ZASTOSOWANIA MATEMATYKI
APPLICATIONES MATHEMATICAE
XIII, 2 (1972)

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DUAL CONTROL FOR DISTURBANCES WITH POISSON DISTRIBUTION

1. Introduction. The purpose of this paper is to show how filtering theory based on the Bayesian approach may be used to solve the problem of optimal controlling a linear system to which an additive Poissonian input is applied. Bellman's dynamic programming was used to find the analytical solution of the feedback control law.

2. Statement of the problem. Let us define the discrete linear system with additive disturbances and exact observations

$$x_{n+1} = x_n + u_n + v_n, \quad x_0 = c,$$

where x_n is the state variable, u_n is the control, and $v_1, v_2, \ldots$ is the sequence of independent random variables with the same known distribution dependent on some parameters for which an *a priori* distribution is given.

Given the initial state c and the *a priori* distribution of unknown parameters, there must be chosen a control u_n, $n = 0, 1, \ldots, N-1$, based on all available data $X_n = (x_0, x_1, \ldots, x_n)$ and $U_{n-1} = (u_0, u_1, \ldots, u_{n-1})$, such that

$$\varkappa_n = \mathrm{E}\Big[\sum_{i=n}^{N-1} (x_i^2 + k u_i^2) \,|\, X_n, U_{n-1}\Big] \tag{1}$$

reaches its minimum.

3. Disturbances with Poisson distribution. Let us suppose that the random variables v_n are distributed according to the Poissonian law

$$\mathrm{P}(v_n = ai \,|\, \lambda) = \frac{\lambda^i}{i!} e^{-\lambda}, \quad i = 0, 1, 2, \ldots, \tag{2}$$

where the *a priori* distribution of parameter λ is given by the gamma density

$$g(\lambda \,|\, a, r) = \frac{a^r}{\Gamma(r)} \lambda^{r-1} e^{-a\lambda}. \tag{3}$$

When such an *a priori* distribution is assigned to λ, the object of filtering is to produce an *a posteriori* density for λ after any new observation of x. We change the control after obtaining new data.

The probability of measuring x, given λ, is

$$p\big(x_1(i)\,|\,\lambda\big) \overset{\text{def}}{=} \mathrm{P}(x_1 = x_0 + u_0 + ai\,|\,\lambda) = \mathrm{P}(v_0 = ai\,|\,\lambda) = \frac{\lambda^i}{i!}e^{-\lambda}.$$

Having observed x_1, the *a posteriori* density $h(\lambda|X_1)$ may be calculated with the help of the Bayes rule

$$h(\lambda|X_1) = \frac{p\big(x_1(i)\,|\,\lambda\big)g(\lambda\,|\,a, r)}{\int\limits_0^\infty p\big(x_1(i)\,|\,\lambda\big)g(\lambda\,|\,a, r)\,d\lambda} = \frac{(1+a)^{r+i}}{\Gamma(r+i)}\lambda^{r+i}e^{-(1+a)\lambda}$$

$$= \frac{(1+a)^{r+v_0/a}}{\Gamma(r+v_0/a)}\lambda^{r+v_0/a}e^{-(1+a)\lambda} = g(\lambda\,|\,a_1, r_1),$$

where

(4) $$a_1 = 1+a \quad \text{and} \quad r_1 = r+v_0/a.$$

Similarly, after x_n is measured, the *a posteriori* density of λ is

$$h(\lambda|X_n) = \frac{p\big(x_n(i)\,|\,\lambda\big)g(\lambda\,|\,a_{n-1}, r_{n-1})}{\int\limits_0^\infty p\big(x_n(i)\,|\,\lambda\big)g(\lambda\,|\,a_{n-1}, r_{n-1})\,d\lambda}$$

$$= \frac{(1+a_{n-1})^{r_{n-1}+v_{n-1}/a}}{\Gamma(r_{n+1}+v_{n-1}/a)}\lambda^{r_{n-1}+v_{n-1}/a-1}e^{-(a_{n-1}+1)\lambda} = g(\lambda\,|\,a_n, r_n),$$

where

(5) $$a_n = 1+a_{n-1} \quad \text{and} \quad r_n = r_{n-1}+v_{n-1}/a.$$

Given X_n, the conditional distribution of v_n can be calculated as follows:

$$\mathrm{P}(v_n = ai\,|\,X_n) = \int\limits_0^\infty \mathrm{P}(v_n = ai\,|\,\lambda)g(\lambda\,|\,a_n, r_n)\,d\lambda$$

$$= \frac{a_n^{r_n}}{i!\,\Gamma(r_n)}\int\limits_0^\infty \lambda^{r_n+i-1}e^{-(a_n+1)\lambda}\,d\lambda = \frac{1}{i!}\frac{a_n^{r_n}}{(1+a_n)^{r_n+i}}\frac{\Gamma(r_n+i)}{\Gamma(r_n)}.$$

Then the expected value of v_n for given X_n can be determined:

$$\mathrm{E}(v_n\,|\,X_n) = \frac{a_n^{r_n}}{\Gamma(r_n)}\sum_{i=0}^{\infty} ai\frac{\Gamma(r_n+i)}{i!(1+a_n)^{r_n+i}}$$

$$= \frac{aa_n^{r_n}}{\Gamma(r_n)}\sum_{i=1}^{\infty}\frac{\Gamma(r_n+1+i-1)}{(i-1)!(1+a_n)^{r_n+1+i-1}} = \frac{aa_n^{r_n}}{(r_n)}\sum_{i=0}^{\infty}\frac{\Gamma(r_n+1+i)}{i!(1+a_n)^{r_n+1+i}}.$$

But

$$\sum_{i=0}^{\infty} \frac{\Gamma(r+i)}{i!(1+a)^{r+i}} = \frac{\Gamma(r)}{a^r},$$

and thus

$$\mathrm{E}(v_n | X_n) = ar_n/a_n. \tag{6}$$

Similarly,

$$\mathrm{E}\big(v_n(v_n-a) | X_n\big) = a^2 r_n(r_n+1)/a_n^2. \tag{7}$$

Formulae (6) and (7) may be used to calculate

$$\begin{aligned}
\mathrm{E}(x_{n+1} | X_n) &= \mathrm{E}(x_n+u_n+v_n | X_n) = x_n+u_n+ar_n/a_n, \\
\mathrm{E}(r_{n+1} | X_n) &= \mathrm{E}(r_n+v_n/a | X_n) = r_n+r_n/a_n, \\
\mathrm{E}(x_{n+1}^2 | X_n) &= \mathrm{E}\big((x_n+u_n+v_n)^2 | X_n\big) \\
&= (x_n+u_n)^2+2(x_n+u_n)\mathrm{E}(v_n | X_n)+\mathrm{E}(v_n^2 | X_n) \\
&= (x_n+u_n)^2+2(x_n+u_n)ar_n/a_n+a^2 r_n(r_n+a_n+1)/a_n^2, \\
\mathrm{E}(r_{n+1}^2 | X_n) &= \mathrm{E}\big((r_n+v_n/a)^2 | X_n\big) = r_n^2+2r_n^2/a_n+r_n(r_n+a_n+1)/a_n^2, \\
\mathrm{E}(x_{n+1}r_{n+1} | X_n) &= \mathrm{E}\big((r_n+v_n/a)(x_n+u_n+v_n) | X_n\big) \\
&= r_n(x_n+u_n)+(x_n+u_n+ar_n)r_n/a_n+ar_n(r_n+a_n+1)/a_n^2.
\end{aligned} \tag{8}$$

To find the optimal control we write

$$V_n = \min_{u_n,\dots,u_{N-1}} \varkappa_n = \min_{u_n,\dots,u_{N-1}} \mathrm{E}\Big[\sum_{i=n}^{N-1} (x_i^2+ku_i^2) | X_n, U_{n-1}\Big].$$

Then

$$V_{N-1} = \min_{u_{N-1}} \varkappa_{N-1} = \min_{u_{N-1}} (x_{N-1}^2+ku_{N-1}^2) = x_{N-1}^2$$

and the optimal $u_{N-1} = 0$.

Moreover, by application of Bellman's dynamic programming optimality principle, we obtain

$$\begin{aligned}
V_n &= \min_{u_n,\dots,u_{N-1}} \mathrm{E}\Big[\sum_{i=n}^{N-1} (x_i^2+ku_i^2) | X_n, U_{n-1}\Big] \\
&= \min_{u_n}\Big[(x_n^2+ku_n^2)+\min_{u_{n+1},\dots,u_{N-1}} \mathrm{E}\Big[\sum_{i=n+1}^{N-1} (x_i^2+ku_i^2) | X_n, U_{n-1}\Big]\Big].
\end{aligned}$$

Since

$$\min_{u_{n+1},\ldots,u_{N-1}} \mathrm{E}\Big[\sum_{i=n+1}^{N-1}(x_i^2+ku_i^2)\,|\,X_n,\,U_{n-1}\Big]$$
$$= \min_{u_{n+1},\ldots,u_{N-1}} \mathrm{E}\Big[\mathrm{E}\Big[\sum_{i=n+1}^{N-1}(x_i^2+ku_i^2)\,|\,X_{n+1},\,U_n\Big]\Big|\,X_n,\,U_{n-1}\Big]$$
$$= \mathrm{E}(V_{n+1}|X_n,\,U_{n-1}),$$

we obtain

$$V_n = \min_{u_n}[x_n^2+ku_n^2+\mathrm{E}(V_{n+1}|X_n,\,U_{n-1})]. \tag{9}$$

But $V_{N-1} = x_{N-1}^2$, and from (8) it follows that $\mathrm{E}(V_{N-1}|X_{N-2},\,U_{N-3})$ is a positively defined square function of $x_{N-2}+u_{N-2}$. Thus the optimal u_{N-2} is linear in x_{N-2}, and V_{N-2} is a positively defined square function of x_{N-2}. By inductive argument, $\mathrm{E}(V_{n+1}|X_n,\,U_{n-1})$ is a positively defined square function of x_n+u_n, optimal u_n is linear in x_n, and V_n is a positively defined square function of x_n. For determining the optimal control this yields the equation

$$2ku_n+\frac{\partial}{\partial u_n}\mathrm{E}(V_{n+1}|X_n,\,U_{n-1}) = 0.$$

But, for given X_n and U_{n-1}, we have linear dependence between x_{n+1} and u_n. Therefrom it follows that

$$2ku_n^*+\mathrm{E}\left[\frac{\partial V_{n+1}}{\partial x_{n+1}}\,|X_n,\,U_{n-1}\right] = 0, \tag{10}$$

where u_n^* is the optimal control.

We will now show that V_n is of the form

$$V_n = A_nx_n^2+2B_nx_nr_n+C_nr_n^2+D_nr_n. \tag{11}$$

For $n = N-1$ it holds and $A_{N-1} = 1$, $B_{N-1} = C_{N-1} = D_{N-1} = 0$. Then, assuming (11) to be true, we get

$$\frac{\partial V_{n+1}}{\partial x_{n+1}} = 2A_{n+1}x_{n+1}+2B_{n+1}r_{n+1},$$

and from (8)

$$\mathrm{E}\left[\frac{\partial V_{n+1}}{\partial x_{n+1}}\,|X_n,\,U_{n-1}\right] = 2A_{n+1}(x_n+u_n^*+ar_n/a_n)+2B_{n+1}\frac{a_n+1}{a_n}r_n. \tag{12}$$

By (10) and (12) we obtain

$$ku_n^* + A_{n+1}(x_n + u_n^* + ar_n/a_n) + B_{n+1}\frac{a_n+1}{a_n}r_n = 0$$

or

(13) $$u_n^* = -\frac{A_{n+1}}{A_{n+1}+k}x_n - \frac{aA_{n+1}+(a_n+1)B_{n+1}}{(A_{n+1}+k)a_n}r_n.$$

Moreover, from (8) and (11)

(14) $$\begin{aligned}\mathrm{E}(V_{n+1}|X_n) &= A_{n+1}\mathrm{E}(x_{n+1}^2|X_n) + 2B_{n+1}\mathrm{E}(x_{n+1}r_{n+1}|X_n) + \\ &\quad + C_{n+1}\mathrm{E}(r_{n+1}^2|X_n) + D_{n+1}\mathrm{E}(r_{n+1}|X_n) \\ &= A_{n+1}[(x_n+u_n^*)^2 + 2(x_n+u_n^*)ar_n/a_n + \\ &\quad + (ar_n + a + a)ar_n/a_n^2] + 2B_{n+1}[(x_n+u_n^*)r_n + \\ &\quad + (x_n+u_n^*+ar_n)r_n/a_n + (ar_n + a + a\,a_n)r_n/a_n^2] + \\ &\quad + C_{n+1}[r_n^2 + 2r_n^2/a_n + (ar_n + a + a\,a_n)r_n/a\,a_n^2] + \\ &\quad + D_{n+1}\frac{a_n+1}{a_n}r_n.\end{aligned}$$

On the other hand, by (9) and (11), the same value may be expressed as follows:

$$\mathrm{E}(V_{n+1}|X_n) = A_n x_n^2 + 2B_n x_n r_n + C_n r_n^2 + D_n r_n - x_n^2 - ku_n^{*n}.$$

By the substitution of (13) instead of u_n^* in (14) and succesive comparison of terms containing $x_n^2, x_n r_n, r_n^2, r_n$, we obtain the recursive formulae

(15) $$A_n = 1 + \frac{kA_{n+1}}{A_{n+1}+k},$$

(16) $$B_n = \frac{k\big(aA_{n+1}+(a_n+1)B_{n+1}\big)}{A_{n+1}+k},$$

$$C_n = \big(a^2A_{n+1} + a(a_n+1)B_{n+1} + (a_n+1)^2C_{n+1}\big)/a_n^2 - \frac{A_n+k}{k^2}B_n^2,$$

$$D_n = (a^2A_{n+1} + 2aB_{n+1} + C_{n+1} + a_nD_{n+1})(1+a_n)/a_n^2$$

with the boundary conditions $A_{N-1} = 1$ and $B_{N-1} = C_{N-1} = D_{N-1} = 0$.

Thus, all the coefficients A_n, B_n, C_n and D_n can be computed recursively and the exact analytical solution for optimal control u_n^*, given by (13), may be obtained. Notice that only A_n and B_n are necessary for optimal control remaining constant, needed for computation of Bayes' risks V_n.

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Received on 16. 4. 1971

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DUALNE STEROWANIE PRZY ZAKŁÓCENIACH O ROZKŁADZIE POISSONA

STRESZCZENIE

Zdefiniujmy dyskretny liniowy układ o addytywnych zakłóceniach i dokładnych obserwacjach

$$x_{n+1} = x_n + u_n + v_n, \quad x_0 = c,$$

gdzie x jest zmienną stanu, u_n – sterowaniem, $v_1, v_2, \ldots$ zaś ciągiem niezależnych zmiennych losowych o rozkładzie Poissona (2). Załóżmy, że rozkład *a priori* parametru λ jest określony za pomocą gęstości rozkładu gamma (3). Dla danego stanu początkowego c i danego rozkładu *a priori* parametru trzeba wyznaczyć sterowanie u_n, zależne od zaobserwowanych wartości $X_n = (x_0, x_1, \ldots, x_n)$ oraz $U_{n-1} = (u_0, u_1, \ldots, u_{n-1})$, tak aby wielkość (1) była możliwie mała.

W pracy udowodniono, że optymalne sterowanie u_n^* określone jest wzorem (13), gdzie stałe a_n i r_n oblicza się z rekurencyjnych wzorów (4) i (5), a stałe A_n i B_n – z wzorów (15), (16) i warunków brzegowych $A_{N-1} = 1$, $B_{N-1} = 0$,

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