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ESTIMATION OF $P\{Y < X\}$ IN THE EXPONENTIAL CASE

1. INTRODUCTION

Let X and Y be independent negative exponentially distributed random variables (r.v.'s) with parameters λ and μ , respectively. Tong ([5] and [6]) obtained the uniformly minimum variance unbiased estimator (UMVUE) of

$$P = P\{Y < X\} = \mu(\lambda + \mu)^{-1}$$

on the basis of two independent random samples X_1, X_2, \dots, X_n and Y_1, Y_2, \dots, Y_m . Estimation of this probability may be useful in applications in which it is desired to estimate the probability that one element fails prior to another element of some device. Therefore, it seems reasonable to find estimators which are based on censored or truncated samples obtained from some life tests. The most frequently applied life testing plans are the following (see Mann et al. [4]):

(a) without replacement (of the failed items) and (with duration of the observation) until the moment of the r -th failure ($r \leq n$, where n is the number of all tested items) (*type II censoring without replacement*);

(b) with replacement and until the moment of the r -th failure (*type II censoring with replacement*);

(c) with replacement and until the fixed moment t_0 (*type I censoring with replacement*);

(d) without replacement and until the fixed moment t_0 (*type I censoring without replacement*).

Let us assume that n identical elements X and m identical elements Y having the negative exponential life distributions with parameters λ and μ , respectively, are independently tested. In this paper we derive the unbiased estimators of P being functions of the minimal sufficient statistics on the basis of observations obtained in the life tests (a)-(d). In cases (a), (b) and (c) these estimators are the UMVUE's of P . In case (d)

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we do not know whether the estimator has the same optimal property, although it is known (see [1]) that the minimal sufficient statistics is not complete.

2. ESTIMATION FOR TYPE II CENSORING

2.1. Testing without replacement. Let $X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(r)}$ denote the first r life-times of elements X ($r \leq n$) and let $Y_{(1)} \leq Y_{(2)} \leq \dots \leq Y_{(q)}$ denote the first q life-times of elements Y ($q \leq m$). The independent r.v.'s

$$S = \sum_{i=1}^r X_{(i)} + (n-r)X_{(r)} \quad \text{and} \quad Z = \sum_{j=1}^q Y_{(j)} + (m-q)Y_{(q)},$$

the accumulated life-times of X 's and Y 's, are the complete sufficient statistics for λ and μ , respectively. The r.v. $2\lambda S$ has the χ^2 -distribution with $2r$ degrees of freedom (χ_{2r}^2) and the r.v. $2\mu Z$ has the χ_{2q}^2 -distribution (see Epstein and Sobel [3]). The independent r.v.'s $nX_{(1)}$ and $mY_{(1)}$ are negative exponentially distributed with parameters λ and μ , respectively, and hence the statistics

$$(1) \quad \tilde{\theta} = \begin{cases} 0 & \text{if } nX_{(1)} \leq mY_{(1)}, \\ 1 & \text{if } nX_{(1)} > mY_{(1)} \end{cases}$$

is an unbiased estimator of P . Taking the conditional expectation

$$(2) \quad \hat{P} = E[\tilde{\theta} | S, Z] = P\{nX_{(1)} > mY_{(1)} | S, Z\} \\ = \int_0^{\infty} P\left\{X_{(1)} > \frac{m}{n}y | S\right\} g(y | Z) dy = \int_0^{\infty} g(y | Z) \int_{(m/n)y}^{\infty} f(x | S) dx dy$$

we obtain, by virtue of the Rao-Blackwell theorem, the UMVUE of P . It is easy to check that the conditional probability density function (c.p.d.f.) of $X_{(1)}$, given S , takes the form

$$(3) \quad f(x | S) = \begin{cases} \frac{n(r-1)}{S} \left(1 - \frac{nx}{S}\right)_+^{r-2} & \text{if } r \geq 2, \\ \delta_{S/n}(x) & \text{if } r = 1, \end{cases}$$

where $a_+^k = [\max(0, a)]^k$ for $k > 0$, $a_+^0 = 0$ for $a \leq 0$ and $a_+^0 = 1$ for $a > 0$, $\delta_a(x)$ is the probability density function of the r.v. with value a and with probability 1.

The expression similar to (3) can be obtained for the c.p.d.f. $g(y|Z)$. By (2) and (3), in the case $r \geq 2$ and $q \geq 2$ we have

$$\begin{aligned}
 (4) \quad \hat{P} &= \int_0^\infty \frac{m(q-1)}{Z} \left(1 - \frac{my}{Z}\right)_+^{q-2} \int_{(m/n)y}^\infty \frac{n(r-1)}{S} \left(1 - \frac{nx}{S}\right)_+^{r-2} dx dy \\
 &= \frac{m(q-1)}{Z} \int_0^\infty \left(1 - \frac{my}{Z}\right)_+^{q-2} \left(1 - \frac{my}{S}\right)_+^{r-1} dy \\
 &= \frac{m(q-1)}{Z} \int_0^\zeta \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \left(\frac{my}{S}\right)^i \left(1 - \frac{my}{Z}\right)^{q-2} dy \\
 &= (q-1) \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \left(\frac{Z}{S}\right)^i B_\xi(i+1, q-1),
 \end{aligned}$$

where $\zeta = \min(Z/m, S/m)$, $\xi = \min(1, S/Z)$, and $B_\xi(i+1, q-1)$ is the incomplete beta-function, i.e.

$$B_\xi(i+1, q-1) = \int_0^\xi u^i (1-u)^{q-2} du.$$

It is easy to verify that (4) may be written in the form

$$\hat{P} = \begin{cases} \sum_{i=0}^{r-1} (-1)^i \frac{(r-1)!(q-1)!}{(r-1-i)!(q-1+i)!} \left(\frac{Z}{S}\right)^i & \text{if } Z \leq S, \text{ i.e. } \xi = 1, \\ 1 - \sum_{i=0}^{q-1} (-1)^i \frac{(q-1)!(r-1)!}{(q-1-i)!(r-1+i)!} \left(\frac{S}{Z}\right)^i & \text{if } S > Z, \text{ i.e. } \xi = \frac{S}{Z}. \end{cases}$$

This formula is also valid in the case $r = 1$ and $q = 1$. If $r = n$ and $q = m$, we obtain Tong's result ([5] and [6]).

2.2. Testing with replacement. We assume now that the failed items are instantly replaced by new ones having the same life distribution. Let $X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(r)}$ and $Y_{(1)} \leq Y_{(2)} \leq \dots \leq Y_{(q)}$ be independent samples of the first r life-times of X 's and the first q life-times of Y 's, respectively. The r.v.'s $X_{(r)}$ and $Y_{(q)}$ are the complete sufficient statistics for λ and μ , respectively. The r.v. $2n\lambda X_{(r)}$ is a χ_{2r}^2 -variate and the r.v. $2m\mu Y_{(q)}$ is a χ_{2q}^2 -variate. In this case an unbiased estimator of P is also of form (1). Hence, in a similar manner as above, we obtain the UMVUE of P which takes the form

$$\hat{P} = \mathbb{E}[\tilde{\theta} | X_{(r)}, Y_{(q)}]$$

$$= \begin{cases} \sum_{i=0}^{r-1} (-1)^i \frac{(r-1)!(q-1)!}{(r-1-i)!(q-1+i)!} \left(\frac{mY_{(q)}}{nX_{(r)}} \right)^i & \text{if } mY_{(q)} \leq nX_{(r)}, \\ 1 - \sum_{i=0}^{q-1} (-1)^i \frac{(q-1)!(r-1)!}{(q-1-i)!(r-1+i)!} \left(\frac{nX_{(r)}}{mY_{(q)}} \right)^i & \text{otherwise.} \end{cases}$$

3. ESTIMATION FOR TYPE I CENSORING

3.1. Testing with replacement. Let us assume that n elements X are placed simultaneously on life test with replacement until the time t , and m elements Y are placed on a similar test until the time τ . Let $X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(D(t))}$ be the moments of failure of X 's until t , and let $Y_{(1)} \leq Y_{(2)} \leq \dots \leq Y_{(K(\tau))}$ be the moments of failure of Y 's until τ . It is easy to see that these random moments form two independent Poisson processes observed during the times t and τ , respectively. The r.v.'s $D(t)$ and $K(\tau)$ are Poisson variables with means $n\lambda t$ and $m\mu\tau$, respectively. They are the complete sufficient statistics for λ and μ , respectively. The statistics (1) is also an unbiased estimator of P in this case. The c.p.d.f. of $X_{(1)}$, given $D(t) = d$, is of the form

$$f(x|d) = \begin{cases} \delta_t(x) & \text{if } d = 0, \\ \frac{d}{t} \left(1 - \frac{x}{t}\right)_+^{d-1} & \text{if } d \geq 1. \end{cases}$$

A similar expression can be obtained for the c.p.d.f. $g(y|k)$ of $Y_{(1)}$, given $K(\tau) = k$. Similarly as above, taking the conditional expectation $\mathbb{E}[\tilde{\theta} | D(t) = d, K(\tau) = k]$, we obtain the UMVUE of P being of the form

$$(5) \quad \hat{P} = \begin{cases} \sum_{i=0}^d (-1)^i \frac{d!k!}{(k+i)!(d-i)!} \left(\frac{m\tau}{nt} \right)^i & \text{if } m\tau \leq nt, \\ 1 - \sum_{i=0}^k (-1)^i \frac{d!k!}{(d+i)!(k-i)!} \left(\frac{nt}{m\tau} \right)^i & \text{otherwise.} \end{cases}$$

3.2. Testing without replacement. Let $X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(D(t))}$ and $Y_{(1)} \leq Y_{(2)} \leq \dots \leq Y_{(K(\tau))}$ be the moments of failure of elements X and Y observed until the moments t and τ , respectively. The vectors $(D(t), S)$ and $(K(\tau), Z)$, where

$$S = \sum_{i=1}^{D(t)} X_{(i)} + (n - D(t))t \quad \text{and} \quad Z = \sum_{j=1}^{K(\tau)} Y_{(j)} + (m - K(\tau))\tau,$$

are the minimal sufficient statistics for λ and μ , respectively. However, these statistics are not complete except for the trivial case $n = m = 1$ (see [1]). The estimator (1) is also an unbiased estimator of P in this case. The conditional expectation

$$(6) \quad \hat{P} = E[\tilde{\theta} | D(t), S, K(\tau), Z]$$

is unbiased for P , having the variance smaller than $\tilde{\theta}$, but we do not know whether it is the UMVUE of P .

In order to derive the expectation (6) we ought to have the c.p.d.f.'s $f(x_1 | D(t), S)$ and $g(y_1 | K(\tau), Z)$. It follows from [1] that the p.d.f. of the vector $(D(t), S)$ is

$$(7) \quad p(d, s) = \begin{cases} e^{-\lambda t} \delta_{nt}(s) & \text{if } d = 0, \\ \lambda^d \binom{n}{d} e^{-\lambda s} & \\ \frac{\lambda^d \binom{n}{d}}{(d-1)!} V_d^n(s, t) & \text{if } d \geq 1, \end{cases}$$

where

$$V_d^n(s, t) = \sum_{j=0}^d (-1)^j \binom{d}{j} [s - (n - d + j)t]_+^{d-1}.$$

The joint p.d.f. of the vector $(X_{(1)}, D(t), S)$ was derived in paper [2]. It is of the form

$$(8) \quad p(x, d, s) = \begin{cases} \exp[-\lambda t] \delta_{nt}(s) & \text{if } d = 0, \\ n \lambda \exp[-\lambda(x + (n-1)t)] \delta_{x+(n-1)t}(s) & \text{if } d = 1, \\ \frac{\lambda^d \binom{n}{d} \exp[-\lambda s]}{(d-2)!} V_{d-1}^{n-1}(s - nx, t - x) & \text{if } d \geq 2. \end{cases}$$

From (7) and (8) we obtain the c.p.d.f. of $X_{(1)}$, given $D(t) = d$ and $S = s$,

$$f(x | d, s) = \begin{cases} \delta_{s-(n-1)t}(x) & \text{if } d = 0, 1, \\ \frac{V_{d-1}^{n-1}(s - nx, t - x)}{d(d-1) V_d^n(s, t)} & \text{if } d \geq 2. \end{cases}$$

In the same way one can obtain a similar expression for $g(y | k, z)$, the c.p.d.f. of $Y_{(1)}$, given $K(\tau) = k$ and $Z = z$. Hence we have

$$\hat{P} = P\{nX_{(1)} > mY_{(1)} | d, s, k, z\} = \int_0^\infty g(y | k, z) \int_{(m/n)y}^\infty f(x | d, s) dx dy.$$

Let us derive \hat{P} in the case where $d \geq 2$ and $k \geq 2$. We have

$$\hat{P} = \frac{d(d-1)k(k-1)}{V_d^n(s, t) V_k^m(z, \tau)} \int_0^\infty V_{k-1}^{m-1}(z - my, \tau - y) \int_{(m/n)y}^\infty V_{d-1}^{n-1}(s - nx, t - x) dx dy.$$

Putting $a = m/n$, $a_j = s - (n - d + j)t$ and $c_j = d - j$, we calculate the inner integral

$$\begin{aligned} \int_{ay}^\infty V_{d-1}^{n-1}(s - nx, t - x) dx &= \sum_{j=0}^{d-1} (-1)^j \binom{d-1}{j} \int_{ay}^\infty [a_j - c_j x]_+^{d-2} dx \\ &= \sum_{j=0}^{d-1} (-1)^j \binom{d-1}{j} \int_{ay}^{\max[ay, a_j/c_j]} [a_j - c_j x]^{d-2} dx \\ &= \sum_{j=0}^{d-1} (-1)^j \binom{d-1}{j} \frac{1}{c_j} \int_{\min[ay, a_j/c_j]}^{a_j - c_j ay} u^{d-2} du \\ &= \sum_{j=0}^{d-1} (-1)^j \binom{d-1}{j} \frac{1}{c_j(d-1)} [a_j - c_j ay]_+^{d-1}. \end{aligned}$$

Hence we obtain

$$\begin{aligned} (9) \quad \hat{P} &= \frac{k(k-1)}{V_d^n(s, t) V_k^m(z, \tau)} \int_0^\infty V_{k-1}^{m-1}(z - my, \tau - y) V_d^n\left(s - my, t - \frac{m}{n}y\right) dy \\ &= \frac{k(k-1)}{V_d^n(s, t) V_k^m(z, \tau)} \sum_{i=0}^{k-1} (-1)^i \binom{k-1}{i} \sum_{j=0}^{d-1} (-1)^j \binom{d}{j} I_{ij}, \end{aligned}$$

where

$$I_{ij} = \int_0^\infty [z - (m - k + i)\tau - (k - i)y]_+^{k-1} \left[s - (n - d + j)t - (d - j)\frac{m}{n}y \right]_+^{d-1} dy.$$

Let us put $a_j = s - (n - d + j)t$ and $b_i = z - (m - k + i)\tau$. Thus we can write

$$\begin{aligned} (10) \quad I_{ij} &= a_j^{d-1} b_i^{k-2} \int_0^{\zeta_{ij}} \left(1 - \frac{k-i}{b_i} y\right)^{k-2} \left(1 - \frac{d-j}{a_j} \frac{m}{n} y\right)^{d-1} dy \\ &= \frac{a_j^{d-1} b_i^{k-1}}{k-i} \sum_{r=0}^{d-1} (-1)^r \binom{d-1}{r} \left(\frac{m}{n} \frac{d-j}{k-i} \frac{b_i}{a_j}\right)^r B_{\zeta_{ij}}(k-1, r+1), \end{aligned}$$

where

$$\zeta_{ij} = \left[\min \left(\frac{b_i}{k-i}, \frac{n}{m} \frac{a_j}{d-j} \right) \right]_+, \quad \xi_{ij} = \left[\min \left(1, \frac{n}{m} \frac{k-i}{d-j} \frac{a_j}{b_i} \right) \right]_+,$$

and $B_{\xi_{ij}}(k-1, r+1)$ is the incomplete beta-function.

Notice that $I_{ij} = 0$ only if $a_j > 0$ and $b_i > 0$.

From (9) and (10) we obtain the estimator of P in the case where $d \geq 2$ and $k \geq 2$:

$$\begin{aligned} \hat{P} &= \frac{k-1}{V_d^n(s, t) V_k^m(z, \tau)} \sum_{r=0}^{d-1} (-1)^r \binom{d-1}{r} \sum_{i=0}^{k-1} (-1)^i \binom{k}{i} b_i^{k-1+r} \times \\ &\quad \times \sum_{j=0}^{d-1} (-1)^j \binom{d}{j} a_j^{d-1-r} \left[\frac{m(d-j)}{n(k-i)} \right]^r B_{\xi_{ij}}(k-1, r+1). \end{aligned}$$

If $d = 0$ and $k = 0$, we have

$$\hat{P} = \begin{cases} 1 & \text{if } m\tau \leq nt, \\ 0 & \text{if } m\tau > nt. \end{cases}$$

This follows immediately from (5).

It is easy to verify that if $d = 0, 1$ and $k \geq 1$, then

$$\hat{P} = \begin{cases} 1 & \text{if } z - (m-k)\tau < \frac{m}{n}(s - (n-1)t), \\ 1 - \frac{V_k^m(z - n(s - (n-1)t), \tau - (n/m)(s - (n-1)t))}{V_k^m(z, \tau)} & \text{otherwise.} \end{cases}$$

Similarly, if $k = 0, 1$ and $d \geq 1$, then

$$\hat{P} = \begin{cases} 0 & \text{if } s - (n-d)t \leq \frac{m}{n}(z - (m-1)\tau), \\ 1 - \frac{V_d^n(s - m(z - (m-1)\tau), t - (m/n)(z - (m-1)\tau))}{V_d^n(s, t)} & \text{otherwise.} \end{cases}$$

Remark. By the same method one can derive the estimators of P when the samples are obtained from two different life testing plans. These estimators are UMVUE's except those cases where a sample is obtained from plan (d). For example, the UMVUE of P , when the elements X are tested according to plan (c) and the elements Y are tested according to plan (b), is of the form

$$\hat{P} = \begin{cases} \sum_{j=0}^d (-1)^j \frac{d!(q-1)!}{(d-j)!(q-1+j)!} \left[\frac{mY_{(q)}}{nt} \right]^j & \text{if } mY_{(q)} \leq nt, \\ 1 - \sum_{j=0}^{q-1} (-1)^j \frac{(q-1)!d!}{(q-1-j)!(d+j)!} \left[\frac{nt}{mY_{(q)}} \right]^j & \text{otherwise.} \end{cases}$$

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ESTYMACJA $P\{Y < X\}$ W PRZYPADKU WYKŁADNICZYM

STRESZCZENIE

W pracy rozpatrzony jest problem nieobciążonej estymacji prawdopodobieństwa $P\{Y < X\}$, gdzie X i Y są niezależnymi zmiennymi losowymi o rozkładach wykładniczych z parametrami odpowiednio λ i μ , na podstawie prób otrzymanych w czterech planach badania: (a) bez odnowy i z czasem badania trwającym do chwili r -tej awarii, (b) z odnową i do chwili r -tej awarii, (c) z odnową i do ustalonej chwili t_0 , (d) bez odnowy i do ustalonej chwili t_0 . Otrzymano estymatory nieobciążone, będące funkcjami minimalnych statystyk dostatecznych. W przypadkach (a), (b) i (c) są one estymatorami nieobciążonymi z jednostajnie minimalną wariancją.