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AN EXTENSION OF CHEBYSHEV'S INEQUALITY

1. Introduction. Let $F(x)$ be a probability distribution function with $\int_{-\infty}^{\infty} x dF(x) = 0$, $\int_{-\infty}^{\infty} x^2 dF(x) = 1$, then, according to the well-known inequality of Chebyshev [3] for any $t > 0$ holds

$$1 \geqslant F(t) - F(-t) \geqslant 1 - 1/t^2.$$

Several improvements of Chebyshev's inequality may be found in the literature concerning restricted classes of distribution functions ([1], [2], [5], [8], [9], [10], [11]).

Among them, two papers of Barlow and Marshall ([1], [2]) give extensions for the so-called Pólya density functions of order 2. A density function $f(x)$ is said to be a Pólya density function of order 2 or strongly unimodal if $\log f(x)$ is convex over a finite or infinite interval (the support of the distribution) and 0 otherwise. The distributions having Pólya densities of order 2 form an important class of probability distributions: they include the normal, exponential etc. distributions and some other distributions of practical importance. Several papers, e. g. [6], pp. 332-392, [4], pp. 15-34, [7], [12], [13] deal with various properties of this class of distributions.

In this paper we suppose that $F(x)$ has a density $f(x)$ which is of Pólya type of order 2.

Barlow and Marshall in their paper gave bounds for a subclass of that class of distributions characterized by the condition $F(0) = 0$. Our paper does not contain this restriction.

In [1] and [2] the essential part of the solution of the problems consisted in finding the extremal families of distributions. If $\mathscr{H}\big(F(x)\big)$ is a functional defined on a family $\mathscr{F}$ of probability distributions, then $\mathscr{G} \subset \mathscr{F}$ is called extremal with respect to $\mathscr{H}$ if $\sup_{F \in \mathscr{F}} \mathscr{H}(F) = \sup_{F \in \mathscr{G}} \mathscr{H}(F)$ or $\inf_{F \in \mathscr{F}} \mathscr{H}(F) = \inf_{F \in \mathscr{G}} \mathscr{H}(F)$. The extremal families were, in the mentioned cases, sufficiently small, thus the bounds could be evaluated by numerical methods. Similar methods had been used by Royden [11] and Mallows ([9], [10]).

In this paper we derive some extremal families. In general, these families are larger than the mentioned ones and thus the computation seems more difficult. We do not give the bounds except in some trivial cases.

2. Bounds. Throughout the paper, probability distribution functions and their density functions will be denoted by the corresponding capital and small case letters, respectively.

Let us denote by $\mathscr{F}$ the family of all distributions with Pólya density functions of order 2 (PF_2) having the expectation 0 and variance 1. We may suppose, without loss of generality, that, if $F \in \mathscr{F}$, $f(x) = F'(x)$ is continuous from inside in the finite endpoints of the distribution (if such endpoints exist). Note that $f(x)$ is continuous anywhere else and $f'(x)$ is piecewise continuous. We remark for the sake of definiteness and simplicity that we may assume that $f'(x)$ is continuous from the left, i.e. $f'(x) = f'(x-0)$; this does not restrict the generality.

Let the families $\mathscr{G}_1$, $\mathscr{G}_2$, $\mathscr{G}_3$ and $\mathscr{G}_4$ be characterised by the following relations:

$G \in \mathscr{G}_1$, if $G \in \mathscr{F}$ and $\log g(x)$ is linear between $-\infty$ and y_0, and between y_0 and ∞.

$G \in \mathscr{G}_2$, if $G \in \mathscr{F}$ and $\log g(x)$ is linear between y_1 and $y_2 (-\infty \leqslant y, < y_2 \leqslant \infty)$, and if $G(y_1) = 0$, $G(y_2) = 1$.

$G \in \mathscr{G}_3$, if $G \in \mathscr{F}$ and $\log g(x)$ is linear between $-\infty$ and y_1, between y_1 and y_2, and between y_2 and ∞ $(y_1 \leqslant y_2)$.

$G \in \mathscr{G}_4$ if $G \in \mathscr{F}$ and $\log g(x)$ is linear between y_1 and y_2 and between y_2 and y_3 $(-\infty \leqslant y_1 \leqslant y_2 \leqslant y_3 \leqslant \infty)$, and if $G(y_1) = 0$, $G(y_3) = 1$. (For some y_i, $i = 0, 1, 2, 3$.)

In the above definitions, linearity of $\log f(x)$ on an interval includes the cases where $f(x)$ vanishes on the interval identically.

Our main result is expressed in the following theorems:

THEOREM 1. *Let* $0 \leqslant t$, *then if* $F \in \mathscr{F}$ *holds*

$$\inf_{G \in \mathscr{G}_2} G(t) \leqslant F(t) \leqslant \sup_{G \in \mathscr{G}_1} G(t),$$

THEOREM 2. *Let* $t_1 \leqslant 0 \leqslant t_2$, *then if* $F \in \mathscr{F}$ *holds*

$$\inf_{G \in \mathscr{G}_4} [G(t_2) - G(t_1)] \leqslant F(t_2) - F(t_1) \leqslant \sup_{G \in \mathscr{G}_3} [G(t_2) - G(t_1)].$$

a. Before proving these theorems let us formulate two simple lemmas in which $f(x)$ and $g(x)$ are piecewise continuous nonnegative functions in (a, b) with

$$\int_a^b f(x)\,dx = \int_a^b g(x)\,dx.$$

LEMMA 1. *If $f(x)$ crosses $g(x)$ from below in their only crossing point x_0 between a and b, in other words if*

$$f(x) \leqslant g(x), \quad \text{for} \quad a < x < x_0,$$
$$f(x) \leqslant g(x), \quad \text{for} \quad x_0 < x < b,$$

then

$$\int_a^b xf(x)\,dx \geqslant \int_a^b xg(x)\,dx.$$

LEMMA 2. *If $\int_a^b xf(x)\,dx = \int_a^b xg(x)\,dx$ and $f(x)$ crosses $g(x)$ exactly twice in the interval (a, b) and in the left crossing point $f(x)$ crosses $g(x)$ from below, in other words if for some x_1, x_2 we have*

$$f(x) \leqslant g(x) \quad \text{for} \quad a < x < x_1,$$
$$f(x) \geqslant g(x) \quad \text{for} \quad x_1 < x < x_2,$$
$$f(x) \leqslant g(x) \quad \text{for} \quad x_2 < x < b,$$

then

$$\int_a^b x^2 f(x)\,dx \leqslant \int_a^b x^2 g(x)\,dx.$$

In both lemmas the equality sign occurs if and only if $f(x) = g(x)$ a.e.

Both lemmas express simple and known statements; the second one is a consequence, e.g. of the fact that the function $(x-x_1)(x-x_2)(f(x)-g(x))$ is nonpositive on the whole interval (a, b).

b. We begin with the proof of Theorem 2. First we consider the right hand side inequality in the case when

$$f(t_1) > 0, \quad f(t_2) > 0.$$

We determine the numbers a_1 and a_2 to fulfil the following relations:

$$(1) \qquad f(t_1)\int_{-\infty}^{t_1} e^{a_1(x-t_1)}\,dx + f(t_2)\int_{t_2}^{\infty} e^{a_2(x-t_2)}\,dx = \int_{-\infty}^{t_1} f(x)\,dx + \int_{t_2}^{\infty} f(x)\,dx,$$

$$(2) \qquad f(t_1)\int_{-\infty}^{t_1} xe^{a_1(x-t_1)}\,dx + f(t_2)\int_{t_2}^{\infty} xe^{a_2(x-t_2)}\,dx = \int_{-\infty}^{t_1} xf(x)\,dx + \int_{t_2}^{\infty} xf(x)\,dx.$$

Let us denote the left-hand sides of (1) and (2) by $P_1(a_1, a_2)$ and $M_1(a_1, a_2)$ the right-hand sides by p and μ respectively. Let us denote the solution of the equation $P_1(a_1, -\infty) = p$ by a_{11} and if $a_{11} \leqslant a_1$ the solution of the equation $P_1(a_1, a_2) = p$ for a_2 by $a_2 = \varphi_1(a_1)$ i.e. $P_1(a_1, \varphi_1(a_1)) \equiv p$. The function $\varphi_1(a_1)$ and consequently $M_1(a_1) = M_1(a_1,$

$\varphi_1(a_1)\big)$ are monotonically increasing functions of a_1 and — according to Lemma 1 $M_1(a_{11}) \leqslant \mu$, $M_1(+\infty) \geqslant \mu$, therefore (1) and (2) can be satisfied by the appropriate choice of a_1 and a_2. If, for these a_1, a_2, the relations

$$a_1 \geqslant \frac{f'(t_1)}{f(t_1)} \quad \text{and} \quad a_2 \leqslant \frac{f'(t_2)}{f(t_2)}$$

hold we define the function $g(x)$ as follows:

$$g(x) = \begin{cases} f(t_1)\,e^{a_1(x-t_1)} & \text{if} \quad x < T_1, \\ f(t_1)\,e^{a(x-T_1)+a_1(T_1-t_1)} & \text{if} \quad T_1 \leqslant x \leqslant T_2, \\ f(t_2)\,e^{a_2(x-t_2)} & \text{if} \quad T_2 < x, \end{cases}$$

where

$$a = \frac{\log f(t_2) - \log f(t_1) + a_2(T_2 - t_2) - a_1(T_1 - t_1)}{T_2 - T_1}.$$

The constants T_1 and T_2 will be determined by the following relations

(3) $$\int_{t_1}^{t_2} g(x)\,dx = \int_{t_1}^{t_2} f(x)\,dx,$$

(4) $$\int_{t_1}^{t_2} x g(x)\,dx = \int_{t_1}^{t_2} x f(x)\,dx.$$

Let us denote the left-hand side expressions of these equations — which are functions of T_1 and T_2 — by $\bar{P}_1(T_1, T_2)$ and $\bar{M}_1(T_1, T_2)$, respectively.

If $t_1 \leqslant T_1 \leqslant T$ and $T \leqslant T_2 \leqslant t_2$ where

$$T = \frac{\log f(t_2) - \log f(t_1) + a_1 t_1 - a_2 t_2}{a_1 - a_2}$$

(T is the abscissa of the crossing point of the curves $f(t_1)e^{a_1(x-t_1)}$ and $f(t_2)e^{a_2(x-t_2)}$) then $\bar{P}_1(T_1, T_2)$ is monotonically increasing in T_1, monotonically decreasing in T_2 and continuous in both.

$$\bar{P}_1(T, T_2) = \bar{P}_1(T_1, T) \geqslant 1 - p$$

since

$$a_1 \geqslant \frac{f'(t_1)}{f(t_1)}, \quad a_2 \leqslant \frac{f'(t_2)}{f(t_2)}$$

and

$$f(x) < g(x) \quad \text{if} \quad t_1 \leqslant x \leqslant t_2,$$

and, by similar arguments,

$$\bar{P}_1(t_1, t_2) \leqslant 1 - p.$$

Therefore, let T_{11} be a value for which

$$\bar{P}_1(T_{11}, t_2) = 1-p,$$

and if $t_1 \leqslant T_1 \leqslant T_{11}$, then let $\varphi_{11}(T_1)$ be such a value for which

$$\bar{P}_1\big(T_1, \varphi_{11}(T_1)\big) = 1-p.$$

Naturally $\varphi_{11}(T_1)$ is a monotonically increasing function of T_1 and by Lemma 1 $M_1\big(T_1, \varphi_{11}(T_1)\big)$ is a monotonically decreasing function and

$$\bar{M}_1\big(t_1\varphi_{11}(t_1)\big) \geqslant -\mu \geqslant \bar{M}_1(T_{11}, t_1)$$

i.e. the equation $\bar{M}_1\big(T_1, \varphi_{11}(T_1)\big) = -\mu$ has a solution and thus, $g(x)$ can be determined according to the requirements.

If

$$a_1 < \frac{f'(t_1)}{f(t_1)} \tag{5}$$

then we define $g(x)$ as follows:

$$g(x) = \begin{cases} f(t_1)\, e^{f'(t_1)(x-t_1)/f(t_1)} & \text{if} \quad x < T_1', \\ f(t_1)\, e^{f'(t_1)(T_1'-t_1)/f(t_1)+a'(x-T_1)} & \text{if} \quad T_1' \leqslant x \leqslant T_2', \\ f(t_1)\, e^{f'(t_1)(T_1'-t_1)+f(t)+a'(T_2'-T_1')+a_2'(x-T_2)} & \text{if} \quad T_2' < x, \end{cases}$$

where the constants a' and T_1' are determined to fulfil the equations (3) and (4) and in the second step a_2' and $T_2' > t_2$ will be determined in such a way that the equations

$$\int_{-\infty}^{t_1} g(x)\,dx + \int_{t_2}^{\infty} g(x)\,dx = p, \tag{6}$$

$$\int_{-\infty}^{t_1} xg(x)\,dx + \int_{t_2}^{\infty} xg(x)\,dx = \mu \tag{7}$$

be fulfilled.

Now we prove possibility of the first step.

Let us denote the left hand side integrals of (3) and (4) by $P_1^*(T_1', a')$ and $M_1^*(T_1', a')$. Let us define T_{11}' by the equation

$$f(t_1) \int_{t_1}^{T_{11}'} e^{f'(t_1)(x-t_1)/f(t_1)}\,dx = 1-p,$$

and for any T_1' with $t_1 \leqslant T_1' \leqslant T_{11}'$, the function $\varphi_1^*(T_1')$ such that $P_1^*\big(T_1', \varphi_1^*(T_1')\big) = 1-p$. Again, with help of Lemma 1 we may conclude that

$$M_1^*\big(t_1, \varphi_1(t_1')\big) \geqslant -\mu \geqslant M_1^*(T_{11}', -\infty).$$

$M_1(T_1', \varphi_1^*(T_1'))$ is a monotonic function, therefore (3) and (4) can be satisfied. It follows from Lemma 1 that $g(t_2) \geqslant f(t_2)$.

Now the left hand side expressions of (6) and (7) can be considered as functions of T_2' and a_2', let us denote them by $\bar{P}_1^*(T_2', a_2')$ and $\bar{M}_1^*(T_2', a_2')$. The condition (5) implies that $\bar{P}_1^*(\infty, a_2') \geqslant p$. We define the function $\bar{\varphi}_1^*(T_2')$ in such a way that $\bar{P}_1^*(T_2', \bar{\varphi}^*(T_2')) = p$. If $\bar{M}_1^*(t_2, \bar{\varphi}_1^*(t_2)) \leqslant m$, we can prove by the same argument as before, that (6) and (7) can be satisfied. If $\bar{M}_1^*(t_2, \bar{\varphi}_1^*(t_2)) > m$ then we change the last definition of $g(x)$ by putting $T_2' < t_2$ instead of $T_2' \geqslant t_2$. In this case we determine first the constants a_2' and $c = f(t_1) \times \exp(f'(t_1)(T_1' - t_1)/f(t_1) + a'(T_2' - T_1') + a_2'(t_2 - T_2'))$ from (6) and (7) easily. The determination of the remaining constants with help of (3) and (4) can be performed in the same way as in the first case.

The case $a_2 > f'(t_2)/f(t_2)$ can be treated similarly.

c. Let us now suppose $f(t_1) = 0$ and by virtue of the log-concavity $F(t_1) = 0$ (and vice versa). First we choose the constants c, a to fulfil the equations

$$c \int_{t_1}^{t_2} e^{a(x-t)} dx = \int_{t_1}^{t_2} f(x)\, dx,$$

$$c \int_{t_1}^{t_2} x e^{a(x-t)} dx = \int_{t_1}^{2} x f(x)\, dx.$$

By Lemma 1 we have $c \geqslant f(t_2)$.

If $F(t_2) = 1$ then we define

$$g(x) = \begin{cases} 0 & \text{if} \quad x > t_1, \\ c e^{a(x-t_2)} & \text{if} \quad t_1 \leqslant x \leqslant t_2, \\ 0 & \text{if} \quad x > t_2. \end{cases}$$

If $F(t_2) < 1$ we choose the constant a_{21} to fulfil the equation

$$c \int_{t_2}^{\infty} e^{a_{21}(x-t_2)} dx = \int_{t_2}^{\infty} f(x)\, dx.$$

If

$$c \int_{t_2}^{\infty} x e^{a_{21}(x-t_2)} dx \geqslant \int_{t_2}^{\infty} x f(x)\, dx$$

then let us define

$$g(x) = \begin{cases} 0 & \text{if} \quad x < t_1, \\ c e^{a(x-t_2)} & \text{if} \quad t_1 \leqslant x < T_2, \\ c e^{a_2(x-T_2) + a(T_2 - t_2)} & \text{if} \quad T_2 < x, \end{cases} \tag{8}$$

where T_2, a_2 are chosen to fulfil the relationships (6) and (7). (Here $t_2 \leqslant T_2$.)

In the opposite case the definition of $g(x)$ is the same with different constants a', a_2', T_2', c'. In this case first c' and a_2' are to be determined from (3) and (4) (again, Lemma 1 implies that $c' > f(t_2)$) and in the second step the values of a' and T_2' are obtained from (6) and (7). In this case $t_2 > T_2'$.

The possibility of the mentioned choices follows in the same manner as in the previous steps of the proof.

d. The case $f(t_1) > 0$, $f(t_2) = 0$ i.e. $F(t_2) = 1$ can be treated in the same way.

e. Now applying Lemma 2 separately in the interval (t_1, t_2) and the complementary set of the real line and summing the inequalities we obtain for each of the above cases

$$\sigma_1^2 = \int_{-\infty}^{\infty} x^2 g(x)\,dx \geqslant \int_{-\infty}^{\infty} x^2 f(x)\,dx = 1.$$

Since — being $t_1 < 0$ —

$$\begin{aligned} F(t_2)-F(t_1) &= G(t_2)-G(t_1) \leqslant G(\sigma_1 t_2)-G(\sigma_1 t_1) \\ &= G_3(t_2)-G_3(t_1) \end{aligned}$$

and $G_3 \in \mathscr{G}_3$ the second inequality of Theorem 2 is proven. Note that in the case $F(t_1) = 0$ we have $G(t_1) = G(\sigma_1 t_1) = 0$ simultaneously.

f. To prove the left-hand side inequality of Theorem 2, let us first suppose $f(t_1) > 0$, $f(t_2) > 0$. In this case we try to define a function $g(x)$ in the following way:

$$g(x) = \begin{cases} 0 & \text{if} \quad x < T_1, \\ f(t_1)e^{a_1(x-t_1)} & \text{if} \quad T_1 \leqslant x \leqslant T_2, \\ f(t_2)e^{a_2(x-t_2)} & \text{if} \quad T_2 \leqslant x \leqslant T_3, \\ 0 & \text{if} \quad T_3 < x \end{cases} \tag{9}$$

with $T_1 \leqslant t_1 \leqslant T_2 \leqslant t_2 \leqslant T_3$, where

$$T_2 = \frac{\log f(t_2) - \log f(t_1) + a_1 t_1 - a_2 t_2}{a_1 - a_2}$$

and the constants a_1, a_2, T_1, T_3 are to be determined by the relations (3), (4), (6) and (7).

In the first step we try to satisfy (3) and (4) by the appropriate choice of a_1 and a_2. Let us now denote the left-hand side expressions of (3) and (4) by $\overline{P}_2(a_1, a_2)$ and $\overline{M}_2(a_1, a_2)$.

Again, $\overline{P}_2(a_1, a_2)$ is a continuous function of ist arguments, and it is monotonically increasing in a_1 and decreasing in a_2 if

$$\frac{\log f(t_2) - \log f(t_1)}{t_2 - t_1} \leqslant a_1 \leqslant \frac{f'(t_1)}{f(t_1)}$$

and

$$\frac{f'(t_2)}{f(t_2)} \leqslant a_2 \leqslant \frac{\log f(t_2)-\log f(t_1)}{t_2-t_1}.$$

Similarly as before, we obtain the conclusions

$$\bar{P}_2\left(\frac{\log f(t_2)-\log f(t_1)}{t_2-t_1}, a_2\right) = \bar{P}_2\left(a_1, \frac{\log f(t_2)-\log f(t_1)}{t_2-t_1}\right) \leqslant 1-p,$$

where $1-p$ denotes, as before, the right-hand side integral of (3) and

$$\bar{P}_2\left(\frac{f'(t_1)}{f(t_1)}, \frac{f'(t_2)}{f(t_2)}\right) \geqslant 1-p.$$

Again let us define a_{11} in such a way that $1-p = \bar{P}_2(a_{11}, f'(t_2)/f(t_2))$, and define $\varphi_2(a_1)$ for

$$a_{11} \leqslant a_1 \leqslant \frac{f'(t_1)}{f(t)}$$

so that

$$\bar{P}_2(a_1, \varphi_2(a_1)) \equiv 1-p.$$

$\varphi_2(a_1)$ turns out to be monotonically increasing and, using again Lemma 1, $\bar{M}_2(a_1) = \bar{M}_2(a_1, \varphi_2(a_1))$ is a monotonically decreasing continuous function. Moreover, we obtain

$$\bar{M}_2(a_{11}) \leqslant -\mu \leqslant \bar{M}_2\left(\frac{f'(t_1)}{f(t_1)}\right)$$

and thus (3) and (4) can be satisfied.

The solvability of the remaining two equations (1) and (2) can be shown in a similar way: we define the function

$$\bar{g}(x) = \begin{cases} f(t_1)e^{a_1(x-t_1)} & \text{if} \quad x \leqslant t_1, \\ f(t_2)e^{a_2(x-t_2)} & \text{if} \quad t_2 \leqslant x, \\ 0 & \text{otherwise}. \end{cases}$$

Then, similarly as before, we conclude that two limits T_1, T_3 can be found in such a way that

$$\int_{T_1}^{T_3} \bar{g}(x)\,dx = p, \qquad \int_{T_1}^{T_3} x\bar{g}(x)\,dx = \mu.$$

If $T_1 \leqslant t_1$ and $t_2 \leqslant T_2$, then the function $g(x)$ defined by (9) is appropriate for further investigations.

In the opposite case and in the cases $F(t_1) = 0$ or $F(t_2) = 1$ we define $g(x)$, instead of (9), in the following way:

If $T_1 > t_1$ or if $F(t_1) = 0$ and $F(t_2) = 1$, $g(x)$ will be defined as follows:

$$g(x) = \begin{cases} 0 & \text{if} \quad x < T_1', \\ ce^{a_1'(x-t_2)} & \text{if} \quad T_1' \leqslant x \leqslant t_2, \\ ce^{a_2'(x-t_2)} & \text{if} \quad t_2 \leqslant x \leqslant T_2', \\ 0 & \text{if} \quad T_2' < x, \end{cases}$$

where given c with $0 < c \leqslant f(t_1)$, the constants a_1', a_2', T_1', T_2' are to be determined by the relations (6), (7), (3) and (4). (We dispose on the constant c later.) Clearly this is possible. It follows from Lemma 1 that $t_1 \leqslant T_1'$.

Choosing $c = f(t_2)$, we have $a_1' \geqslant a_2'$. When c tends to 0, then $a_1' \to -\infty$ and $a_2' \to \infty$. For reasons of continuity the relation $a_1' = a_2'$ must hold for some c_0. In this case the definition of our function reduces to the form

$$g(x) = \begin{cases} 0 & \text{if} \quad x < T_1', \\ c_0 e^{a'(x-t_2)} & \text{if} \quad T_1' \leqslant x \leqslant T_2', \\ 0 & \text{if} \quad T_2' < x. \end{cases} \tag{10}$$

The cases $T_2 < t_2$ and $F(t_2) = 1$, $F(t_1) > 0$ can be reduced to the above case by placing $1-F(x)$, $-t_2$ and $-t_1$ instead of $F(x)$, t_1 and t_2, respectively.

In each of the mentioned cases we can again apply Lemma 2 and obtain that

$$\sigma_2^2 = \int_{-\infty}^{\infty} x^2 g(x)\,dx \leqslant \int_{-\infty}^{\infty} x^2 f(x)\,dx = 1,$$

from which the first inequality of Theorem 2, except the case $F(t_1) = 0$ and $F(t_2) = 1$, can be proved in the same way as the first one.

In the case of $F(t_1) = 0$ and $F(t_2) = 1$ let us define $g_4^*(x)$ as follows:

$$g_4^*(x) = \begin{cases} \dfrac{e^{ax}}{e^{aT} - e^{at_1}} & \text{if} \quad t_1 \leqslant x \leqslant T, \\ 0 & \text{otherwise}, \end{cases} \tag{11}$$

where the constants a and T are to be determined in such a way that $G_4^* \in \mathscr{G}_2 \subset \mathscr{G}_4$. Clearly this is possible.

Lemma 2 proves that $T < t_2$ cannot occur. Therefore $G_4^*(t_2) - G_4^*(t_1) = 1 = F(t_2) - F(t_1)$ and the proof of Theorem 2 is completed.

g. It remains to prove Theorem 1. Let us put in equations (8), (10) and (11) $t_1 = -\infty$, $t_2 = t$; the proofs remain valid and the distribution function $G_3(x)$ will be an element of $\mathscr{G}_1$ and in the same way G_4 and G_4^* become elements of $\mathscr{G}_2$. This proves Theorem 1.

3. Remarks.

Remark 1. The bounds given in Theorems 1 and 2 are sharp, since $\mathscr{G}_i \subset \mathscr{F}$ $(i = 1, 2, 3, 4)$. Moreover, if $F \epsilon \mathscr{F} - \mathscr{G}_i$ then the corresponding inequality is strict.

Remark 2. Applying Theorem 1 to $F(0)$ and $1 - F(0)$ and taking Remark 1 into account we obtain the following exact bounds for $F(0)$ $(F \epsilon \mathscr{F})$:

$$e^{-1} \leqslant F(0) \leqslant 1 - e^{-1}.$$

Here $\mathscr{G}_1 \cap \mathscr{G}_2$ consists of the two distributions

$$G_1(x) = \begin{cases} 0 & \text{if} \quad x < -1, \\ 1 - e^{-x-1} & \text{if} \quad -1 \leqslant x \end{cases}$$

and

$$G_2(x) = 1 - G_1(-x).$$

Remark 3. If $t \geqslant 1$ and $F \epsilon \mathscr{F}$. then the bound

$$F(t) \leqslant 1$$

is sharp. Similarly, if $t_1 \leqslant -3^{1/2}$, $t_2 \geqslant 3^{1/2}$ then the bound

$$F(t_2) - F(t_1) \leqslant 1$$

is sharp, or in general, it is sharp if $t_1 \leqslant m_1$, $t_2 \geqslant m_2$, where m_1, m_2 are such that the distribution

$$\frac{\int\limits_{m_1}^{x} e^{az}\,dz}{\int\limits_{m_1}^{m_2} e^{az}\,dz}, \qquad m_1 \leqslant x \leqslant m_2$$

has the first two moments equal to 0 and 1.

Remark 4. Every of the families $\mathscr{G}_1$ and $\mathscr{G}_2$ occurring in Theorem 1 may be parametrized by a single parameter; consequently the evaluation of the corresponding bounds can be carried out relatively simply. However, this is not the case with Theorem 2. The question arises whether it is possible to replace $\mathscr{G}_3$ and $\mathscr{G}_4$ in Theorem 2 by narrower families. The authors will make efforts in this direction.

Remark 5. Pólya density functions of order 2 represent an important family of density functions both from the point of view of practical applications and with respect to the results of the present paper. These results, however, can be extended to other families.

Let $\psi(x)$ be a strictly monotonic, continuous function defined on the interval $(0, \infty)$; let be $\lim\limits_{x \to +0} \psi(x) = -\infty$, $\lim\limits_{x \to \infty} \psi(x) = \infty$.

Let us modify the definition of the family $\mathscr{F}$ in such a way that, instead of $\ln f(x)$, $\psi(f(x))$ should be convex over the support of F. Otherwise, the definition of $\mathscr{F}$ should be left unaltered.

Similar modifications are to be made in the definitions of the families $\mathscr{G}_1, \mathscr{G}_2, \mathscr{G}_3, \mathscr{G}_4$.

Evidently, theorems 1 and 2 are valid in this case too. In the proof, the function e^x is to be replaced by the inverse function of $\psi(x)$.

REMARK 6. (cf. Remark 5). Let $\psi(x)$ be a strictly monotone, continuous function defined on the interval $[0, \infty)$; let $\psi(0)$ be finite and let $\lim_{x\to\infty} \psi(x) = \infty$.

Let the definition of $\mathscr{F}$ be the same as in the case of Remark 5. (Distributions of this type have always finite support.) Similarly, the same modifications concerning the definitions of $\mathscr{G}_1, \mathscr{G}_2, \mathscr{G}_3, \mathscr{G}_4$ are to be applied, and in addition, in the definition of $\mathscr{G}_1$ and $\mathscr{G}_3$ "linearity between $-\infty$ and y_i" and "linearity between y_i and ∞" are to be replaced by "linearity between the left end point of the distribution and y_i" and "linearity between y_i and the right end point of the distribution", respectively. In these cases, additionally, continuity of the density function in the involved end point is to be postulated.

Our theorems are valid in this case.

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