascamp and Lieb [2] who proved that for a gaussian distribution M_n is at most equal to the absolute central moment of order n of the non-truncated distribution. Our statements concern essentially right truncated distributions but it is easy to obtain similar results for left and two-sided truncated ones. We shall see that these results are narrowly related to log-concavity of some functions or measures so we recall at first some definitions and properties.

Let P be a probability measure on borelian subsets of $\boldsymbol{R}^n$, and F the associated d.f.

Definition 2. We say that P is *log-concave* if and only if for every non-empty borelian A_0 and A_1 and real α, $0 < \alpha < 1$, we have

$$P\big(\alpha A_0 + (1-\alpha)A_1\big) \geqslant \big(P(A_0)\big)^{\alpha}\big(P(A_1)\big)^{1-\alpha},$$

where

$$\alpha A_0 + (1-\alpha)A_1 = \{\alpha a_0 + (1-\alpha)a_1\colon\ a_0 \in A_0,\ a_1 \in A_1\}.$$

Definition 3. A function $f\colon \boldsymbol{R}^n \to [0, \infty]$ is *log-concave on a convex subset D of* $\boldsymbol{R}^n$ if and only if $\forall \alpha \in]0, 1[$, $\forall (x, y) \in D \times D$,

$$f\big(\alpha x + (1-\alpha)y\big) \geqslant \big(f(x)\big)^{\alpha}\big(f(y)\big)^{1-\alpha}.$$

It is clear that if P is log-concave, then F is also log-concave but the converse is false; however, if P has a Lebesgue density f, P is log-concave if and only if f is log-concave (see [11] and [13]).

In this work we shall frequently use the following simple lemma:

Lemma 1. *Let F be the d.f. of some r.v.; then F is log-concave if and only if $\forall (x, y, z, t) \in \boldsymbol{R}^4$ with $x < y \leqslant z < t$ and $x + t = y + z$ we have*

$$F(y)F(z) \geqslant F(x)F(t). \tag{1}$$

Proof. Assume first that F is log-concave; then there exists a real number α, $0 < \alpha < 1$, such that

$$y = \alpha x + (1-\alpha)t \quad \text{and} \quad z = x + t - y = (1-\alpha)x + \alpha t.$$

Since F is log-concave, we have

$$F(y) \geqslant F^{\alpha}(x)F^{1-\alpha}(t) \quad \text{and} \quad F(z) \geqslant F^{1-\alpha}(x)F^{\alpha}(t),$$

so

$$F(y)F(z) \geqslant F(x)F(t),$$

that is, (1) holds true.

Conversely, suppose that (1) is true. If $F(x) = 0$, then obviously

$$F(\alpha x+(1-\alpha)y) \geqslant (F(x))^{\alpha}(F(y))^{1-\alpha}, \tag{2}$$

so we may assume that $(x, y, z, t) \in S^4$. It is easy to see that (1) implies that F is continuous on S. Taking first α of the form $\alpha = 2^{-n}k$, $n \in N$, $k \in \{1, \ldots, 2^n - 1\}$, we can prove by induction that (2) is true. The fact that the set of such α's is dense in $]0, 1[$ and that F is continuous on S implies that F satisfies (2) for every α in $]0, 1[$, that is F is log-concave.

2. Truncation and translation: a property of the expectation. For $(\theta, x) \in \boldsymbol{R}^2$ let

$${}^{\theta}F(x) = F(x-\theta).$$

The family $\mathscr{F} = \{{}^{\theta}F\colon \theta \in \boldsymbol{R}\}$ is stochastically increasing. Let a be fixed in S and $\Theta =]-\infty, a-d[$. We can establish a necessary and sufficient condition for the family

$$\mathscr{F}_a = \{({}^{\theta}F)_a\colon \theta \in \Theta\}$$

to have the same property.

THEOREM 1. (i) *$\mathscr{F}_a$ is stochastically increasing if and only if F is log-concave.*
(ii) *If F is log-concave, then* $\forall \theta \in [0, a-d[$

$$\mathrm{E}(X_a) \leqslant \mathrm{E}(X+\theta)_a \quad \textit{and} \quad \mathrm{E}(X_a) \leqslant \mathrm{E}(X_{a+\theta}) \leqslant \mathrm{E}(X_a)+\theta.$$

Proof. (i) Let $\theta < \theta'$, $x \in \boldsymbol{R}$. We have

$$\forall x \geqslant a,\ ({}^{\theta}F)_a(x) = ({}^{\theta'}F)_a(x) = 1,$$

$$\forall x < a,\ ({}^{\theta}F)_a(x) \geqslant ({}^{\theta'}F)_a(x)$$

$$\Leftrightarrow \frac{F(x-\theta)}{F(a-\theta)} \geqslant \frac{F(x-\theta')}{F(a-\theta')} \Leftrightarrow F \text{ is log-concave by Lemma 1.}$$

(ii) The first relation is a simple consequence of (i).
$\mathrm{E}(X_a) \leqslant \mathrm{E}(X_{a+\theta})$ because $F_{a+\theta} \leqslant F_a$ (even if F is not log-concave).
Integrating by parts we obtain

$$\mathrm{E}(X_a) = a - \frac{1}{F(a)} \int_{-\infty}^{a} F(x)dx$$

and

$$\mathrm{E}(X_{a+\theta}) - \mathrm{E}(X_a) = \theta - \int_{-\infty}^{a} \left(\frac{F(x+\theta)}{F(a+\theta)} - \frac{F(x)}{F(a)}\right)dx \leqslant \theta$$

if F is log-concave, i.e., the second part of (ii) holds true.

Remarks. 1. For a negative exponential distribution, right truncation reduces to translation.

2. We cannot deduce any general relation between $E((X+\theta)_a)$ and $E(X_{a+\theta})$.

For example, if we take for F the d.f. Φ of the gaussian distribution $\mathcal{N}(0;\ 1)$, we have

$$E(X_a) = -\frac{\Phi'(a)}{\Phi(a)},$$

$$E((X+\theta)_a) - E(X_{a+\theta}) = \theta + \frac{\Phi'(a+\theta)}{\Phi(a+\theta)} - \frac{\Phi'(a-\theta)}{\Phi(a-\theta)};$$

the sign of this expression depends on a (e.g., numerical calculations for $\theta = 2$ lead to the values -1.368 if $a = -2$ and $+1.202$ if $a = +2$).

The gaussian case is interesting: we shall prove later (Section 4) that the mapping $a \to E(X_a)$ is concave. We can also prove that

$$\lim_{a \to -\infty} (a - E(X_a)) = 0.$$

In fact (see [12]), we have: $\forall a < -1,\ \exists \lambda \in]0,\ 1[$ such that

$$\Phi(a) = -\frac{\Phi'(a)}{a(1+\lambda/a^2)},$$

$$E(X_a) = -\frac{\Phi'(a)}{\Phi(a)} = a\left(1+\frac{\lambda}{a^2}\right).$$

Nevertheless, we may compare $E((X+\theta)_a)$ and $E(X_a)$ in the case of a log-concave density. This result appears as a corollary to a more general one connected with the Fortet–Mourier distance between two distributions.

Definition 4. Let P and Q be two real probability measures with respective d.f.'s F and G and finite expectation. The *Fortet–Mourier distance* between P and Q is defined to be the number

$$d(P,\ Q) = \int_{-\infty}^{+\infty} |F(x) - G(x)|\, dx$$

(also denoted by $d(F,\ G)$).

Remark. This distance has been defined and widely studied especially by Dudley [3] and [4]; for a definition and properties of this distance in more general spaces, see [14].

Let f and φ be two functions $\boldsymbol{R} \to \boldsymbol{R}^+$ such that for every real θ the product $\varphi(t) f(t-\theta)$ is Lebesgue-integrable and consider the function $H\colon \boldsymbol{R} \times \boldsymbol{R} \to \boldsymbol{R}$ defined by

$$H(\theta,\ x) = \frac{\int_{-\infty}^{x} \varphi(t) f(t-\theta)\, dt}{\int_{-\infty}^{+\infty} \varphi(t) f(t-\theta)\, dt}.$$

Note that if $\varphi = 1_{]-\infty,a[}$ and f is the density of a probability measure, then $H(\theta, \cdot)$ is the d.f. of $(X+\theta)_a$.

THEOREM 2. *If f and φ are log-concave, then*

$$d(H(0,\cdot), H(\theta, \cdot)) \leqslant |\theta|.$$

Proof. We may suppose that f and φ are differentiable. The general case may be reduced to this one: convolution with log-concave functions can regularize them.

Let

$$\Psi(\theta) = d(H(0, \cdot), H(\theta, \cdot)) = \int_{-\infty}^{+\infty} |H(0, x)-H(\theta, x)|dx,$$

$$Z(\theta) = \int_{-\infty}^{+\infty} \varphi(t)f(t-\theta)dt.$$

First, suppose that $\theta \geqslant 0$. We shall prove that

$$H(0, \cdot)-H(\theta, \cdot) \geqslant 0 \quad \text{and} \quad \Psi'(\theta) \leqslant 1.$$

As $\Psi(0) = 0$, we conclude that $\Psi(\theta) \leqslant \theta$. We have

$$\frac{\partial}{\partial\theta}H(\theta, x) = \frac{1}{Z^2(\theta)}\{-Z(\theta) \int_{-\infty}^{x} \varphi(t)f'(t-\theta)dt + \int_{-\infty}^{x} \varphi(t)f(t-\theta)dt \int_{-\infty}^{+\infty} \varphi(t)f'(t-\theta)dt\}.$$

The numerator of this expression may be decomposed as the sum I_1+I_2, where

$$I_1 = \int_{-\infty}^{x} \varphi(u)[\int_{-\infty}^{x} \varphi(t)(f(t-\theta)f'(u-\theta)-f(u-\theta)f'(t-\theta))dt]du,$$

$$I_2 = \int_{x}^{+\infty} \varphi(u)[\int_{-\infty}^{x} \varphi(t)(f(t-\theta)f'(u-\theta)-f(u-\theta)f'(t-\theta))dt]du.$$

An easy calculation gives $I_1 = 0$ and

$$I_2 = \int_{x}^{+\infty} \int_{-\infty}^{x} \varphi(u)\varphi(t)[f'(u-\theta)f(t-\theta)-f(u-\theta)f'(t-\theta)]dtdu \leqslant 0$$

since f is log-concave. Hence

$$\frac{\partial}{\partial\theta}H(\theta, \cdot) \leqslant 0 \quad \text{and} \quad H(0, \cdot)-H(\theta, \cdot) \geqslant 0.$$

To prove that $\Psi'(\theta) \leqslant 1$ we integrate by parts:

$$\int_{x}^{+\infty} \varphi(u)f'(u-\theta)du = -\varphi(x)f(x-\theta)- \int_{x}^{+\infty} \varphi'(u)f(u-\theta)du$$

and

$$\int_{-\infty}^{x} \varphi(t) f'(t-\theta)dt = \varphi(x) f(x-\theta) - \int_{-\infty}^{x} \varphi'(t) f(t-\theta)dt;$$

then

$$\begin{aligned} I_2 &= -\varphi(x) f(x-\theta) \int_{-\infty}^{+\infty} \varphi(t) f(t-\theta)dt \\ &\quad + \int_{x}^{+\infty} \Big[\int_{-\infty}^{x} f(u-\theta) f(t-\theta)\big(\varphi(u)\varphi'(t) - \varphi(t)\varphi'(u)\big)dt \Big] du \\ &\geqslant -\varphi(x) f(x-\theta) \int_{-\infty}^{+\infty} \varphi(t) f(t-\theta)dt \end{aligned}$$

since φ is log-concave. Hence

$$\begin{aligned} \Psi'(\theta) &= \int_{-\infty}^{+\infty} \left(-\frac{\partial}{\partial\theta} H(\theta, x)dx \right) \\ &\leqslant \frac{1}{Z^2(\theta)} \int_{-\infty}^{+\infty} \varphi(x) f(x-\theta)dx \int_{-\infty}^{+\infty} \varphi(t) f(t-\theta)dt \leqslant 1, \end{aligned}$$

$$\Psi(\theta) \leqslant \theta \quad \text{for every } \theta \geqslant 0.$$

For $\theta < 0$ the first part of our proof remains valid:

$$\frac{\partial}{\partial\theta} H(\theta, \cdot) \leqslant 0 \quad \text{and} \quad H(0, \cdot) - H(\theta, \cdot) \leqslant 0,$$

$$\begin{aligned} \Psi'(\theta) &= \int_{-\infty}^{+\infty} \frac{\partial}{\partial\theta} H(\theta, x)dx \\ &\geqslant -\frac{1}{Z^2(\theta)} \int_{-\infty}^{+\infty} \varphi(x) f(x-\theta)dx \int_{-\infty}^{+\infty} \varphi(t) f(t-\theta)dt. \end{aligned}$$

Then $0 \geqslant \Psi'(\theta) \geqslant -1$ and $\Psi(\theta) \leqslant |\theta|$.

COROLLARY 1. *Let X be an r.v. with d.f. F, log-concave density and finite expectation. Then* $\forall a \in S$, $\forall \theta \in [0, a-d[$,

$$\mathrm{E}\big((X+\theta)_a\big) \leqslant \mathrm{E}(X_a) + \theta.$$

Proof. Take $\varphi = 1_{]-\infty, a[}$ in Theorem 2. It is easy to see that

$$\mathrm{E}\big((X+\theta)_a\big) - \mathrm{E}(X_a) = \int_{-\infty}^{a} \left| \frac{F(x)}{F(a)} - \frac{F(x-\theta)}{F(a-\theta)} \right| dx = d({}^0F_a, {}^\theta F_a).$$

Hence, with the hypothesis of log-concavity of the density f (which implies log-concavity of F), we have the following relations as consequences of Theorems 1 and 2: $\forall \theta \in [0, a-d[$,

$$\mathrm{E}(X_a) \leqslant \mathrm{E}\big((X+\theta)_a\big) \leqslant \mathrm{E}(X_a) + \theta,$$

$$\mathrm{E}(X_a) \leqslant \mathrm{E}(X_{a+\theta}) \leqslant \mathrm{E}(X_a) + \theta.$$

3. Truncation and dispersive ordering.

DEFINITION 5. Let X and Y be two r.v.'s with respective d.f.'s F_X and F_Y. We say that X is *less than* Y for dispersive ordering and we write

$$X \overset{\text{disp}}{\leqslant} Y \quad \text{or} \quad F_X \overset{\text{disp}}{\leqslant} F_Y$$

if and only if $\forall(\alpha, \beta)$, $0 < \alpha \leqslant \beta < 1$,

$$F_X^{-1}(\beta) - F_X^{-1}(\alpha) \leqslant F_Y^{-1}(\beta) - F_Y^{-1}(\alpha),$$

where $F^{-1}(u) = \inf\{t\colon F(t) \geqslant u\}$, $u \in]0, 1[$.

This partial ordering, defined by Lewis and Thomson [7], has been studied especially by Bartoszewicz [1], Lynch et al. [8], Saunders and Moran [15], Shaked [16]. Following Karlin and Novikov [5], Shaked stated that if $X \overset{\text{disp}}{\leqslant} Y$ and if Ψ is a convex function from $\boldsymbol{R}$ to $\boldsymbol{R}$, then

$$\mathrm{E}(\Psi(X - \mathrm{E}(X))) \leqslant \mathrm{E}(\Psi(Y - \mathrm{E}(Y)))$$

(if these expectations exist).

THEOREM 3. *If F is log-concave, then the family $\mathscr{F} = \{F_a\colon a \in S\}$ is increasing for dispersive ordering.*

COROLLARY 2. *Let F be a log-concave d.f. and Ψ a convex function from $\boldsymbol{R}$ to $\boldsymbol{R}$. The mapping*

$$a \to \int_{-\infty}^{+\infty} \Psi(x - M(a))\, dF_a(x)$$

defined on S is increasing. This is, in particular, the case for M_n, the central absolute moment of order n.

The proof of this theorem and Corollary 2 can be found in [10].

We can also deduce a property of two-sided truncations when F has a log-concave density.

By left and right truncation of F at a and b, respectively, such that $F(b) > F(a)$ we define the d.f. $F_{a,b}$ by

$$F_{a,b}(x) = \begin{cases} 0, & x \in]-\infty, a], \\ \dfrac{F(x) - F(a)}{F(b) - F(a)}, & x \in]a, b], \\ 1, & x \in]b, +\infty[. \end{cases}$$

Let

$$M(a, b) = \int_{-\infty}^{+\infty} x\, dF_{a,b}(x) \quad \text{and} \quad W(a, b) = \int_{-\infty}^{+\infty} \Psi(x - M(a, b))\, dF_{a,b}(x),$$

where Ψ is a convex function from $\boldsymbol{R}$ to $\boldsymbol{R}$.

COROLLARY 3. *If* $]a_2, b_2[\subset]a_1, b_1[$ *and if the density* f *is log-concave, then*

(i) $$F_{a_2,b_2} \overset{\text{disp}}{\leqslant} F_{a_1,b_1},$$

(ii) $$W(a_2, b_2) \leqslant W(a_1, b_1).$$

Proof. Since f is log-concave, F and $1-F$ are also log-concave. Let G_{a_1} and G_{a_2} be the left truncated d.f.'s of F at a_1 and a_2, respectively. A similar proof to that of Theorem 3 permits us to establish that

$$G_{a_2} \overset{\text{disp}}{\leqslant} G_{a_1};$$

f being log-concave, so are G_{a_1} and G_{a_2}. We can then apply Theorem 3 to the right truncation of G_{a_1} at b_1 and G_{a_2} at b_2, that is F_{a_1,b_1} and F_{a_2,b_2}, to obtain

$$F_{a_2,b_2} \overset{\text{disp}}{\leqslant} F_{a_1,b_2} \overset{\text{disp}}{\leqslant} F_{a_1,b_1}$$

and (ii).

4. The special case of the variance of a truncated distribution. Theorem 3 and Corollary 2 give a sufficient condition for M_n to be increasing, $n \geqslant 1$. For $n = 2$ we establish now a necessary and sufficient condition for this result in the case of continuous F. We also give a sufficient condition for a lattice distribution, this case not being included in Section 3.

Suppose that F is the d.f. of some r.v. with finite variance. Let $F_{[1]}$ and $F_{[2]}$ be the first and the second primitives of F vanishing at $-\infty$ (these primitives exist with the hypothesis on F). Integrating by parts one can easily establish that

$$\int_{-\infty}^{a} x dF(x) = aF(a) - F_{[1]}(a),$$

$$\int_{-\infty}^{a} x^2 dF(x) = 2F_{[2]}(a) - 2aF_{[1]}(a) + a^2 F(a),$$

so that

(3) $$M(a) = a - \frac{F_{[1]}(a)}{F(a)},$$

(4) $$M_2(a) = \frac{2F_{[2]}(a)}{F(a)} - \frac{F_{[1]}^2(a)}{F^2(a)}.$$

4.1. *The continuous case.* Suppose here that F is continuous. The following three lemmas are useful to state Theorem 4 and applications.

LEMMA 2. *Let Y be an r.v. with values in an interval $[a, b]$ and continuous d.f. G. Let φ be a real function of class $\mathscr{C}^1$ on $[G(a), G(b)]$. Then $\forall(\alpha, \beta)\in[a, b]\times[a, b]$, $\alpha \leqslant \beta$,*

$$\int_{\alpha}^{\beta} \varphi'(G(y))dG(y) = \varphi(G(\beta))-\varphi(G(\alpha)).$$

The proof of this lemma is obvious.

LEMMA. 3 *If F is continuous, then*

$$M_2(b)-M_2(a) = 2\int_{a}^{b}(F_{[1]}^2(x)-F(x)F_{[2]}(x))F^{-3}(x)dF(x)$$

whenever $b \geqslant a > d$.

Proof. It suffices to apply Lemma 2 to the functions $G(y) = F(y)$, $\varphi(y) = y^{-2}$ and $\varphi(y) = y^{-1}$, then to compare the integral on the right-hand side to the expression $M_2(b)-M_2(a)$ obtained by (4).

LEMMA 4. *Let $\tilde{X} = (X_1, \ldots, X_n)$ be a sample of size n of an r.v. X with log-concave density and Ψ a convex function from $\boldsymbol{R}^n$ to $\boldsymbol{R}$; then the r.v. $\Psi(\tilde{X})$ has a log-concave d.f. F.*

Proof. We must prove that $\forall\alpha\in]0, 1[$, $\forall(x, y)\in\boldsymbol{R}^2$,

$$F(\alpha x+(1-\alpha)y) \geqslant (F(x))^{\alpha}(F(y))^{1-\alpha}.$$

Following the definition of F we have

$$F(\alpha x+(1-\alpha)y) = P(\tilde{X}\in\Psi^{-1}(]-\infty, \alpha x+(1-\alpha)y[)).$$

The convexity of Ψ implies that

$$\Psi^{-1}(]-\infty, \alpha x+(1-\alpha)y[) \supset \alpha\Psi^{-1}(]-\infty, x[)+(1-\alpha)\Psi^{-1}(]-\infty, y[).$$

Hence

$$P_{\tilde{X}}(]-\infty, \alpha x+(1-\alpha)y[) \geqslant P_{\tilde{X}}(\alpha\Psi^{-1}(]-\infty, x[)+(1-\alpha)\Psi^{-1}(]-\infty, y[)).$$

The density of X being log-concave, so is the probability $P_{\tilde{X}}$ (see [13]). Taking the borelian subsets $A_0 =]-\infty, x[$ and $A_1 =]-\infty, y[$ in the definition of log-concavity we obtain the result.

THEOREM 4. *Assume that F is continuous. M_2 is increasing if and only if $F_{[2]}$ is log-concave on $I = \{0 < F < 1\}$. M_2 is decreasing if and only if $F_{[2]}$ is log-convex on I.*

Proof. Let $H = F_{[1]}^2-FF_{[2]}$. The condition "$F_{[2]}$ log-concave on I" is equivalent to "$H(x) \geqslant 0$ for every $x\in I$". If $H \geqslant 0$ on I, then M_2 is increasing on I, and also on S by Lemma 3.

Conversely, assume that M_2 is increasing and that there exists x_0 in I such that $H(x_0) < 0$.

Being continuous, H is not greater than a strictly negative number on an interval $]x_0-\varepsilon,\ x_0+\varepsilon[\ \subset I$. On this interval F must be constant (else Lemma 3 would lead to $M_2(x_0+\varepsilon) < M_2(x_0-\varepsilon)$, contradicting our hypothesis). Let $]\alpha,\ \beta[\ \subset I$ be the largest open interval containing x_0 on which F is constant.

We have

$$F(x_0) = F(\alpha), \qquad F_{[1]}(x_0) = F_{[1]}(\alpha)+(x_0-\alpha)F(\alpha),$$

$$F_{[2]}(x_0) = \frac{(x_0-\alpha)^2}{2}F^2(\alpha)+(x_0-\alpha)F_{[1]}(\alpha)+F_{[2]}(\alpha).$$

Then

$$H(\alpha) = H(x_0)-\frac{(x_0-\alpha)^2}{2}F^2(\alpha)-(x_0-\alpha)F_{[1]}(\alpha)F(\alpha) < 0.$$

Accordingly, there exists an interval $[\alpha-\eta,\ \alpha]$ on which H is strictly negative and F is not constant; hence, by Lemma 3, $M_2(\alpha-\eta) > M_2(\alpha)$, contradicting our hypothesis "M_2 increasing".

The second part of Theorem 4 can be proved in an analogous way.

COROLLARY 4. *Let F be a continuous d.f.*

(i) *If F or $F_{[1]}$ is log-concave, then M_2 is increasing.*

(ii) *If F has a log-concave density, then M_2 is increasing.*

Proof. It suffices to apply a well-known result (see, e.g., [11]): if a real function is log-concave, then its primitive which vanishes at $-\infty$ is also log-concave. Thus we find again a result of Section 3.

EXAMPLES. (a) As a special case we can give the gaussian one, the density being obviously log-concave (but mixtures of gaussian distributions do not have this property, in general). In this case we have

$$M_2(a) = 1-a\frac{\Phi'(a)}{\Phi(a)}-\frac{\Phi'^2(a)}{\Phi^2(a)} = 1-M'(a).$$

M_2 being increasing, we find that M is concave.

(b) Consider now two independent r.v.'s X and Y with respective densities f and g both assumed to be log-concave; then the density of $X+Y$ is also log-concave and Theorems 3 and 4 apply for this sum.

(c) Let $\tilde{X} = (X_1,\dots,X_n)$ be a gaussian vector with independent components $\mathcal{N}(m_i;\ \sigma_i)$ and let Ψ be the mapping from $\boldsymbol{R}^n$ to $\boldsymbol{R}$ defined by

$$\Psi(x_1,\dots,x_n) = \sum_{i=1}^{n} x_i^2.$$

Following lemma 4 the r.v. $Y = \Psi(\tilde{X})$ has a log-concave d.f. Thus, as a special case we find a χ^2-distribution with n degrees of freedom (we may notice that for such a distribution the density is log-concave for $n \geqslant 2$ and log-convex for $n = 1$, but the d.f. is log-concave whatever be n).

(d) *Statistical application.* Let $\tilde{X} = (X_1, \ldots, X_n)$ be a sample of size n of an r.v. X with log-concave density f. Applying once again Lemma 4 to various convex functions Ψ we can see that empirical mean, empirical variance, maximum, range, etc. are such that their central absolute moments are increasing functions of the truncation point a.

This property of the maximum X_n^* is interesting for the point-estimator of a. If f is log-concave, then the expectation of the quadratic difference between a and X_n^* is an increasing function of a. In fact, for the d.f. of X_n^* we have

$$(F_a)^n = (F^n)_a \quad \text{and} \quad \mathrm{E}(X_n^* - a)^2 = \mathrm{var}(X_n^*) + (a - \mathrm{E}(X_n^*))^2.$$

Our preceding results prove that $\mathrm{var}(X_n^*)$ is increasing with a since F^n is log-concave with F; $h(a) = a - \mathrm{E}(X_n^*)$ is positive and, following (3),

$$h(a) = \frac{(F^n)_{[1]}(a)}{F^n(a)} \quad \text{with } (F^n)_{[1]}(a) = \int_{-\infty}^{a} F^n(x)\,dx.$$

F^n being log-concave, $(F^n)_{[1]}$ is also log-concave; therefore h and h^2 are increasing. The result about $\mathrm{E}(X_n^* - a)^2$ follows.

4.2. *The case of lattice distributions.* Assume now that the support of the distribution is an interval I of $\boldsymbol{Z}$ (or a lattice of real numbers). Let $\{(k, p_k)\colon k \in \boldsymbol{Z}\}$ and F be the distribution and d.f., respectively. With a hypothesis similar to log-concavity of F one can prove the increasing property of M_2. However, the proof is more technical than that given in the continuous case; it can be found in [9].

THEOREM 5. *Assume that*

$$\forall k \in S, \; F^2(k) \geqslant F(k-1)F(k+1). \tag{5}$$

Then M_2 is increasing.

COROLLARY 5. *Assume that*

$$\forall k \in S, \; p_k^2 \geqslant p_{k-1}p_{k+1}. \tag{6}$$

Then M_2 is increasing.

This corollary follows from the fact that (6) implies (5). Note that relations (6) are stable by convolution (see [6]).

As examples satisfying the assumption of Theorem 5 we have Bernoulli, binomial, Poisson, negative geometric distributions (in this latter case M_2 is constant, as it is also true in the continuous case for negative exponential distributions).

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Received on 1988.10.12