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REDSHIFT IN FRIEDMANN'S COSMOLOGICAL MODEL

In a variety of cosmological models, Friedmann's model, due to the chronology and remarkable simplicity, may be called the most classical one. A three-dimensional sphere, with a radius R depending on time, is the space in this model. To determine the dependence of R on the time T we may obtain, from field equations of the general theory of relativity, the differential equation

$$\left(\frac{dR}{dT}\right)^2 = \frac{R_0-R}{R} \tag{1}$$

where R_0 is a constant (the so called maximum radius of the universe). Every solution of this equation (with the exception of the singular ones) is a function of time, first increasing from zero to the maximum value R_0 and then decreasing again to zero. So the appropriate curve consists of two monotonic arcs; on each of them there exists a one-to-one correspondence between the radius and time. Therefore, the radius may be accepted as a measure of time which for computational reasons is very convenient. To avoid the ambiguity due to the described behaviour of the function $R(T)$ we will distinguish, as in colloquial time measure, the values of R a. m. (that is the values before summit) and the values of R p.m. (after the summit).

Metric relations in Friedmann's space depend on time. This requires the specification of the distance. There are many useful and essentially equivalent ways for defining this notion. Here we accept as the distance the time required for light to travel between the given two points. This depends of course on the moment of start of the light ray; so that the defined distance changes in time.

Either classical methods, or those specific for some relativistic models based on redshift, may be used to determine the distance between objects in Friedmann's universum. By the classical methods we understand here such well known astronomical methods as the evaluation of distance either on the measurements of linear and angular sizes of the observed object or on the comparison of absolute and apparent brightness of the object. Of course, both classical methods have to be adapted to Friedmann's model since here the geometry depends on time and it is no longer a plane one. However, classical methods of determining the distance not always give unique results in Friedmann's model. In particular, it may occur that the evaluation of distance on the measurements of the linear and the angular diameters of an object yields as much as four different results. This will happen if light emitted by the object before "the noon" is received by the observer after "the noon". For some values of the ratio of linear and angular diameters of the object we then obtain two different values of the distance and for other values of the ratio four different values of the distance. This follows from a minimax theorem for functions of two variables as it is shown in [3]. The author is quite sure that sometimes in the future such pairs and quadruplets of objects (each of them being in different distance from the observer but having the same ratio of linear and angular diameters) will be discovered. For this observation we have to wait perhaps many many millions of years. It will be probably the longest period in the history of science between the theoretical statement and its empirical verification; this prospect making the author especially satisfied.

Now, we will draw our attention to the determination of distance based on redshift. From the original paper by Friedmann [1] it follows that the ratio of the emitted wave lenght λ and the received wave lenght λ_1 equals to the ratio of the radius R of the universe at the emission time and the radius R_1 of the universe at the reception time.

$$\frac{\lambda}{\lambda_1} = \frac{R}{R_1}. \tag{2}$$

Formula (2) may be generalised (see Heckmann [2]) for the case when the source of light and the observer are moving so that the distance between them changes in time not only due to the changes of geometry but also as an effect of additional factors (the movement). In [2] a special case of movements is discussed in which only the radial velocities of the two objects are taken into account; i.e. the velocities are supposed to have non-zero components only along the geodetic between the source and the observer. Under such restriction the generalised formula (2) takes the form

$$\frac{\lambda}{\lambda_1} = \frac{R\sqrt{1-v_\varphi^2}(1-u_\varphi)}{R_1\sqrt{1-u_\varphi^2}(1-v_\varphi)} \tag{3}$$

where u_φ and v_φ are radial velocities of the source and the observer, respectively.

We turn now to the general case of a movement with arbitrary velocities. It occurs that the redshift in this case depends as well on the radial components as on the moduli of complete velocities

$$\frac{\lambda}{\lambda_1} = \frac{R\sqrt{1-v^2}(1-u_\varphi)}{R_1\sqrt{1-u^2}(1-v_\varphi)}; \tag{4}$$

the other components of velocities do not occur in an explicit form.

Now, let us define a system of spherical coordinates φ, ψ, ϑ on the three-dimensional sphere. This may be done in a solid way so that the points on the sphere will not change their coordinates with the change of the radius of the sphere. As a velocity we understand the ratio of infinitesimal displacement and the corresponding time period. The displacement here is the product of the angular shift and the radius of the universe $R(T)$ at a given moment of time.

From the pole A with the coordinates $(0, 0, 0)$ a light signal is emitted at the moment T towards the point B with the coordinates $(\varphi_1, 0, 0)$. This signal will be received there at the moment T_1. Simultanously, the source of light is moving with a velocity u so that at the moment $T+\Delta T$ it reaches the point $C(\Delta\varphi, \Delta\psi, 0)$. Then, the light signal is emitted towards the point $D(\varphi_1+\Delta\varphi_1, \Delta\psi_1, \Delta\vartheta_1)$ which reaches its destination at the moment $T_1+\Delta T_1$. The observer is also moving with the velocity v so that being in the point B at the moment T_1 it will arrive to D at the moment $T_1+\Delta T_1$. Here some of the coordinates of the points A, B, C, B were chosen to be equal to zero this, however, brings no restriction on their relative configuration.

The angular distance a between two points $(\varphi_1, \psi_1, \vartheta_1)$ and $(\varphi_2, \psi_2, \vartheta_2)$ on a three-dimensional sphere may be found from the equation

$$\cos a = \cos\varphi_1\cos\varphi_2+\sin\varphi_1\sin\varphi_2[\cos\psi_1\cos\psi_2+ \\ +\sin\psi_1\sin\psi_2\cos(\vartheta_1-\vartheta_2)]. \tag{5}$$

The velocity u of the moving object is equal to

$$u = R(T)\,\frac{da}{dT}$$

and the angular shift of the object moving with the velocity u while the radius of the universe has changed from the value R_1 to R_2, taking into account equation (1), may be expressed in the form

$$a = u\arccos\frac{2R-R_0}{R_0}\Bigg|_{R_1}^{R_2}. \tag{6}$$

Starting from (5) and (6) we may come, after some tedious calculations, to the following condition for the observer and the light signal to meet in the point D at the moment $T_1+\Delta T_1$

$$(7)\quad 2\Delta R_1 = \frac{2\sqrt{R_1(R_0-R_1)}\,\Delta R+(\Delta\varphi-\Delta\varphi_1)\left[(2R_1-R_0)\,\Delta R-2\sqrt{R(R_0-R)R_1(R_0-R_1)}\right]}{\sqrt{R(R_0-R)}}.$$

In (7) the velocity of light has been accepted as equal to 1 and

$$R=\frac{dR}{dT}\Delta T,\qquad R_1=\frac{dR}{dT}\Delta T_1.$$

Now, let t and t_1 be the times in the local inertial coordinate systems moving with the source and the observer, respectively. Then, we have

$$(8)\qquad \Delta T=\frac{\Delta t}{\sqrt{1-u^2}},\qquad T_1=\frac{\Delta t_1}{\sqrt{1-v^2}}.$$

Let u_φ, v_φ be the components of u and v with respect to the geodetic between A and $B(\psi,\vartheta$ are constants)

$$(9)\qquad u_\varphi=R(T)\frac{\Delta\varphi}{\Delta T},\qquad v_\varphi=R(T_1)\frac{\Delta\varphi_1}{\Delta T_1}.$$

Using (8) and (9), after further calculations, the condition (7) may be reduced to the form

$$(10)\qquad \Delta t=\frac{R(T)\sqrt{1-v^2}(1-u_\varphi)}{R(T_1)\sqrt{1-u^2}(1-v_\varphi)}\Delta t_1.$$

Formula (10) expresses the relation between the time interval Δt spacing the two emitted signals (measured in the proper time of the source of light) and the time interval Δt_1 spacing their reception (measured in the proper time of the observer).

If ΔT is the emission time of a single wave of lenght λ, then ΔT_1 is the reception time of the observed single wave. Since $\lambda=\Delta T$ and $\lambda_1=\Delta T_1$ for the velocity of light equal to 1, then formula (10) gives also the relation between the length of emitted wave and that of the observed one

$$\frac{\lambda}{\lambda_1}=\frac{R(T)\sqrt{1-v^2}(1-u_\varphi)}{R(T_1)\sqrt{1-u^2}(1-v_\varphi)}.$$

References

[1] A. Friedmann, *Über die Krümmung des Raumes*, Zeitschrift für Physik 10 (1922), pp. 377-386.

[2] O. Heckmann and E. Schücking, *Encyclopaedia of Physics*, Vol. LIII, Berlin 1959, p. 510.

[3] A. Zięba, *Remarks on the ambiguity of distance determination in Friedmann's cosmological model*, Zastosow. Matem. (in print).

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